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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
Submitted to *Mathematics of Operations Research*
ISSN 0364-765X, EISSN 1526-5471

Well-posedness and Sensitivity Analysis of a Fluid Model for Multiclass Many-Server Queues with Abandonment Under Global FCFS Discipline

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Abstract. In this paper, under mild conditions on the arrival, service and patience time distributions, we establish the well-posedness of the fluid model of a multiclass many-server queueing model with differentiated service and patience times operated under the global FCFS service discipline. In particular, the well-posedness of the fluid model is established through the study of the existence and uniqueness of fixed points of certain functional map of Volterra type. In addition, by showing a local Lipschitz property of this functional map as a functional of the initial data to the fluid model, we also perform a sensitivity analysis on the fluid model.

Key words: multiclass many-server queues, $G/GI/N+GI$ queue, global FCFS service discipline, fluid model equations, abandonment, well-posedness, sensitivity analysis

1. Introduction In the past twenty years, there has been tremendous attention to the study of many-server queueing systems with customer abandonment due to its applications to telephone contact centers and (more generally) customer contact centers; see, e.g., [B(2005)], [GKM(2003)], [GMR(2002)], [G(2006)], and references therein. In this paper, we consider a multiclass many-server queueing system, also known as multiclass $G_t/GI/N+GI$ model. In the system, there are N parallel identical servers, and K classes of customers arrive with (possibly) time-dependent differentiated arrival rates and customers from each class require i.i.d. service times, and have i.i.d. patience times. The service times and the patience times of all customer classes are assumed to be mutually independent with possibly differentiated distributions and are also independent of the arrival process of each customer class. Customers are assumed to abandon from the system if the time spent waiting in queue reaches their patience time. The service discipline here is global first-come-first-serve (global FCFS), that is, a server will serve the oldest customer waiting in queue irrespective of its customer class at the moment when it becomes available, and non-idling, that is, no server will idle whenever there is a customer of any class in queue. Such many-server systems are typically studied under the so-called, many-server heavy-traffic regime, where the arrival rates and the number of servers get large, while service and patience time distributions are fixed. When the mean arrival rates and the number of servers grow proportionally (a law-of-large-numbers scaling), the scaling limit of the system state descriptor, which represents the state of the system, is described in terms of a set of so-called *fluid model equations*. To justify the solution to the fluid model equations as the scaling limit of the system state descriptor, it is important to establish the well-posedness of fluid model equations first, that is, to show that the fluid model equations admit a unique solution.

In the single customer class setting, the system is simply the $G_t/GI/N+GI$ model that was proposed in Whitt [WW(2006)] and later studied in Kang and Ramanan [KR(2010)] and then in Wash-Zuñiga [Z(2014)]. In those papers, the existence of solutions to the fluid model equations was established indirectly by first showing a sequence of fluid scaled measure-valued state descriptors for the $G_t/GI/N+GI$ system is tight and then verifying any weak limit of the sequence satisfies the fluid model equations. In Kang [Kang(2014)], under mild conditions on the arrival rate and the service and patience time distributions, a direct proof of the existence and uniqueness of solutions to the fluid model equations is established using two non-linear functional integral equations, one of which is of the Volterra type. In the multiclass setting, the system operated under a fixed non-preemptive priority policy instead of the global FCFS policy was considered in Atar, Kaspi and Shimkin [AKS(2014)]. In that paper, the uniqueness of solutions to the fluid model equations was proved using a certain Skorokhod map and the existence of solutions was also established in a similar manner as in Kang and Ramanan [KR(2010)].

Motivated by Kang [Kang(2014)], in this paper, we focus on the (measure-valued) fluid model equations (see Definition 1) as the fluid analog of the multiclass $G_t/GI/N+GI$ model under the global FCFS policy. Any solution $(\overline{X},\overline{\nu},\overline{\eta})$ to the fluid model equations comprises three components, where $\overline{X}=(\overline{X}_k,k\in\mathcal{K})$, $\overline{\nu}=(\overline{\nu}^k,k\in\mathcal{K})$, and $\overline{\eta}=(\overline{\eta}^k,k\in\mathcal{K})$ (here $\mathcal{K}\doteq\{1,2,\cdots,K\}$ represents the set of customer class indexes and K is the total number of different customer classes). For each $t\geq 0$ and $k\in\mathcal{K}$, $\overline{X}_k(t)$ represents the fluid analog of the total number of customers of class k in the system at time t, $\overline{\nu}^k_t$ is the fluid analog of a measure-valued process at time t that keeps track of the age (the amount of time elapsed since the customer entered service) of customers of class

k in service, $\overline{\eta}_t^k$ is the fluid analog of a measure-valued process at time t that keeps track of the times elapsed since entry into the system of all customers of class k (whether or not they have entered service), and not only of customers of class k currently in the queue. This fluid model equations is a natural extension of the fluid model equations for a single class $G_t/GI/N+GI$ queue in Kang and Ramanan [KR(2010)], but the analysis of its well-posedness is more involved than the analysis in Kang [Kang(2014)] for the single customer class setting not only due to the coupling between the service dynamics and queue dynamics of each customer class, but also due to the coupling across all customer classes. So the non-idling condition (see (2.10)) holds not at each customer class level, but at the global level due to the global FCFS policy. Our first main result is the well-posedness of the fluid model equations (see Theorem 3.1 and Theorem 3.2). We explore the connection between $\sum_{k\in\mathcal{K}}\overline{X}_k$, the fluid analog of the total number of customers of all classes in the system, and a functional map Λ of the Volterra type (see (3.31)) and establish that the functional map Λ admits a unique fixed point. This enables us to derive the uniqueness of solutions to the fluid model equations. On the other hand, one would expect that the unique fixed point of Λ will naturally lead to a solution to the fluid model equations, hence gives us the existence result. Unfortunately, such an argument is not straightforward. The main issue is that certain processes defined from the unique fixed point of Λ do not readily have the monotonicity property needed (see the discussion above Proposition 3). We overcome this issue and hence establish the existence of solutions to the fluid model equations under an additional assumption on the initial data (see Theorem 3.2). By using the well-posedness result established here, one can justify rigorously using a similar argument as in Kang and Ramanan [KR(2010)] that the fluid model equations stated in this paper is indeed the fluid limit of the multiclass $G_t/GI/N+GI$ model under the global FCFS policy. Also when $K=1$, that is, only one customer class present in the system, the proof of the well-posedness result here provides an alternative approach to the results in Kang [Kang(2014)].

The second main result in this paper is the sensitivity analysis on the fluid model equations (see Theorem 4.3). We would like to know how the unique solution to the fluid model equations reacts to small perturbations on the input data to the fluid model equations, which includes the arrival rates of customers of all classes and the initial state of the system. For this, we establish a local Lipschitz property on the functional map Λ (see Proposition 5). This property gives us insight on the impact of small perturbations on the input data to $\sum_{k\in\mathcal{K}}\overline{X}_k$, the fluid analog of the total number of customers of all classes in the system, which, in turn, on the unique solution to the fluid model equations.

The two main results established in this paper can potentially be used to establish well-posedness of fluid model equations for many-server networks with multiclass $G_t/GI/N + GI$ queues as building blocks. For example, in Kang and Pang [KP(2024)], a fluid model for a non-Markovian many-server queueing network with customer abandonment and Markov routing is considered. In that network model, there are a fixed number of service stations, each of which has either finitely or infinitely many parallel servers, a single queue and its own designated customer class. Customers enter the system at a service station, and receive service immediately if there is a free server at the station, and join the queue at the station otherwise. Upon service completion, a customer is immediately routed to one of the service stations or leaves the system following a Markovian routing mechanism, independent of other customers. Customers can be out of patience and leave the system (without reentry) when they are waiting in the queue before receiving service. Externally arrived and internally routed customers at each service station are served in the non-idling, First-Come-First-Serve (FCFS) discipline. Thus, each service station behaves like a $G_t/GI/N+GI$ queue with two customer classes under the global FCFS policy. We believe that our results here can be used to give an alternative proof of the well-posedness of fluid model in Kang and Pang [KP(2024)] under a different set of assumptions on the arrival processes and service and patience time distributions.

1.1. Notation and Terminology The following notation will be used throughout the paper. $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{R}$ is set of real numbers, $\mathbb{R}_+$ is the set of non-negative real numbers. For $a, b \in \mathbb{R}$, $a \vee b$ denotes the maximum of a and b, $a \wedge b$ the minimum of a and b and the short-hand a^+ is used for $a \vee 0$. Given a set B, $\mathbb{1}_B$ denotes the indicator function of the set B (that is, $\mathbb{1}_B(x) = 1$ if $x \in B$ and $\mathbb{1}_B(x) = 0$ otherwise). The constant functions $f \equiv 1$ and $f \equiv 0$ will be represented by the symbols $\mathbf{1}$ and $\mathbf{0}$, respectively. Given a non-decreasing, right continuous function f having left limits on $\mathbb{R}_+$, f^{-1} denotes the inverse function of f in the sense that

$$f^{-1}(y) = \inf\{x \geq 0 : f(x) \geq y\}, \tag{1.1}$$

with the convention that infimum over an empty set is ∞. The space of Radon measures on a Polish space E, endowed with the Borel σ-algebra, is denoted by $\mathcal{M}(E)$, while $\mathcal{M}_F(E)$ is the subspace of finite non-negative measures in $\mathcal{M}(E)$. The symbol δ_x will be used to denote the measure with unit mass at the point x and, with some abuse of notation, we will also use $\mathbf{0}$ to denote the identically zero Radon measure on E. When E is an interval, say $[0, H)$, for notational conciseness, we will often write $\mathcal{M}_F[0, H)$ instead of $\mathcal{M}_F([0, H))$. We say a measure μ is continuous at x if and only if

$\mu(\{x\}) = 0$ and μ is continuous on $\mathbb{R}_+$ if μ is continuous at each $x \in \mathbb{R}_+$. For any Borel measurable function $f : [0,H) \to \mathbb{R}$ that is integrable with respect to $\xi \in \mathcal{M}[0,H)$, we often use the short-hand notation

$$\langle f, \xi \rangle \doteq \int_{[0,H)} f(x)\, \xi(dx).$$

Let $\mathcal{I}_0(\mathbb{R}_+)$ be the set of non-decreasing, right continuous functions f having left limits on $\mathbb{R}_+$ with $f(0) = 0$. Let $C(\mathbb{R}_+)$ be the set of continuous functions on $\mathbb{R}_+$, $C_b(\mathbb{R}_+)$ be the subset of $C(\mathbb{R}_+)$ of functions that are bounded, $C[a,b]$ be the set of continuous functions on $[a,b]$.

2. Fluid Model Equations

In this section we state fluid model equations as a fluid analog of the multiclass $G_t/GI/N + GI$ queues and prove some properties of three key auxiliary processes stated in the definition of the fluid model equations.

For a probability cumulative distribution function G on $\mathbb{R}_+$ with density g, the right end of the support H of g is defined as $H \doteq \sup\{x \in \mathbb{R}_+ : \ g(x) > 0\}$, then the hazard rate function h on $\mathbb{R}_+$ is defined as $h(x) \doteq g(x)/\bar{G}(x)$ with the convention that $0/0$ is interpreted as 0 when $x \geq H$ if $H < \infty$, where $\bar{G}(x) \doteq 1 - G(x)$. For each $k \in \mathcal{K}$, let G_k^r with density g_k^r and G_k^s with density g_k^s denote the probability cumulative distribution of the patience times and the probability cumulative distribution of the service times of customers of class k, respectively, and H_k^r and H_k^s denote the right ends of the supports of g_k^r and g_k^s, respectively.

Define the following space of feasible input data for the fluid model equations

$$\mathcal{S}_0 \doteq \left\{ \begin{array}{l} (e,x,\nu,\eta) \in \mathcal{I}_0(\mathbb{R}_+)^K \times \mathbb{R}_+^K \times \Pi_{k\in\mathcal{K}} \mathcal{M}_F[0,H_k^s) \times \Pi_{k\in\mathcal{K}} \mathcal{M}_F[0,H_k^r) : \\ 1 - \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \nu^k \rangle = [1 - \sum_{k\in\mathcal{K}} x_k]^+, \ [\sum_{k\in\mathcal{K}} x_k - 1]^+ \leq \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \eta_0^k \rangle \end{array} \right\}. \tag{2.1}$$

Definition 1. **(Fluid Model Equations)** The càdlàg function $(\overline{X}, \overline{\nu}, \overline{\eta})$ defined on $\mathbb{R}_+$ such that for each $t \in \mathbb{R}_+$, $\overline{X}(t) = (\overline{X}_k(t), k \in \mathcal{K}) \in \mathbb{R}_+^K$, $\overline{\nu}_t = (\overline{\nu}_t^k, k \in \mathcal{K}) \in \Pi_{k\in\mathcal{K}} \mathcal{M}_F[0,H_k^s)$, and $\overline{\eta}_t = (\overline{\eta}_t^k, k \in \mathcal{K}) \in \Pi_{k\in\mathcal{K}} \mathcal{M}_F[0,H_k^r)$ is said to solve the fluid model equations associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0) \in \mathcal{S}_0$ and the hazard rate functions $h_k^r \doteq g_k^r/\bar{G}_k^r$ and $h_k^s \doteq g_k^s/\bar{G}_k^s$, $k \in \mathcal{K}$, if and only if for every $t \in \mathbb{R}_+$ and $k \in \mathcal{K}$,

$$\int_0^t \langle h_k^r, \overline{\eta}_s^k \rangle\, ds < \infty, \qquad \int_0^t \langle h_k^s, \overline{\nu}_u^k \rangle\, du < \infty, \tag{2.2}$$

and the following relations are satisfied: for every $f \in C_b(\mathbb{R}_+)$,

$$\int_{[0,H_k^s)} f(x)\, \overline{\nu}_t^k(dx) = \int_{[0,H_k^s)} f(x+t) \frac{\bar{G}_k^s(x+t)}{\bar{G}_k^s(x)}\, \overline{\nu}_0^k(dx) + \int_0^t f(t-s) \bar{G}_k^s(t-s)\, d\overline{L}_k(s), \tag{2.3}$$

where

$$\overline{L}_k(t) = \langle \mathbf{1}, \overline{\nu}_t^k \rangle - \langle \mathbf{1}, \overline{\nu}_0^k \rangle + \int_0^t \langle h_k^s, \overline{\nu}_u^k \rangle \, du; \tag{2.4}$$

$$\int_{[0,H_k^r)} f(x)\, \overline{\eta}_t^k(dx) = \int_{[0,H_k^r)} f(x+t) \frac{\bar{G}_k^r(x+t)}{\bar{G}_k^r(x)} \overline{\eta}_0^k(dx) + \int_0^t f(t-s)\bar{G}_k^r(t-s)\, d\overline{E}_k(s); \tag{2.5}$$

$$\overline{Q}_k(t) = \overline{X}_k(t) - \langle \mathbf{1}, \overline{\nu}_t^k \rangle; \tag{2.6}$$

$$\overline{R}_k(t) = \int_0^t \left(\int_0^{\sum_{k\in\mathcal{K}} \overline{Q}_k(w)} h_k^r((\overline{F}_w)^{-1}(u)) d\overline{F}_w^k((\overline{F}_w)^{-1}(u)) \right) dw, \tag{2.7}$$

where $\overline{F}_w^k(x) \doteq \overline{\eta}_w^k[0,x]$, $\overline{F}_w(x) \doteq \sum_{k\in\mathcal{K}} \overline{F}_w^k(x)$*;*

$$\overline{Q}_k(t) = \overline{F}_t^k \left((\overline{F}_t)^{-1} \left(\sum_{k\in\mathcal{K}} \overline{Q}_k(t) \right) \right); \tag{2.8}$$

$$\overline{X}_k(t) = \overline{X}_k(0) + \overline{E}_k(t) - \int_0^t \langle h_k^s, \overline{\nu}_u^k \rangle \, du - \overline{R}_k(t); \tag{2.9}$$

and the global non-idling condition

$$1 - \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \overline{\nu}_t^k \rangle = \left[1 - \sum_{k\in\mathcal{K}} \overline{X}_k(t) \right]^+. \tag{2.10}$$

From the definition of the fluid model equations, we obtain the following two additional balance equations: from (2.6) and (2.10),

$$\sum_{k\in\mathcal{K}} \overline{Q}_k(t) = \left[\sum_{k\in\mathcal{K}} \overline{X}_k(t) - 1 \right]^+, \tag{2.11}$$

and from (2.4), (2.6) and (2.9), for each $k \in \mathcal{K}$,

$$\overline{Q}_k(0) + \overline{E}_k(t) = \overline{Q}_k(t) + \overline{L}_k(t) + \overline{R}_k(t). \tag{2.12}$$

Remark 1. Note that (2.3) and (2.5) are required to be satisfied only for bounded continuous functions in Definition 1. But by using a standard approximation argument, namely representing indicators of finite open intervals in $\mathbb{R}_+$ as monotone limits of continuous functions with compact support and appealing to the monotone class theorem, it follows that both equations in fact hold for any bounded measurable or nonnegative measurable f. In particular, these equations hold with $f = h_k^s$ in (2.3) and $f = h_k^r$ in (2.5). The latter fact is used several times in this paper.

We first state a simple result on the action of time-shifts on solutions to the fluid equations. For this, we need the following notation: for any $t \in \mathbb{R}_+$,

$$\overline{E}^{[t]} \doteq \overline{E}(t+\cdot) - \overline{E}(t), \qquad \overline{L}^{[t]} \doteq \overline{L}(t+\cdot) - \overline{L}(t), \qquad \overline{X}^{[t]} \doteq \overline{X}(t+\cdot), \qquad \overline{\nu}^{[t]} \doteq \overline{\nu}_{t+\cdot},$$

$$\overline{R}^{[t]} \doteq \overline{R}(t+\cdot) - \overline{R}(t), \qquad \overline{\eta}^{[t]} \doteq \overline{\eta}_{t+\cdot}, \qquad \overline{Q}^{[t]} \doteq \overline{Q}(t+\cdot).$$

LEMMA 1. *Suppose the càdlàg function $(\overline{X}, \overline{\nu}, \overline{\eta})$ defined on $\mathbb{R}_+$ and taking values in $\mathbb{R}_+^K \times \Pi_{k\in\mathcal{K}}\mathcal{M}_F[0,H_k^s) \times \Pi_{k\in\mathcal{K}}\mathcal{M}_F[0,H_k^r)$ solves the fluid equations associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0) \in \mathcal{S}_0$, then $(\overline{X}^{[t]}, \overline{\nu}^{[t]}, \overline{\eta}^{[t]})$ solves the fluid equations associated with $(\overline{E}^{[t]}, \overline{X}(t), \overline{\nu}_t, \overline{\eta}_t) \in \mathcal{S}_0$, where $\overline{L}^{[t]}, \overline{R}^{[t]}, \overline{Q}^{[t]}$ are the corresponding processes that satisfy (2.4), (2.6), (2.7), (2.8) with $\overline{\nu}^{[t]}$, $\overline{\eta}^{[t]}$ and $\overline{X}^{[t]}$ in place of $\overline{\nu}$, $\overline{\eta}$ and $\overline{X}$.*

Proof Fix $t \in \mathbb{R}_+$. It is easy to see that $(\overline{X}^{[t]}, \overline{\nu}^{[t]}, \overline{\eta}^{[t]})$ satisfies (2.2), (2.4), (2.6), (2.7), (2.8), (2.9) and (2.10) by a rewriting of those fluid equations and an application of change of variables. For the rest of the fluid equations, fix $s \in \mathbb{R}_+$ and $k \in \mathcal{K}$. For each $f \in C_b(\mathbb{R}_+)$,

$$\begin{aligned}\int_{[0,H_k^s)} f(x)\,\overline{\nu}_s^{[t],k}(dx) &= \int_{[0,H_k^s)} f(x)\,\overline{\nu}_{t+s}^k(dx) \\ &= \int_{[0,H_k^s)} f(x+t+s)\frac{\bar{G}_k^s(x+t+s)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx) \\ &\quad + \int_0^{t+s} f(t+s-u)\bar{G}_k^s(t+s-u)\,d\overline{L}_k(u). \end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned}&\int_{[0,H_k^s)} f(x+s)\frac{\bar{G}_k^s(x+s)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^{[t],k}(dx) + \int_0^s f(s-u)\bar{G}_k^s(s-u)\,d\overline{L}_k^{[t]}(u) \qquad (2.14)\\ &= \int_{[0,H_k^s)} f(x+s)\frac{\bar{G}_k^s(x+s)}{\bar{G}_k^s(x)}\,\overline{\nu}_t^k(dx) + \int_0^s f(s-u)\bar{G}_k^s(s-u)\,d\overline{L}_k(t+u).\end{aligned}$$

Since the function $f(s+\cdot)\dfrac{\bar{G}_k^s(s+\cdot)}{\bar{G}_k^s(\cdot)} \in C_b(\mathbb{R}_+)$, it follows from (2.3) for $\overline{\nu}_t^k$ with $f(s+\cdot)\dfrac{\bar{G}_k^s(s+\cdot)}{\bar{G}_k^s(\cdot)}$ in place of $f(\cdot)$ that the first term on the right-hand side of (2.14) satisfies

$$\begin{aligned}
&\int_{[0,H_k^s)} f(x+s)\frac{\bar{G}_k^s(x+s)}{\bar{G}_k^s(x)}\,\overline{\nu}_t^k(dx)\\
&=\int_{[0,H_k^s)} f(x+t+s)\frac{\bar{G}_k^s(x+t+s)}{\bar{G}_k^s(x+t)}\frac{\bar{G}_k^s(x+t)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)\\
&\qquad+\int_0^t f(t-u+s)\frac{\bar{G}_k^s(t-u+s)}{\bar{G}_k^s(t-u)}\bar{G}_k^s(t-u)\,d\overline{L}_k(u)\\
&=\int_{[0,H_k^s)} f(x+t+s)\frac{\bar{G}_k^s(x+t+s)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)+\int_0^t f(t-u+s)\bar{G}_k^s(t-u+s)\,d\overline{L}_k(u).
\end{aligned}$$

Applying a change of variables on the second term on the right-hand side of (2.14), we have that

$$\int_0^s f(s-u)\bar{G}_k^s(s-u)\,d\overline{L}_k(t+u)=\int_t^{t+s} f(s+t-u)\bar{G}_k^s(s+t-u)\,d\overline{L}_k(u).$$

Adding the above two displays together and comparing the sum with (2.13) and (2.14), we see that $(\overline{X}^{[t]},\overline{\nu}^{[t]},\overline{\eta}^{[t]})$ satisfies (2.3). A similar argument shows that $(\overline{X}^{[t]},\overline{\nu}^{[t]},\overline{\eta}^{[t]})$ satisfies (2.5). □

We now define a function $\overline{\chi}(\cdot)$ on $\mathbb{R}_+$ as

$$\overline{\chi}(t)\doteq(\overline{F}_t)^{-1}\left(\sum_{k\in\mathcal{K}}\overline{Q}_k(t)\right),\quad t\in\mathbb{R}_+. \tag{2.15}$$

For each time $t\in\mathbb{R}_+$, the quantity $\overline{\chi}(t)$ can be interpreted as the fluid analog of the waiting time of the oldest customer among all classes of customers in queue at time t. We establish a basic property of $\overline{\chi}(\cdot)$.

LEMMA 2. *Suppose that $\overline{E}(\cdot)$ is absolutely continuous with a.e. derivative $\overline{\lambda}(\cdot)=(\overline{\lambda}_1(\cdot),\ldots,\overline{\lambda}_K(\cdot))$, then for each $0\le s<t$, $\overline{\chi}(t)\le\overline{\chi}(s)+(t-s)$.*

Proof By using Lemma 1, without loss of generality, we may assume $s=0$, that is, we show that $\overline{\chi}(t)\le\overline{\chi}(0)+t$ for each $t>0$. For this, by (2.15), it is sufficient to show that

$$\sum_{k\in\mathcal{K}}\overline{Q}_k(t)\le\overline{F}_t(\overline{\chi}(0)+t)\text{ for each }t>0.$$

Consider the function

$$f(t) \doteq F_t(\overline{\chi}(0)+t) - \sum_{k\in\mathcal{K}} \overline{Q}_k(t), \; t \in \mathbb{R}_+. \tag{2.16}$$

Note that $f(0) = F_0(\overline{\chi}(0)) - \sum_{k\in\mathcal{K}} \overline{Q}_k(0) \geq 0$ by (2.15) and the definition of $(\overline{F}_0)^{-1}$ in (1.1) with $\overline{F}_0$ defined in (2.7) in place of f. Since $\overline{E}(\cdot)$ is assumed to be absolutely continuous, then $\overline{X}(\cdot)$ and hence $\sum_{k\in\mathcal{K}} \overline{Q}_k(\cdot)$ and $\sum_{k\in\mathcal{K}} \overline{L}_k(\cdot)$ are also absolutely continuous by (2.9), (2.7), (2.11) and (2.12). In particular, by using the same argument as in the proof of Theorem 3.5 of [KaR(2011)] (cf. the proof of (3.12) of [KaR(2011)]) using the global non-idling condition (2.10), the a.e derivative $\left(\sum_{k\in\mathcal{K}} \overline{L}\right)'(\cdot)$ of $\left(\sum_{k\in\mathcal{K}} \overline{L}\right)(\cdot)$ satisfies for a.e. $t \in \mathbb{R}_+$,

$$\left(\sum_{k\in\mathcal{K}} \overline{L}_k\right)'(t) = \begin{cases} \sum_{k\in\mathcal{K}} \overline{\lambda}_k(t) & \text{if } \sum_{k\in\mathcal{K}} \overline{X}_k(t) < 1, \\ \sum_{k\in\mathcal{K}} \overline{\lambda}_k(t) \wedge \sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_t^k \rangle & \text{if } \sum_{k\in\mathcal{K}} \overline{X}_k(t) = 1, \\ \sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_t^k \rangle & \text{if } \sum_{k\in\mathcal{K}} \overline{X}_k(t) > 1. \end{cases} \tag{2.17}$$

On the other hand, by the definition of $\overline{F}_t$ in (2.7), (2.5) and Remark 1,

$$\begin{aligned} \overline{F}_t(\overline{\chi}(0)+t) = \sum_{k\in\mathcal{K}} \overline{\eta}_t^k[0, \overline{\chi}(0)+t] &= \sum_{k\in\mathcal{K}} \int_{[0,H_k^r)} \mathbf{1}_{[0,\overline{\chi}(0)]}(x) \frac{\bar{G}_k^r(x+t)}{\bar{G}_k^r(x)} \overline{\eta}_0^k(dx) \\ &\quad + \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^r(t-s) \overline{\lambda}_k(s)\, ds. \end{aligned}$$

Thus $\overline{F}_t(\overline{\chi}(0)+t)$, as a function of t, is absolutely continuous with a.e derivative

$$\begin{aligned} \left(\overline{F}_t(\overline{\chi}(0)+t)\right)' &= -\sum_{k\in\mathcal{K}} \int_{[0,\overline{\chi}(0)]} \frac{g_k^r(x+t)}{\bar{G}_k^r(x)} \overline{\eta}_0^k(dx) + \sum_{k\in\mathcal{K}} \overline{\lambda}_k(t) - \sum_{k\in\mathcal{K}} \int_0^t g_k^r(t-s) \overline{\lambda}_k(s)\, ds \\ &= \sum_{k\in\mathcal{K}} \overline{\lambda}_k(t) - \sum_{k\in\mathcal{K}} \int_{[0,\overline{\chi}(0)+t]} h_k^r(x) d\overline{F}_t^k(x), \end{aligned} \tag{2.18}$$

where the last equality follows from (2.5) with $f = \mathbf{1}_{[0,\overline{\chi}(0)+t]} h_k^r$ for each $k \in \mathcal{K}$. Then f in (2.16) is also absolutely continuous by its definition.

We now show that f has the following property: for each $t > 0$,

$$f(t) \leq 0 \text{ implies } f'(t) \geq 0. \tag{2.19}$$

To prove this property, fix $t > 0$ such that $f(t) \le 0$, then

$$\sum_{k\in\mathcal{K}} \overline{Q}_k(t) \ge \overline{F}_t(\overline{\chi}(0)+t) > 0. \tag{2.20}$$

Thus, by (2.11), $\sum_{k\in\mathcal{K}} \overline{X}_k(t) = \sum_{k\in\mathcal{K}} \overline{Q}_k(t) + 1 > 1$. Thus, $\left(\sum_{k\in\mathcal{K}} \overline{X}_k\right)'(t) = \left(\sum_{k\in\mathcal{K}} \overline{Q}_k\right)'(t)$ and $\left(\sum_{k\in\mathcal{K}} \overline{L}_k\right)'(t) = \sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_u^k\rangle$ by (2.17). Using (2.9), (2.20), an application of change of variables and (2.18), we have that

$$\begin{aligned}
\left(\sum_{k\in\mathcal{K}} \overline{X}_k\right)'(t) &= \sum_{k\in\mathcal{K}} \left(\overline{\lambda}_k(t) - \langle h_k^s, \overline{\nu}_u^k\rangle - \int_0^{\sum_{k\in\mathcal{K}} \overline{Q}_k(t)} h_k^r((\overline{F}_t)^{-1}(u))d\overline{F}_t^k((\overline{F}_t)^{-1}(u))\right) \\
&\le \sum_{k\in\mathcal{K}} \left(\overline{\lambda}_k(t) - \langle h_k^s, \overline{\nu}_u^k\rangle - \int_0^{\overline{F}_t(\overline{\chi}(0)+t)} h_k^r((\overline{F}_t)^{-1}(u))d\overline{F}_t^k((\overline{F}_t)^{-1}(u))\right) \\
&= \sum_{k\in\mathcal{K}} \left(\overline{\lambda}_k(t) - \sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_u^k\rangle - \int_{[0,\overline{\chi}(0)+t]} h_k^r(x)d\overline{F}_t^k(x)\right) \\
&= -\sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_u^k\rangle + \left(\overline{F}_t(\overline{\chi}(0)+t)\right)',
\end{aligned}$$

and then

$$\left(\overline{F}_t(\overline{\chi}(0)+t)\right)' - \left(\sum_{k\in\mathcal{K}} \overline{Q}_k\right)'(t) = \left(\overline{F}_t(\overline{\chi}(0)+t)\right)' - \left(\sum_{k\in\mathcal{K}} \overline{X}_k\right)'(t) \ge \sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_u^k\rangle.$$

It follows that $f'(t) \ge \sum_{k\in\mathcal{K}} \langle h_k^s, \overline{\nu}_u^k\rangle \ge 0$. Thus, we proved the property (2.19). For each $t > 0$, if $f(t) < 0$, then since $f(0) \ge 0$, let $\tau \doteq \sup\{u \le t : f(u) \ge 0\}$, then $0 \le \tau < t$ by the continuity of f. Then $f(\tau) = 0$ and $f(u) < 0$ for all $u \in (\tau, t]$. It follows that $f(t) = f(\tau) + \int_\tau^t f'(s)ds \ge f(\tau) = 0$, which is a contradiction. Thus, for all $t > 0$, $f(t) \ge 0$, that is, $\sum_{k\in\mathcal{K}} \overline{Q}_k(t) \le \overline{F}_t(\overline{\chi}(0)+t)$. □

We close this section by showing that for each $k \in \mathcal{K}$, $\overline{L}_k(\cdot)$ in Definition 1 is non-decreasing.

Lemma 3. *Suppose that $\overline{E}(\cdot)$ is absolutely continuous with a.e. derivative $\overline{\lambda}(\cdot) = (\overline{\lambda}_1(\cdot), \ldots, \overline{\lambda}_K(\cdot))$, then for each $k \in \mathcal{K}$, $\overline{L}_k(\cdot)$ in Definition 1 is non-decreasing and hence is absolutely continuous on $\mathbb{R}_+$.*

Proof Fix $k \in \mathcal{K}$ and $0 \le s < t < \infty$. Let $h \doteq t - s$. It follows from Lemma 2 that $\overline{\chi}(s+u) \le \overline{\chi}(s) + u$ for each $u \in [0, h]$. Since $\overline{\chi}(s+h) \le \overline{\chi}(s) + h$, then $\overline{\eta}_{s+h}^k(\overline{\chi}(s+h), \overline{\chi}(s)+h] \ge 0$. On the other hand, by Lemma 1, $(\overline{X}^{[s]}, \overline{\nu}^{[s]}, \overline{\eta}^{[s]})$ solves the fluid equations associated with $(\overline{E}^{[s]}, \overline{X}(s), \overline{\nu}_s, \overline{\eta}_s) \in \mathcal{S}_0$. Then, by Remark 1, $\overline{\eta}^k_{s+h} = \overline{\eta}^{[s],k}_h$ satisfies (2.5) with h, $\mathbf{1}_{[0,\overline{\chi}(s)+h]}(\cdot)$, $\overline{\eta}^k_s$ and $\overline{E}^{[s]}_k$ in place of t, f, $\overline{\eta}^k_0$ and $\overline{E}_k$, respectively. Combining this with (2.8), we have that

$$
\begin{aligned}
&\overline{\eta}^k_{s+h}(\overline{\chi}(s+h), \overline{\chi}(s)+h] \qquad (2.21)\\
&= \overline{\eta}^k_{s+h}[0, \overline{\chi}(s)+h] - \overline{\eta}^k_{s+h}[0, \overline{\chi}(s+h)]\\
&= \int_{[0,H^r_k)} \mathbf{1}_{[0,\overline{\chi}(s)+h]}(x+h) \frac{\bar{G}^r_k(x+h)}{\bar{G}^r_k(x)} \overline{\eta}^k_s(dx)\\
&\quad + \int_0^h \mathbf{1}_{[0,\overline{\chi}(s)+h]}(h-u) \bar{G}^r_k(h-u) \overline{\lambda}_k(s+u)\, du - \overline{Q}_k(s+h)\\
&= \int_{[0,H^r_k)} \mathbf{1}_{[0,\overline{\chi}(s)]}(x) \frac{\bar{G}^r_k(x+h)}{\bar{G}^r_k(x)} \overline{\eta}^k_s(dx) + \int_0^h \bar{G}^r_k(h-u) \overline{\lambda}_k(s+u)\, du - \overline{Q}_k(s+h)\\
&= \overline{Q}_k(s) - \int_{[0,H^r_k)} \mathbf{1}_{[0,\overline{\chi}(s)]}(x) \frac{G^r_k(x+h) - G^r_k(x)}{\bar{G}^r_k(x)} \overline{\eta}^k_s(dx)\\
&\quad + (\overline{E}_k(s+h) - \overline{E}_k(s)) - \int_0^h G^r_k(h-u) \overline{\lambda}_k(s+u)\, du - \overline{Q}_k(s+h).
\end{aligned}
$$

For each $u \in [0,h]$, since $\overline{\eta}^k_{s+u} = \overline{\eta}^{[s],k}_u$ also satisfies (2.5) with u, $\mathbf{1}_{[0,\overline{\chi}(s)+u]}(\cdot) h^r_k(\cdot)$, $\overline{\eta}^k_s$ and $\overline{E}^{[s]}_k$ in place of t, f, $\overline{\eta}^k_0$ and $\overline{E}_k$, respectively, we have that

$$
\begin{aligned}
&\int_{[0,H^r_k)} \mathbf{1}_{[0,\overline{\chi}(s)+u]}(x) h^r_k(x) \overline{\eta}^k_{s+u}(dx)\\
&= \int_{[0,H^r_k)} \mathbf{1}_{[0,\overline{\chi}(s)+u]}(x+u) \frac{g^r_k(x+u)}{\bar{G}^r_k(x)} \overline{\eta}^k_s(dx) + \int_0^u g^r_k(u-w) \overline{\lambda}_k(s+w)\, dw\\
&= \int_{[0,H^r_k)} \mathbf{1}_{[0,\overline{\chi}(s)]}(x) \frac{g^r_k(x+u)}{\bar{G}^r_k(x)} \overline{\eta}^k_s(dx) + \int_0^u g^r_k(u-w) \overline{\lambda}_k(s+w)\, dw.
\end{aligned}
$$

From this, (2.7) and an application of a change of variables, we can see that

$$
\begin{aligned}
&\overline{R}_k(s+h)-\overline{R}_k(s) &(2.22)\\
&=\int_0^h\int_{[0,H_k^r)}\mathbf{1}_{[0,\overline{\chi}(s+u)]}(x)h_k^r(x)\overline{\eta}_{s+u}^k(dx)du\\
&\le\int_0^h\int_{[0,H_k^r)}\mathbf{1}_{[0,\overline{\chi}(s)+u]}(x)h_k^r(x)\overline{\eta}_{s+u}^k(dx)du\\
&=\int_0^h\left(\int_{[0,H_k^r)}\mathbf{1}_{[0,\overline{\chi}(s)]}(x)\frac{g_k^r(x+u)}{\bar{G}_k^r(x)}\,\overline{\eta}_s^k(dx)\right)du+\int_0^h\left(\int_0^u g_k^r(u-w)\overline{\lambda}_k(s+w)\,dw\right)du\\
&=\int_{[0,H_k^r)}\mathbf{1}_{[0,\overline{\chi}(s)]}(x)\frac{G_k^r(x+h)-G_k^r(x)}{\bar{G}_k^r(x)}\,\overline{\eta}_s^k(dx)+\int_0^h G_k^r(h-u)\overline{\lambda}_k(s+u)\,du,
\end{aligned}
$$

where the last equality follows from the change of order of integration. Using (2.12), we have that

$$\overline{E}_k(s+h)-\overline{E}_k(s)=(\overline{Q}_k(s+h)-\overline{Q}_k(s))+(\overline{L}_k(s+h)-\overline{L}_k(s))+(\overline{R}_k(s+h)-\overline{R}_k(s)).$$

Combining this with (2.21) and (2.22) yield that

$$0\le\overline{\eta}_{s+h}^k(\overline{\chi}(s+h),\overline{\chi}(s)+h]\le\overline{L}_k(s+h)-\overline{L}_k(s).$$

Thus, $\overline{L}_k(\cdot)$ is non-decreasing on $\mathbb{R}_+$. Since $\sum_{k\in\mathcal{K}}\overline{L}_k(\cdot)$ is absolutely continuous and is non-decreasing on $\mathbb{R}_+$, then $\overline{L}_k$ is also absolutely continuous on $\mathbb{R}_+$. □

3. Well-posedness of Solutions to the Fluid Model Equations

In this section we establish the existence and uniqueness of solutions to the fluid model equations associated with $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)\in\mathcal{S}_0$ in Definition 1 under the following assumption.

ASSUMPTION 1. *The arrival process $\overline{E}=(\overline{E}_1,\dots,\overline{E}_K)$ is absolutely continuous with a.e. derivative $\overline{\lambda}(\cdot)=(\overline{\lambda}_1(\cdot),\dots,\overline{\lambda}_K(\cdot))$, for each $k\in\mathcal{K}$, $\overline{\eta}_0^k(\{x\})=0$ for all $x\in\mathbb{R}_+$, the hazard rate functions $\{h_k^r,\ k\in\mathcal{K}\}$ of the patience time distributions $\{G_k^r,\ k\in\mathcal{K}\}$ are a.e. locally bounded and the densities $\{g_k^s,\ k\in\mathcal{K}\}$ of the service time distributions $\{G_k^s,\ k\in\mathcal{K}\}$ satisfy that for each $k\in\mathcal{K}$, there is an integer $q_k\ge 1$ such that for each $S>0$,*

$$\int_0^S|g_k^s(s+h)-g_k^s(s)|^{q_k}ds\to 0\text{ as }h\downarrow 0. \tag{3.23}$$

Moreover, if h_k^r is unbounded on $[0,H_k^r)$ for some $k\in\mathcal{K}$, it is assumed that

$$\overline{\chi}(0)=\left(\overline{F}_0\right)^{-1}\left(\left[\sum_{k\in\mathcal{K}}\overline{X}_k(0)-1\right]^+\right)<\infty. \tag{3.24}$$

Remark 2. Since the hazard rate function of any distribution is only locally integrable and never integrable over its support, then when we assume that the hazard rate functions $\{h_k^r,\ k\in\mathcal{K}\}$ are a.e. locally bounded in Assumption 1, we implicitly assume that $H_k^r=\infty$ for all $k\in\mathcal{K}$. The condition (3.23) on the service time densities $\{g_k^s,\ k\in\mathcal{K}\}$ is not too restrictive. For example, if $\{g_k^s,\ k\in\mathcal{K}\}$ are right continuous, then they satisfies (3.23) with $q_k=1$, by a simple application of dominated convergence theorem. The condition (3.24) is assumed so that the hazard rate functions $\{h_k^r,\ k\in\mathcal{K}\}$ are bounded on $\overline{\chi}(0)+t$ for each $t\in\mathbb{R}_+$.

Fix $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)\in\mathcal{S}_0$ that satisfies Assumption 1. Suppose that $(\overline{X},\overline{\nu},\overline{\eta})$ is a solution to the fluid model equations associated with $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)$.

It follows from (2.11) and Lemma 2 that, for each $t\in\mathbb{R}_+$,

$$\left(\overline{F}_t\right)^{-1}\left(\left[\sum_{k\in\mathcal{K}}\overline{X}_k(t)-1\right]^+\right)\le\overline{\chi}(0)+t. \tag{3.25}$$

Note that by (2.9), for each $t\in\mathbb{R}_+$,

$$\sum_{k\in\mathcal{K}}\overline{X}_k(t)=\sum_{k\in\mathcal{K}}\overline{X}_k(0)+\sum_{k\in\mathcal{K}}\overline{E}_k(t)-\sum_{k\in\mathcal{K}}\int_0^t\langle h_k^s,\overline{\nu}_u^k\rangle\,du-\sum_{k\in\mathcal{K}}\overline{R}_k(t).$$

For each $k\in\mathcal{K}$ and $t\in\mathbb{R}_+$, by (2.3), Remark 1, an application of changing the order of integration and an application of integration by parts, and (2.12),

$$\begin{aligned}\int_0^t\langle h_k^s,\overline{\nu}_u^k\rangle\,du&=\int_0^t\left(\int_{[0,H_k^s)}\frac{g_k^s(x+u)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)\right)du+\int_0^t\int_0^u g_k^s(u-w)\,d\overline{L}_k(w)du\\&=\int_{[0,H_k^s)}\frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)+\int_0^t\overline{L}_k(u)g_k^s(t-u)du\\&=\int_{[0,H_k^s)}\frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)\\&\quad+\int_0^t(\overline{Q}_k(0)+\overline{E}_k(u)-\overline{Q}_k(u)-\overline{R}_k(u))g_k^s(t-u)du.\end{aligned}$$

From the above two displays, we can see that for each $t \in \mathbb{R}_+$,

$$\begin{aligned}\sum_{k\in\mathcal{K}} \overline{X}_k(t) &= \sum_{k\in\mathcal{K}} \overline{X}_k(0) + \sum_{k\in\mathcal{K}} \overline{E}_k(t) - \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx) \\ &\quad - \sum_{k\in\mathcal{K}} \int_0^t (\overline{Q}_k(0)+\overline{E}_k(u))g_k^s(t-u)du \\ &\quad + \sum_{k\in\mathcal{K}} \int_0^t (\overline{Q}_k(u)+\overline{R}_k(u))g_k^s(t-u)du - \sum_{k\in\mathcal{K}} \overline{R}_k(t).\end{aligned} \tag{3.26}$$

Let $\xi(\cdot)$ be the function on $\mathbb{R}_+$ defined by the input data $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0)$ as

$$\begin{aligned}\xi(t) &\doteq \sum_{k\in\mathcal{K}} \overline{X}_k(0) + \sum_{k\in\mathcal{K}} \overline{E}_k(t) - \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx) \\ &\quad - \sum_{k\in\mathcal{K}} \int_0^t (\overline{Q}_k(0)+\overline{E}_k(u))g_k^s(t-u)du \\ &= \sum_{k\in\mathcal{K}} \overline{X}_k(0) - \sum_{k\in\mathcal{K}} (\overline{X}_k(0) - \langle \mathbf{1}, \overline{\nu}_0^k\rangle)G_k^s(t) + \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u)d\overline{E}_k(u) \\ &\quad - \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx).\end{aligned} \tag{3.27}$$

Then $\xi(\cdot) \in C(\mathbb{R}_+)$, where recall that $C(\mathbb{R}_+)$ denotes the space of real-valued continuous functions on $\mathbb{R}_+$. By (2.7) and an application of integration by parts on the left-hand side of the display below, for each $k \in \mathcal{K}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}&\overline{R}_k(t) - \int_0^t \overline{R}_k(u)g_k^s(t-u)du \\ &= \int_0^t \bar{G}_k^s(t-u)\left(\int_0^{\sum_{k\in\mathcal{K}}\overline{Q}_k(u)} h_k^r((\overline{F}_u)^{-1}(w))d\overline{F}_u^k((\overline{F}_u)^{-1}(w))\right)du.\end{aligned} \tag{3.28}$$

Now, for each $k \in \mathcal{K}$ and $t, x \in \mathbb{R}_+$, define

$$\begin{cases} A^k(t,x) \doteq \overline{F}_t^k\left((\overline{F}_t)^{-1}\left([x-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)\right)\right), \\ B^k(t,x) \doteq \int_0^{[x-1]^+\wedge\overline{F}_t(\overline{\chi}(0)+t)} h_k^r((\overline{F}_t)^{-1}(u))d\overline{F}_t^k((\overline{F}_t)^{-1}(u)). \end{cases} \tag{3.29}$$

Then (3.26), (3.27), (3.28), (2.8), (2.11), (3.29) and (3.25) and Lemma 2 with $s=0$ together imply that for $t\in\mathbb{R}_+$,

$$\begin{aligned}\sum_{k\in\mathcal{K}}\overline{X}_k(t) &= \xi(t)+\sum_{k\in\mathcal{K}}\int_0^t\overline{Q}_k(u)g_k^s(t-u)du\\ &\quad -\sum_{k\in\mathcal{K}}\int_0^t\bar{G}_k^s(t-u)\left(\int_0^{\sum_{k\in\mathcal{K}}\overline{Q}_k(u)}h_k^r((\overline{F}_u)^{-1}(w))d\overline{F}_u^k((\overline{F}_u)^{-1}(w))\right)du\\ &= \xi(t)+\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-u)A^k\left(u,\sum_{k\in\mathcal{K}}\overline{X}_k(u)\right)du\\ &\quad -\sum_{k\in\mathcal{K}}\int_0^t\bar{G}_k^s(t-u)B^k\left(u,\sum_{k\in\mathcal{K}}\overline{X}_k(u)\right)du.\end{aligned}$$

We can see that $\sum_{k\in\mathcal{K}}\overline{X}_k(\cdot)$ is a solution to the following integral equation:

$$f(t)=\xi(t)-\sum_{k\in\mathcal{K}}\int_0^t\bar{G}_k^s(t-u)B^k(u,f(u))du+\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-u)A^k(u,f(u))du. \tag{3.30}$$

Let Λ be the following functional map defined on $C(\mathbb{R}_+)$ by

$$\Lambda(x)(t)\doteq\xi(t)-\sum_{k\in\mathcal{K}}\int_0^t\bar{G}_k^s(t-u)B^k(u,x(u))du+\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-u)A^k(u,x(u))du, \tag{3.31}$$

where $\xi(\cdot)$ is given by (3.27). Then $\sum_{k\in\mathcal{K}}\overline{X}_k(\cdot)$ is a fixed point of the functional map Λ, that is,

$$\sum_{k\in\mathcal{K}}\overline{X}_k=\Lambda\left(\sum_{k\in\mathcal{K}}\overline{X}_k\right).$$

Thus, the existence and uniqueness of solutions to the fluid model equations is linked to the well-posedness of the integral equation (3.30), i.e., the existence and uniqueness of the integral equation (3.30), or equivalently, to the existence and uniqueness of fixed points of the functional map Λ.

3.1. Well-posedness of the integral equation (3.30). In this subsection, we establish that the integral equation (3.30) admits a unique continuous solution $x(\cdot)$ on $\mathbb{R}_+$. To do this, we first establish some properties of A^k and B^k, $k\in\mathcal{K}$, defined in (3.29).

LEMMA 4. *Suppose that Assumption 1 holds. For each $k \in \mathcal{K}$, the functions A^k and B^k satisfy the following properties:*

1. *for each $t \in \mathbb{R}_+$, $A^k(t,x)$ and $B^k(t,x)$ are continuous in $x \in \mathbb{R}_+$,*
2. *for each $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}_+$, $|A^k(t,x) - A^k(t,y)| \leq |x-y|$,*
3. *for each $t \in \mathbb{R}_+$ and each $x, y \in \mathbb{R}_+$, $|B^k(t,x) - B^k(t,y)| \leq C_t^{r,k}|x-y|$, where $C_t^{r,k} \doteq \sup_{0\leq u\leq \overline{\chi}(0)+t} h_k^r(u) < \infty$.*

Proof For each $k \in \mathcal{K}$ and $t \in \mathbb{R}_+$, since, by Assumption 1, $\overline{E}$ is absolutely continuous and $\overline{\eta}_0(\{x\}) = 0$ for all $x \in \mathbb{R}_+$, then functions $\overline{F}_t^k(\cdot)$ and hence $\overline{F}_t(\cdot)$ are continuous, and then for each $x \in \mathbb{R}_+$,

$$\overline{F}_t\left((\overline{F}_t)^{-1}(x \wedge \overline{F}_t(\overline{\chi}(0)+t))\right) = x \wedge \overline{F}_t(\overline{\chi}(0)+t). \tag{3.32}$$

Note that $A^k(t,x)$ is increasing as a function of $x \in \mathbb{R}_+$ by its definition. Then, for each $x, y \in \mathbb{R}_+$ (without loss of generality, assume that $x > y$),

$$\begin{aligned}
&|A^k(t,x) - A^k(t,y)| \\
&= A^k(t,x) - A^k(t,y) \\
&\leq \sum_{k\in\mathcal{K}} \left(\overline{F}_t^k\left((\overline{F}_t)^{-1}\left([x-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)\right)\right) - \overline{F}_t^k\left((\overline{F}_t)^{-1}\left([y-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)\right)\right)\right) \\
&= \overline{F}_t\left((\overline{F}_t)^{-1}\left([x-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)\right)\right) - \overline{F}_t\left((\overline{F}_t)^{-1}\left([y-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)\right)\right) \\
&= [x-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t) - [y-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t) \\
&\leq [x-1]^+ - [y-1]^+ \leq x - y.
\end{aligned}$$

This establishes property (2) and also shows that $A^k(t,x)$ is continuous in x.

Next, note that $B^k(t,x)$ is increasing as a function of $x \in \mathbb{R}_+$ by its definition. For each $k \in \mathcal{K}$, each $t \in \mathbb{R}_+$ and each $x, y \in \mathbb{R}_+$ (without loss of generality, assume that $x > y$),

$$\begin{aligned}
|B^k(t,x) - B^k(t,y)| &= B^k(t,x) - B^k(t,y) \\
&= \int_{[y-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)}^{[x-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t)} h_k^r((\overline{F}_t)^{-1}(u)) d\overline{F}_t^k((\overline{F}_t)^{-1}(u)) \\
&\le C_t^{r,k} \left(\overline{F}_t^k \left((\overline{F}_t)^{-1} \left([x-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t) \right) \right) \right. \\
&\qquad \left. - \overline{F}_t^k \left((\overline{F}_t)^{-1} \left([y-1]^+ \wedge \overline{F}_t(\overline{\chi}(0)+t) \right) \right) \right) \\
&= C_t^{r,k} (A^k(t,x) - A^k(t,y)) \\
&\le C_t^{r,k} (x-y).
\end{aligned}$$

This establishes property (3) and also shows that $B^k(t,x)$ is continuous in x. Since both $A^k(t,x)$ and $B^k(t,x)$ are continuous in x, this establishes property (1). □

LEMMA 5. *Suppose that Assumption 1 holds. The functional map Λ in (3.31) is a mapping from $C(\mathbb{R}_+)$ into $C(\mathbb{R}_+)$.*

Proof Fix $x(\cdot) \in C(\mathbb{R}_+)$. We show that $\Lambda(x)(\cdot) \in C(\mathbb{R}_+)$. Note that $\xi(\cdot) \in C(\mathbb{R}_+)$ and it is also clear that for each $k \in \mathcal{K}$, $\int_0^t \bar{G}_k^s(t-u) B^k(u, x(u)) du$ as a function of t is in $C(\mathbb{R}_+)$. Thus, we just need to show that for each $k \in \mathcal{K}$, $\int_0^t g_k^s(t-u) A^k(u, x(u)) du$ as a function of t is in $C(\mathbb{R}_+)$. Fix $t \in \mathbb{R}_+$ and $k \in \mathcal{K}$. We first show that $\int_0^t g_k^s(t-u) A^k(u, x(u)) du$ as a function of t is right continuous at t. To show this, for each $0 < h < 1$,

$$\begin{aligned}
&\left| \int_0^{t+h} g_k^s(t+h-u) A^k(u,x(u)) du - \int_0^t g_k^s(t-u) A^k(u,x(u)) du \right| \qquad (3.33) \\
&\le \left| \int_0^{t+h} g_k^s(t+h-u) A^k(u,x(u)) du - \int_0^t g_k^s(t+h-u) A^k(u,x(u)) du \right| \\
&\quad + \left| \int_0^t g_k^s(t+h-u) A^k(u,x(u)) du - \int_0^t g_k^s(t-u) A^k(u,x(u)) du \right| \\
&\le \int_t^{t+h} g_k^s(t+h-u) A^k(u,x(u)) du + \int_0^t \left| g_k^s(t+h-u) - g_k^s(t-u) \right| A^k(u,x(u)) du.
\end{aligned}$$

By property (2) of Lemma 4 with $y = 0$, we have that $A^k(u, x(u)) \le |x(u)|$ for each $u \in \mathbb{R}_+$ since $A^k(u, 0) = 0$, then for each $0 < h < 1$,

$$\begin{aligned}\int_t^{t+h} g_k^s(t+h-u)A^k(u,x(u))du &\le \sup_{0\le u\le t+h}|x(u)|\int_t^{t+h} g_k^s(t+h-u)du \\ &\le \sup_{0\le u\le t+h}|x(u)|G_k^s(h).\end{aligned}$$

For the second term at the right-hand side of (3.33), by using Hölder's inequality,

$$\begin{aligned}&\int_0^t \left|g_k^s(t+h-u)-g_k^s(t-u)\right| A^k(u,x(u))du \qquad (3.34)\\ &\le \left(\int_0^t \left|g_k^s(t+h-u)-g_k^s(t-u)\right|^{q_k} du\right)^{1/q_k}\left(\int_0^t A^k(u,x(u))^{p_k}du\right)^{1/p_k}\\ &\le \left(\int_0^t \left|g_k^s(t+h-u)-g_k^s(t-u)\right|^{q_k} du\right)^{1/q_k}\sup_{0\le u\le t}|x(u)|t^{1/p_k}\\ &\le \left(\int_0^t \left|g_k^s(u+h)-g_k^s(u)\right|^{q_k} du\right)^{1/q_k}\sup_{0\le u\le t}|x(u)|t^{1/p_k},\end{aligned}$$

where $1/p_k + 1/q_k = 1$ (when $q_k = 1$, then p_k is understood as ∞ and $1/p_k = 0$). It follows that, for each $0 < h < 1$,

$$\begin{aligned}&\left|\int_0^{t+h} g_k^s(t+h-u)A^k(u,x(u))du - \int_0^t g_k^s(t-u)A^k(u,x(u))du\right| \qquad (3.35)\\ &\le \sup_{0\le u\le t+h}|x(u)|\left[G_k^s(h)+\left(\int_0^t \left|g_k^s(u+h)-g_k^s(u)\right|^{q_k} du\right)^{1/q_k} t^{1/p_k}\right].\end{aligned}$$

Then by using (3.23) and taking the limits on both sides of (3.35) as $h \downarrow 0$, we have that

$$\left|\int_0^{t+h} g_k^s(t+h-u)A^k(u,x(u))du - \int_0^t g_k^s(t-u)A^k(u,x(u))du\right| \to 0 \text{ as } h\downarrow 0,$$

which shows that $\int_0^t g_k^s(t-u)A^k(u,x(u))du$ as a function of t is right continuous at t. We next show that $\int_0^t g_k^s(t-u)A^k(u,x(u))du$ as a function of t is left continuous at t if $t>0$. For this, suppose that $t>0$. For each $0<h<t$,

$$\begin{aligned}
&\left|\int_0^{t-h} g_k^s(t-h-u)A^k(u,x(u))du-\int_0^t g_k^s(t-u)A^k(u,x(u))du\right| \\
&\le \left|\int_0^{t-h} g_k^s(t-h-u)A^k(u,x(u))du-\int_0^{t-h} g_k^s(t-u)A^k(u,x(u))du\right| \\
&\quad +\left|\int_0^{t-h} g_k^s(t-u)A^k(u,x(u))du-\int_0^t g_k^s(t-u)A^k(u,x(u))du\right| \\
&\le \int_{t-h}^t g_k^s(t-u)A^k(u,x(u))du+\int_0^{t-h}\left|g_k^s(t-h-u)-g_k^s(t-u)\right|A^k(u,x(u))du,
\end{aligned}$$

where

$$\int_{t-h}^t g_k^s(t-u)A^k(u,x(u))du \le \sup_{0\le u\le t}|x(u)|\int_{t-h}^t g_k^s(t-u)du \le \sup_{0\le u\le t}|x(u)|G_k^s(h),$$

and

$$\begin{aligned}
&\int_0^{t-h}\left|g_k^s(t-h-u)-g_k^s(t-u)\right|A^k(u,x(u))du \\
&\le \left(\int_0^{t-h}\left|g_k^s(t-h-u)-g_k^s(t-u)\right|^{q_k}du\right)^{1/q_k}\left(\int_0^{t-h}A^k(u,x(u))^{p_k}du\right)^{1/p_k} \\
&\le \left(\int_0^{t-h}\left|g_k^s(t-h-u)-g_k^s(t-u)\right|^{q_k}du\right)^{1/q_k}\sup_{0\le u\le t}|x(u)|t^{1/p_k} \\
&\le \left(\int_0^{t}\left|g_k^s(u+h)-g_k^s(u)\right|^{q_k}du\right)^{1/q_k}\sup_{0\le u\le t}|x(u)|t^{1/p_k}.
\end{aligned}$$

It follows that, for each $0<h<t$,

$$\begin{aligned}
&\left|\int_0^{t-h} g_k^s(t-h-u)A^k(u,x(u))du-\int_0^t g_k^s(t-u)A^k(u,x(u))du\right| \\
&\le \sup_{0\le u\le t}|x(u)|\left[G_k^s(h)+\left(\int_0^t\left|g_k^s(u+h)-g_k^s(u)\right|^{q_k}du\right)^{1/q_k}t^{1/p_k}\right].
\end{aligned} \tag{3.36}$$

Then by using (3.23) and taking the limits on both sides of (3.36) as $h \downarrow 0$, we have that

$$\left| \int_0^{t-h} g_k^s(t-h-u) A^k(u, x(u)) du - \int_0^t g_k^s(t-u) A^k(u, x(u)) du \right| \to 0 \text{ as } h \downarrow 0,$$

which shows that $\int_0^t g_k^s(t-u) A^k(u, x(u)) du$ as a function of t is also left continuous at t if $t > 0$. Thus, for each $k \in \mathcal{K}$, $\int_0^t g_k^s(t-u) A^k(u, x(u)) du$ as a function of t is continuous on $\mathbb{R}_+$ and then Λ is a mapping from $C(\mathbb{R}_+)$ into $C(\mathbb{R}_+)$. □

PROPOSITION 1. *Suppose that Assumption 1 holds. Then there exists an interval* $[0, \sigma']$, $\sigma' > 0$ *such that the equation (3.30) admits a unique continuous solution* $x(\cdot)$ *on* $[0, \sigma']$.

Proof Motivated by Miller and Sell [MS(1968)], we first establish the existence of a solution to (3.30) on $[0, \beta)$ for some $\beta > 0$ by applying Schauder-Tychonoff Fixed Point Theorem. Recall that by Lemma 5, Λ is a mapping from $C(\mathbb{R}_+)$ into $C(\mathbb{R}_+)$.

Fix $\beta > 0$ and let $a > 0$, define a subset $\mathcal{H}[0, \beta]$ of $C[0, \beta]$ as

$$\mathcal{H}[0, \beta] \doteq \left\{ x(\cdot) \in C[0, \beta] : \sup_{0 \le t \le \beta} |x(t) - \xi(t)| \le a \right\}.$$

It is clear that the continuous function ξ restricted on $[0, \beta]$ is in $\mathcal{H}[0, \beta]$, so $\mathcal{H}[0, \beta] \neq \emptyset$. For each $x(\cdot) \in \mathcal{H}[0, \beta]$ and each $t \in [0, \beta]$, $|x(t)| \le |\xi(t)| + a$. Let

$$M_\xi \doteq \sup_{0 \le t \le \beta} |\xi(t)| + a.$$

Then $\sup_{0 \le t \le \beta} |x(t)| \le M_\xi$ for each $x(\cdot) \in \mathcal{H}[0, \beta]$. Property (2) of Lemma 4 with $y = 0$ implies that $A^k(t, x(t)) \le |x(t)| \le M_\xi$ for all $t \in [0, \beta]$. Also property (3) of Lemma 4 with $y = 0$ implies that $B^k(t, x(t)) \le C_t^{r,k} |x(t)| \le C_\beta^{r,k} |x(t)|$, where recall $C_t^{r,k} = \sup_{0 \le u \le \overline{\chi}(0) + t} h_k^r(u)$ and $C_\beta^{r,k} = \sup_{0 \le u \le \overline{\chi}(0) + \beta} h_k^r(u)$. Note that by the definition of Λ in (3.31), for each $x(\cdot) \in \mathcal{H}[0, \beta]$ and each $t \in [0, \beta]$,

$$\begin{aligned}
|\Lambda(x)(t) - \xi(t)| &\le \sum_{k \in \mathcal{K}} \int_0^t \bar{G}_k^s(t-u) B^k(u, x(u)) du + \sum_{k \in \mathcal{K}} \int_0^t g_k^s(t-u) A^k(u, x(u)) du \\
&\le \sum_{k \in \mathcal{K}} C_\beta^{r,k} M_\xi \int_0^t \bar{G}_k^s(t-u) du + \sum_{k \in \mathcal{K}} M_\xi \int_0^t g_k^s(t-u) du \\
&= \sum_{k \in \mathcal{K}} C_\beta^{r,k} M_\xi \int_0^t \bar{G}_k^s(u) du + \sum_{k \in \mathcal{K}} M_\xi G_k^s(t).
\end{aligned}$$

Now choose $\alpha \in (0,\beta)$ such that

$$\sum_{k\in\mathcal{K}} C_\beta^{r,k} M_\xi \int_0^t \bar{G}_k^s(u)du + \sum_{k\in\mathcal{K}} M_\xi G_k^s(t) \le a \text{ for all } t \in [0,\alpha].$$

Consider $\mathcal{H}[0,\alpha]$, the restriction of $\mathcal{H}[0,\beta]$ on $[0,\alpha]$. Note that $\mathcal{H}[0,\alpha]$ is a nonempty bounded convex subset of the Banach space $C[0,\alpha]$ and Λ maps $\mathcal{H}[0,\alpha]$ to itself.

We now show that the operator Λ on $\mathcal{H}[0,\alpha]$ is compact. Since Λ maps $\mathcal{H}[0,\alpha]$ to itself, the set of images $\Lambda(\mathcal{H}[0,\alpha]) \doteq \{\Lambda(x) : x(\cdot) \in \mathcal{H}[0,\alpha]\}$ is bounded. Thus, by the Arzelà–Ascoli Theorem, it suffices to show that $\Lambda(\mathcal{H}[0,\alpha])$ is equi-continuous. For each $t \in [0,\alpha]$, $x(\cdot) \in \mathcal{H}[0,\alpha]$ and any $\varepsilon > 0$, $h \in \mathbb{R}$ such that $t+h \in [0,\alpha]$,

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$$\begin{aligned} |\Lambda(x)(t+h) - \Lambda(x)(t)| \le{}& |\xi(t+h) - \xi(t)| \\ &+ \sum_{k\in\mathcal{K}} \left| \int_0^{t+h} \bar{G}_k^s(t+h-u) B^k(u,x(u))du - \int_0^t \bar{G}_k^s(t-u)B^k(u,x(u))du \right| \\ &+ \sum_{k\in\mathcal{K}} \left| \int_0^{t+h} g_k^s(t+h-u)A^k(u,x(u))du - \int_0^t g_k^s(t-u)A^k(u,x(u))du \right|. \end{aligned} \tag{3.37}$$

Note that by using (3.35) when $h > 0$ and (3.36) when $h < 0$ and using the fact that $t, t+h \in [0,\alpha]$,

$$\begin{aligned} &\sum_{k\in\mathcal{K}} \left| \int_0^{t+h} g_k^s(t+h-u)A^k(u,x(u))du - \int_0^t g_k^s(t-u)A^k(u,x(u))du \right| \\ &\le \sup_{0\le u\le\alpha} |x(u)| \left[\sum_{k\in\mathcal{K}} G_k^s(|h|) + \sum_{k\in\mathcal{K}} \left(\int_0^t |g_k^s(u+|h|) - g_k^s(u)|^{q_k} du \right)^{1/q_k} \alpha^{1/p_k} \right] \\ &\le M_\xi \left[\sum_{k\in\mathcal{K}} G_k^s(|h|) + \sum_{k\in\mathcal{K}} \alpha^{1/p_k} \left(\int_0^t |g_k^s(u+|h|) - g_k^s(u)|^{q_k} du \right)^{1/q_k} \right]. \end{aligned} \tag{3.38}$$

Similarly, for each $k \in \mathcal{K}$, since $B^k(u, x(u)) \le C_\beta^{r,k} M_\xi$ for each $t \in [0, \alpha]$ and $x(\cdot) \in \mathcal{H}[0, \alpha]$, we have that, when $h > 0$,

$$
\begin{aligned}
&\left| \int_0^{t+h} \bar{G}_k^s(t+h-u) B^k(u, x(u)) du - \int_0^t \bar{G}_k^s(t-u) B^k(u, x(u) du \right| \\
&\le \int_t^{t+h} \bar{G}_k^s(t+h-u) B^k(u, x(u)) du + \int_0^t (G_k^s(t+h-u) - G_k^s(t-u)) B^k(u, x(u)) du \\
&\le C_\beta^{r,k} M_\xi \left(\int_t^{t+h} \bar{G}_k^s(t+h-u) du + \int_0^t (G_k^s(t+h-u) - G_k^s(t-u)) du \right) \\
&\le C_\beta^{r,k} M_\xi \left(h + \int_0^t \int_0^h g_k^s(t-u+v) dv du \right) \\
&\le C_\beta^{r,k} M_\xi \left(h + \int_0^h (G_k^s(t+v) - G_k^s(v)) dv \right) \le 2 C_\beta^{r,k} M_\xi h;
\end{aligned}
$$

and when $h < 0$,

$$
\begin{aligned}
&\left| \int_0^{t+h} \bar{G}_k^s(t+h-u) B^k(u, x(u)) du - \int_0^t \bar{G}_k^s(t-u)) B^k(u, x(u) du \right| \\
&\le \int_{t+h}^t \bar{G}_k^s(t-u) B^k(u, x(u)) du + \int_0^{t+h} (G_k^s(t-u) - G_k^s(t+h-u)) B^k(u, x(u)) du \\
&\le C_\beta^{r,k} M_\xi \left(\int_{t+h}^t \bar{G}_k^s(t-u) du + \int_0^{t+h} (G_k^s(t-u) - G_k^s(t+h-u)) du \right) \\
&\le C_\beta^{r,k} M_\xi \left(-h + \int_0^{t+h} \int_0^{-h} g_k^s(t+h-u+v) dv du \right) \\
&\le C_\beta^{r,k} M_\xi \left(-h + \int_0^{-h} (G_k^s(t+h+v) - G_k^s(v)) dv \right) \le 2 C_\beta^{r,k} M_\xi (-h).
\end{aligned}
$$

Then it follows that

$$
\left| \int_0^{t+h} \bar{G}_k^s(t+h-u) B^k(u, x(u)) du - \int_0^t \bar{G}_k^s(t-u)) B^k(u, x(u) du \right| \le 2 C_\beta^{r,k} M_\xi |h|.
$$

Combining this with (3.37) and (3.38), we have that for each $t \in [0,\alpha]$ and $h \in \mathbb{R}$ such that $t+h \in [0,\alpha]$,

$$|\Lambda(x)(t+h) - \Lambda(x)(t)| \le |\xi(t+h) - \xi(t)| + 2C_\beta^{r,k} M_\xi |h| + M_\xi \sum_{k\in\mathcal{K}} G_k^s(|h|) \\ + M_\xi \sum_{k\in\mathcal{K}} \alpha^{1/p_k} \left(\int_0^t \left| g_k^s(u+|h|) - g_k^s(u) \right|^{q_k} du \right)^{1/q_k}.$$

By the uniform continuity of ξ on $[0,\alpha]$ and (3.23), for the given $\varepsilon > 0$, there is a $\delta > 0$ (independent of $x(\cdot) \in \mathcal{H}[0,\alpha]$) such that $|\Lambda(x)(t+h) - \Lambda(x)(t)| < \varepsilon$ for any $t,\ t+h \in [0,\alpha]$ whenever $|h| < \delta$. This establishes the equi-continuity we need and hence Λ on $\mathcal{H}[0,\alpha]$ is compact.

We next show that the operator Λ on $\mathcal{H}[0,\alpha]$ is continuous. Let $\{x_n(\cdot), n \ge 1\}$ be a sequence in $\mathcal{H}[0,\alpha]$ that converges uniformly to $x(\cdot) \in \mathcal{H}[0,\alpha]$ as $n \to \infty$. We need to show that $\Lambda(x_n) \to \Lambda(x)$ uniformly on $[0,\alpha]$ as $n \to \infty$. Since we have proved that the set $\{\Lambda(x_n), n \ge 1\}$ is equi-continuous, It suffices to show that for each $t \in [0,\alpha]$, $\Lambda(x_n)(t) \to \Lambda(x)(t)$ as $n \to \infty$. For this, notice that for each $k \in \mathcal{K}$ and $u \in [0,t]$, $A^k(u, x_n(u)) \to A^k(u, x(u))$ and $B^k(u, x_n(u)) \to B^k(u, x(u))$ as $n \to \infty$ due to property (1) of Lemma 4. Since $g_k^s(t-u)A^k(u, x_n(u)) \le g_k^s(t-u)M_\xi$ and $\bar{G}_k^s(t-u)B^k(u, x_n(u)) \le \bar{G}_k^s(t-u)C_\beta^{r,k} M_\xi$ for each $u \in [0,t]$ and all $n \ge 1$, then by the dominate convergence theorem, we have that $\Lambda(x_n)(t) \to \Lambda(x)(t)$ as $n \to \infty$. This establishes the continuity of Λ.

Since we have established that the operator Λ on $\mathcal{H}[0,\alpha]$ is compact, continuous and maps $\mathcal{H}[0,\alpha]$ to itself, thus by Schauder-Tychonoff Fixed Point Theorem [SMN(1975)], Λ has a fixed point in $\mathcal{H}[0,\alpha]$, that is, there exists a function $x(\cdot) \in \mathcal{H}[0,\alpha]$ such that $\Lambda(x)(\cdot) = x(\cdot)$ on $[0,\alpha]$ and then $x(\cdot)$ is a solution to (3.30) on $[0,\alpha]$.

At last, we show that there is $\alpha' \in (0,\alpha)$ such that $x(\cdot)$ is the only solution to (3.30) on $[0,\alpha']$. Consider two solutions $x_1(\cdot)$ and $x_2(\cdot)$ to (3.30) on $[0,\alpha]$. Property (2) of Lemma 4 implies that $|A^k(u, x_1(u)) - A^k(u, x_2(u))| \le |x_1(u) - x_2(u)|$ for each $u \in [0,\alpha]$. Moreover, property (3) of Lemma 4 implies that for each $u \in [0,\alpha]$,

$$|B^k(u, x_1(u)) - B^k(u, x_2(u))| \le C_\beta^{r,k} |x_1(u) - x_2(u)|.$$

So for each $t \in [0,\alpha]$,

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$$
\begin{aligned}
|x_1(t) - x_2(t)| \le & \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u)|B^k(u,x_1(u)) - B^k(u,x_2(u))|du \\
& + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u)|A^k(u,x_1(u)) - A^k(u,x_2(u))|du \\
\le & \sum_{k\in\mathcal{K}} C_\beta^{r,k} \int_0^t \bar{G}_k^s(t-u)|x_1(u)-x_2(u)|du + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u)|x_1(u)-x_2(u)|du.
\end{aligned}
$$

Choose $\alpha' \in (0,\alpha)$ to be such that $\sum_{k\in\mathcal{K}} C_\beta^{r,k} \int_0^{\alpha'} \bar{G}_k^s(u)du + \sum_{k\in\mathcal{K}} G_k^s(\alpha') < 1$. Then for each $t \in [0,\alpha']$,

$$
|x_1(t) - x_2(t)| \le \sup_{0\le u\le \alpha'} |x_1(u) - x_2(u)| \left(\sum_{k\in\mathcal{K}} C_\beta^{r,k} \int_0^{\alpha'} \bar{G}_k^s(u)du + \sum_{k\in\mathcal{K}} G_k^s(\alpha') \right).
$$

Thus, the above display implies that $\sup_{0\le u\le\alpha'} |x_1(u) - x_2(u)| = 0$, that is, $x_1(\cdot) = x_2(\cdot)$ on $[0,\alpha']$. This shows that the solution to (3.30) is unique on $[0,\alpha']$. □

We now extend the unique solution to (3.30) from $[0,\alpha']$ to $\mathbb{R}_+$.

PROPOSITION 2. *Suppose that Assumption 1 holds. Then the equation (3.30) admits a unique continuous solution $x(\cdot)$ on $\mathbb{R}_+$.*

Proof From Proposition 1, the equation (3.30) admits a unique continuous solution $x(\cdot)$ on $[0,\alpha']$. Consider the following integral equation as the time-shifted version of (3.30):

$$
f(t) = \hat{\xi}(t) - \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u)B^k(u+\alpha', f(u))du + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u)A^k(u+\alpha', f(u))du, \tag{3.39}
$$

where

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$$\hat{\xi}(t) = \xi(\alpha'+t) - \sum_{k\in\mathcal{K}} \int_0^{\alpha'} \bar{G}_k^s(\alpha'+t-u)B^k(u,x(u))du + \sum_{k\in\mathcal{K}} \int_0^{\alpha'} g_k^s(\alpha'+t-u)A^k(u,x(u))du.$$

It is clear that $\xi(\alpha'+t) - \sum_{k\in\mathcal{K}} \int_0^{\alpha'} \bar{G}_k^s(\alpha'+t-u)B^k(u,x(u))du$ as a function of t is continuous in $t\in\mathbb{R}_+$. To show that $\hat{\xi}(t)$ is continuous in $t\in\mathbb{R}_+$, it suffices to show that for each $k\in\mathcal{K}$, $\int_0^{\alpha'} g_k^s(\alpha'+t-u)A^k(u,x(u))du$ as a function of t is continuous in $t\in\mathbb{R}_+$. For this, fix $k\in\mathcal{K}$ and $t\in\mathbb{R}_+$. Let $h\in\mathbb{R}$ be such that $t+h>0$. We then have that if $h>0$,

$$\begin{aligned}
&\left|\int_0^{\alpha'} g_k^s(\alpha'+t+h-u)A^k(u,x(u))du - \int_0^{\alpha'} g_k^s(\alpha'+t-u)A^k(u,x(u))du\right| \\
&\le \left(\int_0^{\alpha'} |g_k^s(\alpha'+t+h-u) - g_k^s(\alpha'+t-u)|^{q_k} du\right)^{1/q_k} \sup_{0\le u\le\alpha'} |x(u)|(\alpha')^{1/p_k} \\
&= \left(\int_t^{\alpha'+t} |g_k^s(u+h) - g_k^s(u)|^{q_k} du\right)^{1/q_k} \sup_{0\le u\le\alpha'} |x(u)|(\alpha')^{1/p_k} \\
&\le \left(\int_0^{\alpha'+t} |g_k^s(u+h) - g_k^s(u)|^{q_k} du\right)^{1/q_k} \sup_{0\le u\le\alpha'} |x(u)|(\alpha')^{1/p_k};
\end{aligned}$$

and if $h<0$ (when $t>0$),

$$\begin{aligned}
&\left|\int_0^{\alpha'} g_k^s(\alpha'+t+h-u)A^k(u,x(u))du - \int_0^{\alpha'} g_k^s(\alpha'+t-u)A^k(u,x(u))du\right| \\
&\le \left(\int_0^{\alpha'} |g_k^s(\alpha'+t+h-u) - g_k^s(\alpha'+t-u)|^{q_k} du\right)^{1/q_k} \sup_{0\le u\le\alpha'} |x(u)|(\alpha')^{1/p_k} \\
&= \left(\int_{t+h}^{\alpha'+t+h} |g_k^s(u-h) - g_k^s(u)|^{q_k} du\right)^{1/q_k} \sup_{0\le u\le\alpha'} |x(u)|(\alpha')^{1/p_k} \\
&\le \left(\int_0^{\alpha'+t} |g_k^s(u-h) - g_k^s(u)|^{q_k} du\right)^{1/q_k} \sup_{0\le u\le\alpha'} |x(u)|(\alpha')^{1/p_k}.
\end{aligned}$$

Taking the limits on both sides of the above two displays and applying (3.23), we have that as $h \longrightarrow 0$,

$$\left| \int_0^{\alpha'} g_k^s(\alpha'+t+h-u)A^k(u,x(u))du - \int_0^{\alpha'} g_k^s(\alpha'+t-u)A^k(u,x(u))du \right| \to 0.$$

It follows that, for each $k \in \mathcal{K}$, $\int_0^{\alpha'} g_k^s(\alpha'+t-u)A^k(u,x(u))du$ as a function of t is continuous in $t \in \mathbb{R}_+$ and then the function $\hat{\xi}(\cdot) \in C(\mathbb{R}_+)$. For each $k \in \mathcal{K}$, note that the functions $A^k(\alpha'+\cdot,\cdot)$ and $B^k(\alpha'+\cdot,\cdot)$ also satisfy the properties stated in Lemma 4. Then by following the same argument as in the proof of Proposition 1 with $\xi(\cdot)$ replaced by $\hat{\xi}(\cdot)$ and $A^k(\cdot,\cdot)$ and $B^k(\cdot,\cdot)$ replaced by $A^k(\alpha'+\cdot,\cdot)$ and $B^k(\alpha'+\cdot,\cdot)$ respectively, there exists an interval $[0,\hat{\sigma}']$, $\hat{\sigma}'>0$ such that the equation (3.39) admits a unique continuous solution $\hat{x}(\cdot)$ on $[0,\hat{\sigma}']$. Then we can extend the solution $x(\cdot)$ from $[0,\sigma']$ to $[0,\sigma'+\hat{\sigma}']$ as follows:

$$x(t) \doteq \begin{cases} x(t) & \text{if } 0 \le t \le \alpha', \\ \hat{x}(t-\alpha') & \text{if } \alpha' \le t \le \alpha'+\hat{\alpha}'. \end{cases}$$

Note that

$$\hat{x}(0) = \hat{\xi}(0) = \xi(\alpha') - \sum_{k\in\mathcal{K}} \int_0^{\alpha'} \bar{G}_k^s(\alpha'-u)B^k(u,x(u))du + \sum_{k\in\mathcal{K}} \int_0^{\alpha'} g_k^s(\alpha'-u)A^k(u,x(u))du = x(\alpha').$$

So the extension of $x(\cdot)$ defined above is the unique continuous solution to the equation (3.30) on $[0,\sigma'+\hat{\sigma}']$. By applying a simple contradiction argument, it is clear that the maximal interval on which the equation (3.30) admits a unique continuous solution has to be $\mathbb{R}_+$. This completes the proof of the proposition. □

3.2. Uniqueness of Solutions to the Fluid Model Equations.

THEOREM 3.1. *Suppose that Assumption 1 holds. Then, given $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0) \in \mathcal{S}_0$, There is at most one continuous solution $(\overline{X},\overline{\nu},\overline{\eta})$ to the fluid model equations as in Definition 1 associated with $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)$.*

Proof Suppose that $(\overline{X}^1,\overline{\nu}^1,\overline{\eta}^1)$ and $(\overline{X}^2,\overline{\nu}^2,\overline{\eta}^2)$ are two solutions to the fluid model equations as in Definition 1 associated with $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0) \in \mathcal{S}_0$. By (2.5), it is obvious that $\overline{\eta}^1 = \overline{\eta}^2$ and then

$\overline{F}^1_t = \overline{F}^2_t$ for each $t \in \mathbb{R}_+$ and $\overline{\chi}^1(0) = \overline{\chi}^2(0)$. We simply denote them as $\overline{F}_t$ for each $t \in \mathbb{R}_+$ and $\overline{\chi}(0)$. By the discussion right after Remark 2, we see that $\sum_{k\in\mathcal{K}} \overline{X}^1_k(\cdot)$ and $\sum_{k\in\mathcal{K}} \overline{X}^2_k(\cdot)$ are two solutions to (3.30). It follows from Proposition 2 that $\sum_{k\in\mathcal{K}} \overline{X}^1_k(\cdot) = \sum_{k\in\mathcal{K}} \overline{X}^2_k(\cdot)$, which in turn implies that $\overline{Q}^1(\cdot) = \overline{Q}^2(\cdot)$ by (2.8) and then $\overline{R}^1(\cdot) = \overline{R}^2(\cdot)$ by (2.7). It follows that $\overline{L}^1(\cdot) = \overline{L}^2(\cdot)$ by (2.12) and then $\overline{\nu}^1 = \overline{\nu}^2$ by (2.3) and $\overline{X}^1(\cdot) = \overline{X}^2(\cdot)$ by (2.9). Thus, $(\overline{X}^1, \overline{\nu}^1, \overline{\eta}^1) = (\overline{X}^2, \overline{\nu}^2, \overline{\eta}^2)$. □

3.3. Existence of Solutions to the Fluid Model Equations. In this section, we establish the existence of solutions to the fluid model equation from the unique continuous solution $x(\cdot)$ to the equation (3.30) ensured by Proposition 2 under an additional assumption on the input data $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0) \in \mathcal{S}_0$ and the service time densities $\{g^s_k,\ k \in \mathcal{K}\}$ of the service time distributions $\{G^s_k,\ k \in \mathcal{K}\}$.

Assumption 2. *The service time densities $\{g^s_k,\ k \in \mathcal{K}\}$ are right continuous on their supports and are absolutely continuous on $[0,\delta]$ for some $\delta > 0$, and one of the following two conditions holds:*

(A) There exists $k \in \mathcal{K}$ such that $\int_{[0,H^s_k)} \frac{g^s_k(x+t)}{\bar{G}^s_k(x)} \overline{\nu}^k_0(dx) > 0$ for all $t \in \mathbb{R}_+$.
(B) For each $k \in \mathcal{K}$, $h^s_k(x) > 0$ for each $x \in [0, H^s_k)$.

Now, for each $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0) \in \mathcal{S}_0$ that satisfies Assumptions 1 and 2, let $x(\cdot)$ be the unique continuous solution to the equation (3.30) associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0)$ ensured by Proposition 2. Now we construct $(\overline{X}, \overline{\nu}, \overline{\eta})$ from the unique continuous solution $x(\cdot)$ to the equation (3.30). For each $k \in \mathcal{K}$, define $\overline{\eta}^k[0,x]$ from the data $(\overline{E}_k, \overline{\eta}^k_0)$ using the right-hand side of (2.5) with $f = \mathbf{1}_{[0,x]}$, $x \in [0, H^r_k)$. For each $t \in \mathbb{R}_+$ and $k \in \mathcal{K}$, define $\overline{Q}_k(t)$, $\overline{R}_k(t)$, $\overline{L}_k(t)$ by

$$\overline{Q}_k(t) \doteq A^k(t, x(t)), \tag{3.40}$$

$$\overline{R}_k(t) \doteq \int_0^t B^k(u, x(u))du, \tag{3.41}$$

$$\overline{L}_k(t) \doteq \overline{Q}_k(0) + \overline{E}_k(t) - \overline{Q}_k(t) - \overline{R}_k(t). \tag{3.42}$$

At last, for each $k \in \mathcal{K}$, define $\overline{\nu}^k$ from the data $(\overline{L}_k, \overline{\nu}_0^k)$ as follows: for each $x \in [0, H_k^s)$,

$$\overline{\nu}^k[0,x] \doteq \int_{[0,H_k^s)} \mathbf{1}_{[0,x]}(y+t)\frac{\bar{G}_k^s(y+t)}{\bar{G}_k^s(y)}\overline{\nu}_0^k(dy) + \overline{L}_k(t) - \bar{G}_k^s(x)\overline{L}_k(t-x\wedge t) - \int_{t-x\wedge t}^t \overline{L}_k(s)g_k^s(t-s)ds. \tag{3.43}$$

At last, for each $t \in \mathbb{R}_+$, define

$$\overline{X}_k(t) \doteq \overline{Q}_k(t) + \langle \mathbf{1}, \overline{\nu}_t^k\rangle, \tag{3.44}$$

where $\langle \mathbf{1}, \overline{\nu}_t^k\rangle = \overline{\nu}^k[0, H_k^s)$ using (3.43). The main result of this section is the following existence of solutions to the fluid model equations.

Theorem 3.2. *Suppose that Assumptions 1 and 2 hold. Given $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0) \in \mathcal{S}_0$, then $(\overline{X}, \overline{\nu}, \overline{\eta})$ constructed from the unique continuous solution $x(\cdot)$ to the equation (3.30) is a continuous solution to the fluid model equations associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0)$ on $\mathbb{R}_+$.*

Note that for each $k \in \mathcal{K}$, $\overline{L}_k$ defined by (3.42) may not be non-decreasing automatically and then $\overline{\nu}^k$ defined by (3.43) may not be a nonnegative measure. If we can show that $\overline{L}_k$, $k \in \mathcal{K}$, is non-decreasing and that $x(t) - 1 \le \overline{F}_t(\overline{\chi}(0) + t)$ for each $t \in \mathbb{R}_+$, then the following proposition shows that $(\overline{X}, \overline{\nu}, \overline{\eta})$ constructed from the unique continuous solution $x(\cdot)$ to the equation (3.30) is a continuous solution to the fluid model equations.

Proposition 3. *Suppose that Assumption 1 holds. Given $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0) \in \mathcal{S}_0$, if for each $k \in \mathcal{K}$, $\overline{L}_k$ defined in (3.42) is non-decreasing on $\mathbb{R}_+$ and $x(t) - 1 \le \overline{F}_t(\overline{\chi}(0) + t)$ for each $t \in \mathbb{R}_+$, then $(\overline{X}, \overline{\nu}, \overline{\eta})$ constructed from the unique continuous solution $x(\cdot)$ to the equation (3.30) is a continuous solution to the fluid model equations associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0, \overline{\eta}_0)$.*

Proof Suppose that for each $k \in \mathcal{K}$, $\overline{L}_k$ defined in (3.42) is non-decreasing on $\mathbb{R}_+$ and $x(t) - 1 \le \overline{F}_t(\overline{\chi}(0) + t)$ for each $t \in \mathbb{R}_+$. Then, for each $k \in \mathcal{K}$, using a simple integration by parts to the last term of the right-hand side of (3.43),

$$\overline{\nu}^k[0,x] = \int_{[0,H_k^s)} \mathbf{1}_{[0,x]}(y+t)\frac{\bar{G}_k^s(y+t)}{\bar{G}_k^s(y)}\overline{\nu}_0^k(dy) + \int_0^t \mathbf{1}_{[0,x]}(t-s)\bar{G}_k^s(t-s)d\overline{L}_k(s).$$

Thus, $\overline{\nu}^k$ defined by (3.43) is a nonnegative measure and then satisfies (2.3) as discussed in Remark 1. We show that $(\overline{X}, \overline{\nu}, \overline{\eta})$ constructed from the unique continuous solution $x(\cdot)$ to the equation (3.30) with its associated processes $\overline{Q}$, $\overline{R}, \overline{L}$ given in (3.40)–(3.42) is a continuous solution to the fluid model equations. It is clear that the constructed processes satisfy (2.3), (2.5), (2.6).

To show that (2.4) holds, note from the construction of $\overline{\nu}^k$, Remark 1, changing the order of integration and an application of integration by parts that, for each $t \in \mathbb{R}_+$ and $k \in \mathcal{K}$,

$$\begin{aligned}\int_0^t \langle h_k^s, \overline{\nu}_u^k \rangle du &= \int_{[0,H_k^s)} \frac{G_k^s(x+t) - G_k^s(x)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \int_0^t \int_0^s g_k^s(s-u)\, d\overline{L}_k(u) ds \\ &= \int_{[0,H_k^s)} \frac{G_k^s(x+t) - G_k^s(x)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \int_0^t G_k^s(t-u)\, d\overline{L}_k(u) \\ &= \int_{[0,H_k^s)} \frac{G_k^s(x+t) - G_k^s(x)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \int_0^t \overline{L}_k(u) g_k^s(t-u)\, du < \infty \end{aligned} \qquad (3.45)$$

and

$$\begin{aligned}\langle \mathbf{1}, \overline{\nu}_t^k \rangle &= \int_{[0,H_k^s)} \frac{\bar{G}_k^s(x+t)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \int_0^t (\bar{G}_k^s(t-s))\, d\overline{L}_k(s) \\ &= \int_{[0,H_k^s)} \frac{\bar{G}_k^s(x+t)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \overline{L}_k(t) - \int_0^t \overline{L}_k(u) g_k^s(t-u)\, du. \end{aligned} \qquad (3.46)$$

Adding the above two displays yields that for each $k \in \mathcal{K}$,

$$\int_0^t \langle h_k^s, \overline{\nu}_u^k \rangle du + \langle \mathbf{1}, \overline{\nu}_t^k \rangle = \langle \mathbf{1}, \overline{\nu}_0^k \rangle + \overline{L}_k(t).$$

Thus, (2.4) holds by arranging terms in the above display. By the construction of $\overline{\eta}^k$, we also have that for each $k \in \mathcal{K}$,

$$\int_0^t \langle h_k^r, \overline{\eta}_u^k \rangle du = \int_{[0,H_k^r)} \frac{G_k^r(x+t) - G_k^r(x)}{\bar{G}_k^r(x)} \overline{\eta}_0^k(dx) + \int_0^t \overline{E}_k(u) g_k^r(t-u)\, du < \infty.$$

Then it is clear that (2.2) holds. From (2.4) and the constructions of $\overline{L}_k$ and $\overline{X}_k$, it is easy to see that (2.9) holds. We next show that (2.10) holds. From the construction of $\overline{Q}_k$ and the assumption that $x(t) - 1 \le \overline{F}_t(\overline{\chi}(0) + t)$ for each $t \in \mathbb{R}_+$, we see that for each $t \in \mathbb{R}_+$,

$$\sum_{k \in \mathcal{K}} \overline{Q}_k(t) = \sum_{k \in \mathcal{K}} \overline{F}_t^k \left((\overline{F}_t)^{-1} \left([x(t) - 1]^+ \right) \right) = \overline{F}_t \left((\overline{F}_t)^{-1} \left([x(t) - 1]^+ \right) \right) = [x(t) - 1]^+.$$

This and the construction of $\overline{X}$ together imply that for each $t \in \mathbb{R}_+$,

$$\sum_{k\in\mathcal{K}} \overline{X}_k(t) = \sum_{k\in\mathcal{K}} \overline{Q}_k(t) + \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \overline{\nu}_t^k \rangle = [x(t)-1]^+ + \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \overline{\nu}_t^k \rangle. \tag{3.47}$$

The construction of $\overline{L}$ and (3.45) together imply that for each $t \in \mathbb{R}_+$,

$$\begin{aligned}
\sum_{k\in\mathcal{K}} \int_0^t \langle h_k^s, \overline{\nu}_u^k \rangle du &= \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \sum_{k\in\mathcal{K}} \int_0^t \overline{L}_k(u) g_k^s(t-u)\, du \\
&= \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) \\
&\quad + \sum_{k\in\mathcal{K}} \int_0^t (\overline{Q}_k(0) + \overline{E}_k(u) - \overline{Q}_k(u) - \overline{R}_k(u)) g_k^s(t-u)\, du.
\end{aligned}$$

Since we have shown that (2.9) holds, this and the above display together imply that for each $t \in \mathbb{R}_+$,

$$\begin{aligned}
\sum_{k\in\mathcal{K}} \overline{X}_k(t) &= \sum_{k\in\mathcal{K}} \overline{X}_k(0) + \sum_{k\in\mathcal{K}} \overline{E}_k(t) - \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t)-G_k^s(x)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) \\
&- \sum_{k\in\mathcal{K}} \int_0^t (\overline{Q}_k(0) + \overline{E}_k(u)) g_k^s(t-u) du \\
&\quad + \sum_{k\in\mathcal{K}} \int_0^t (\overline{Q}_k(u) + \overline{R}_k(u)) g_k^s(t-u) du - \sum_{k\in\mathcal{K}} \overline{R}_k(t).
\end{aligned}$$

Recall the definition of ξ in (3.27), the constructions of $\overline{Q}$ and $\overline{R}$, we then have that for each $t \in \mathbb{R}_+$,

$$\sum_{k\in\mathcal{K}} \overline{X}_k(t) = \xi(t) + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) A^k(t, x(u))\, du - \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u) B^k(t, x(u))\, du.$$

Since $x(\cdot)$ is the unique solution to (3.30), then for each $t \in \mathbb{R}_+$, the above display implies that $\sum_{k\in\mathcal{K}} \overline{X}_k(t) = x(t)$. Then, (3.47) implies that for each $t \in \mathbb{R}_+$,

$$x(t) = [x(t)-1]^+ + \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \overline{\nu}_t^k \rangle.$$

So for each $t \in \mathbb{R}_+$ such that $\sum_{k\in\mathcal{K}} \overline{X}_k(t) = x(t) > 1$, the above display implies that $\sum_{k\in\mathcal{K}} \langle \mathbf{1}, \overline{\nu}_t^k \rangle = 1$. Otherwise, when $\sum_{k\in\mathcal{K}} \overline{X}_k(t) = x(t) \le 1$, the above display implies that $\sum_{k\in\mathcal{K}} \overline{X}_k(t) = x(t) = \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \overline{\nu}_t^k \rangle$. Thus, we have that (2.10) holds. It follows that (2.11) holds as well. Combining this with the fact that $\sum_{k\in\mathcal{K}} \overline{X}_k(t) = x(t)$, the constructions of $\overline{Q}$ and $\overline{R}$ and the assumption that $x(t) - 1 \le \overline{F}_t(\overline{\chi}(0) + t)$ for each $t \in \mathbb{R}_+$, we have that (2.7), (2.8) hold. Then the constructed $(\overline{X}, \overline{\nu}, \overline{\eta})$ with its associated processes is a solution to the fluid model equations. Note that, under Assumption 1, the constructed $(\overline{X}, \overline{\nu}, \overline{\eta})$ with its associated processes are all continuous on $\mathbb{R}_+$. □

In the rest of this section, we prove that $\overline{L}_k$ is non-decreasing for each $k \in \mathcal{K}$ and that $x(t) - 1 \le \overline{F}_t(\overline{\chi}(0) + t)$ for each $t \in \mathbb{R}_+$ under Assumption 2. For this, let

$$\tilde{\chi}(t) = (\overline{F}_t)^{-1}\left([x(t) - 1]^+ \wedge \overline{F}_t(\overline{\chi}(0) + t)\right) \text{ for each } t \in \mathbb{R}_+. \tag{3.48}$$

Then $\tilde{\chi}(0) = \overline{\chi}(0)$ and $\tilde{\chi}(t) \le \overline{\chi}(0) + t$ for each $t \in \mathbb{R}_+$. The main idea to show that $\overline{L}_k$, $k \in \mathcal{K}$, is non-decreasing on $\mathbb{R}_+$ is to show that there exists an interval $[0, T]$ such that $\tilde{\chi}(t + u) \le \tilde{\chi}(t) + u$ for all $0 \le t \le t + u \le T$. From this, a similar argument as in Lemma 3 can be used to show that $\overline{L}_k$, $k \in \mathcal{K}$, is non-decreasing on $[0, T]$. Then a standard extension argument can be used to show that $\overline{L}_k$, $k \in \mathcal{K}$, is non-decreasing on $\mathbb{R}_+$.

We first show that there exists an interval $[0, T]$ such that $x(t) - 1 < \overline{F}_t(\overline{\chi}(0) + t)$ for all $0 < t \le T$. This result also plays very important role in showing the non-decreasing property of $\overline{L}_k$ for each $k \in \mathcal{K}$.

LEMMA 6. *Suppose that Assumption 1 holds, the service time densities* $\{g_k^s,\ k \in \mathcal{K}\}$ *are right continuous and* $\sum_{k\in\mathcal{K}} \int_{[0, H_k^s)} h_k^s(x)\, \overline{\nu}_0^k(dx) > 0$. *Then there exists* $T > 0$ *such that* $x(t) - 1 < \overline{F}_t(\overline{\chi}(0) + t)$ *for all* $0 < t \le T$.

Proof Note that $x(\cdot)$ is the unique continuous fixed point of Λ, a simple application of integration by parts yields that

ATTENTION: The following displayed equation, in its current form, exceeds the column width that will be used in the published edition of your article. Please break or rewrite this equation to fit, including the equation number, within a column width of 470 pt / 165.81 mm / 6.53 in (the width of this red box).

$$\begin{aligned} x(t) &= \xi(t) - \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u) B^k(u,x(u))du + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) A^k(u,x(u))du \\ &= \xi(t) - \sum_{k\in\mathcal{K}} \int_0^t B^k(u,x(u))du + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) \left(A^k(u,x(u)) + \int_0^u B^k(w,x(w))dw \right) du \\ &= \xi(t) - \sum_{k\in\mathcal{K}} \int_0^t B^k(u,x(u))du + g(t), \end{aligned}$$

where

$$g(t) \doteq \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) \left(A^k(u,x(u)) + \int_0^u B^k(w,x(w))dw \right) du.$$

Consider the process $\tilde{x}(\cdot)$ defined as

$$\tilde{x}(t) \doteq \xi(t) - \sum_{k\in\mathcal{K}} \int_0^t B^k(u,x(u))du + \tilde{g}(t),$$

where

$$\begin{aligned} \tilde{g}(t) \doteq \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) \Big(& A^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)) \\ & + \int_0^u B^k(w, 1+\overline{F}_w(\overline{\chi}(0)+w))dw \Big) du. \end{aligned} \tag{3.49}$$

Note that the difference between $g(t)$ and $\tilde{g}(t)$ is that $x(u)$ in the definition of $g(t)$ is replaced by $1+\overline{F}_u(\overline{\chi}(0)+u)$. By the definition of A^k and B^k in (3.29), for each $k\in\mathcal{K}$ and $u\in\mathbb{R}_+$,

$$\begin{cases} A^k(u,x(u)) \le A^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)), \\ B^k(u,x(u)) \le B^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)). \end{cases} \tag{3.50}$$

It follows that for each $t\in\mathbb{R}_+$, $\tilde{g}(t)\ge g(t)$ and then $\tilde{x}(t)\ge x(t)$ and $\tilde{x}(0)=\xi(0)=x(0)$. Now, consider the continuous function h on $\mathbb{R}_+$ defined by

$$h(t) \doteq \overline{F}_t(\overline{\chi}(0)+t) + 1 - \tilde{x}(t), \qquad t\in\mathbb{R}_+.$$

Note that for each $k \in \mathcal{K}$ and $u \in \mathbb{R}_+$, by the definition of A^k and B^k in (3.29), (2.5) and the fact that $H^r_k = \infty$, $k \in \mathcal{K}$ due to Remark 2,

$$A^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)) = \overline{F}^k_u(\overline{\chi}(0)+u)$$
$$= \int_{\mathbb{R}_+} \mathbf{1}_{[0,\overline{\chi}(0)]}(x) \frac{\bar{G}^r_k(x+u)}{\bar{G}^r_k(x)} \overline{\eta}^k_0(dx) + \int_0^u \bar{G}^r_k(u-s)\overline{\lambda}_k(s)ds$$

and

$$B^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)) = \int_{\mathbb{R}_+} \mathbf{1}_{[0,\overline{\chi}(0)+u]}(x) h^r_k(x) \overline{\eta}^k_u(dx)$$
$$= \int_{\mathbb{R}_+} \mathbf{1}_{[0,\overline{\chi}(0)]}(x) \frac{g^r_k(x+u)}{\bar{G}^r_k(x)} \overline{\eta}^k_0(dx) + \int_0^u g^r_k(u-s)\overline{\lambda}_k(s)ds.$$

Note that $A^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u))$ as a function of u is absolutely continuous with a.e. derivative

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$$\frac{\partial}{\partial u} A^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)) = -\int_{\mathbb{R}_+} \mathbf{1}_{[0,\overline{\chi}(0)]}(x) \frac{g^r_k(x+u)}{\bar{G}^r_k(x)} \overline{\eta}^k_0(dx) + \overline{\lambda}_k(u) - \int_0^u g^r_k(u-s)\overline{\lambda}_k(s)ds.$$

It follows that for each $k \in \mathcal{K}$ and $u \in \mathbb{R}_+$,

$$\frac{\partial}{\partial u} A^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)) + B^k(u, 1+\overline{F}_u(\overline{\chi}(0)+u)) = \overline{\lambda}_k(u). \tag{3.51}$$

By an application of change of variables to the right-hand side of (3.49),

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$$\tilde{g}(t) = \sum_{k\in\mathcal{K}} \int_0^t g^s_k(u) \left(A^k(t-u, 1+\overline{F}_{t-u}(\overline{\chi}(0)+t-u)) + \int_0^{t-u} B^k(w, 1+\overline{F}_w(\overline{\chi}(0)+w))dw \right) du.$$

By using (3.51), it follows that $\tilde{g}$ is absolutely continuous with a.e. derivative $\tilde{g}'(t)$ given by

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$$\begin{aligned}
\tilde{g}'(t) &= \sum_{k\in\mathcal{K}} g_k^s(t)A^k(0,1+\overline{F}_0(\overline{\chi}(0)+0)) \\
&\quad + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(u)\left(\frac{\partial}{\partial u}A^k(t-u,1+\overline{F}_{t-u}(\overline{\chi}(0)+t-u)) + B^k(t-u,1+\overline{F}_{t-u}(\overline{\chi}(0)+t-u))\right)du \\
&= \sum_{k\in\mathcal{K}} g_k^s(t)\overline{Q}_k(0) + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(u)\overline{\lambda}_k(t-u)du \\
&= \sum_{k\in\mathcal{K}} g_k^s(t)\overline{Q}_k(0) + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u)\overline{\lambda}_k(u)du.
\end{aligned}$$

It follows from the a.e. derivative of $\overline{F}_t(\overline{\chi}(0)+t)$ in (2.18), the expression of ξ in (3.27) and the a.e. derivative $\tilde{g}'$ of $\tilde{g}$ above that h is absolutely continuous and its a.e. derivative $h'(t)$ is given by

$$\begin{aligned}
h'(t) &= \left(-\sum_{k\in\mathcal{K}} \int_{[0,\overline{\chi}(0)]} \frac{g_k^r(x+t)}{\bar{G}_k^r(x)}\,\overline{\eta}_0^k(dx) + \sum_{k\in\mathcal{K}} \overline{\lambda}_k(t) - \sum_{k\in\mathcal{K}} \int_0^t g_k^r(t-s)\overline{\lambda}_k(s)\,ds\right) \\
&\quad + \left(\sum_{k\in\mathcal{K}} \overline{Q}_k(0)g_k^s(t) - \sum_{k\in\mathcal{K}} \overline{\lambda}_k(t) + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u)\overline{\lambda}_k(u)du\right. \\
&\quad \left. + \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{g_k^s(x+t)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)\right) + \sum_{k\in\mathcal{K}} B^k(t,x(t)) - g'(t) \\
&= \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{g_k^s(x+t)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx) + \sum_{k\in\mathcal{K}} \left(B^k(t,x(t)) - \int_{[0,\overline{\chi}(0)]} \frac{g_k^r(x+t)}{\bar{G}_k^r(x)}\,\overline{\eta}_0^k(dx)\right. \\
&\quad \left. - \int_0^t g_k^r(t-u)\overline{\lambda}_k(u)du\right).
\end{aligned}$$

For each $k \in \mathcal{K}$ and $t \in \mathbb{R}_+$, by the definition of $B^k(t,x)$ in (3.29), (2.5), Remark 1 and the definition of $\tilde{\chi}(t)$ in (3.48), we have that

$$\begin{aligned} B^k(t,x(t)) &= \int_0^{[x(t)-1]^+\wedge \overline{F}_t(\overline{\chi}(0)+t)} h_k^r((\overline{F}_t)^{-1}(u))d\overline{F}_t^k((\overline{F}_t)^{-1}(u)) \qquad (3.52)\\ &= \int_{\mathbb{R}_+} \mathbf{1}_{[0,\tilde{\chi}(t)]}(x)h_k^r(x)d\overline{F}_t^k(x) \\ &= \mathbf{1}_{\{\tilde{\chi}(t)>t\}}\left(\int_{\mathbb{R}_+} \mathbf{1}_{[0,\tilde{\chi}(t)-t]}(x)\frac{g_k^r(x+t)}{\bar{G}_k^r(x)}\overline{\eta}_0^k(dx) + \int_0^t g_k^r(t-s)\overline{\lambda}_k(s)ds\right) \\ &\quad + \mathbf{1}_{\{\tilde{\chi}(t)\le t\}}\int_{t-\tilde{\chi}(t)}^t g_k^r(t-s)\overline{\lambda}_k(s)ds. \end{aligned}$$

We now show that there exists $T>0$ such that $h'(t)>0$ for all $t\in[0,T]$ by considering the following two mutually exclusive cases:

Case 1: $\overline{\chi}(0)=0$. Then for each $k\in\mathcal{K}$, $\int_{[0,\overline{\chi}(0)]}\frac{g_k^r(x+t)}{\bar{G}_k^r(x)}\overline{\eta}_0^k(dx)=0$ since $\overline{\eta}_0^k(\{0\})=0$ by Assumption 1. Also in this case, by the definition of $\tilde{\chi}(t)$ in (3.48), $\tilde{\chi}(t)\le t$ for all $t\in\mathbb{R}_+$. Then, by (3.52), $B^k(t,x(t))=\int_{t-\tilde{\chi}(t)}^t g_k^r(t-u)\overline{\lambda}_k(u)du$ for each $k\in\mathcal{K}$. Thus, it follows that

$$\begin{aligned} &\sum_{k\in\mathcal{K}}\left(B^k(t,x(t)) - \int_{[0,\overline{\chi}(0)]}\frac{g_k^r(x+t)}{\bar{G}_k^r(x)}\overline{\eta}_0^k(dx) - \int_0^t g_k^r(t-u)\overline{\lambda}_k(u)du\right)\\ &= \sum_{k\in\mathcal{K}}\left(\int_{t-\tilde{\chi}(t)}^t g_k^r(t-u)\overline{\lambda}_k(u)du - \int_0^t g_k^r(t-u)\overline{\lambda}_k(u)du\right)\\ &= -\sum_{k\in\mathcal{K}}\int_0^{t-\tilde{\chi}(t)} g_k^r(t-u)\overline{\lambda}_k(u)du\\ &\to 0 \text{ as } t\downarrow 0. \end{aligned}$$

By the assumed right continuity of g_k^s, $k\in\mathcal{K}$, and an application of Fatou's Lemma, we have that

$$0<\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_0^k(dx) \le \liminf_{t\downarrow 0}\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{g_k^s(x+t)}{\bar{G}_k^s(x)}\overline{\nu}_0^k(dx). \qquad (3.53)$$

Thus, it follows that there exists $T>0$ such that $h'(t)>0$ for all $t\in[0,T]$.

Case 2: $\overline{\chi}(0)>0$. Since $\tilde{\chi}(t)\le\overline{\chi}(0)+t$ for each $t>0$, then $\limsup_{t\to 0}\tilde{\chi}(t)\le\overline{\chi}(0)$. We claim that $\liminf_{t\to 0}\tilde{\chi}(t)\ge\overline{\chi}(0)$. Suppose that the claim is not true, then $\liminf_{t\to 0}\tilde{\chi}(t)<\overline{\chi}(0)$. Then there exist a $\delta \in (0, \overline{\chi}(0))$ and a sequence $\{t_n, n \in \mathbb{N}\}$ such that $t_n \to 0$ as $n \to \infty$ and $\tilde{\chi}(t_n) < \overline{\chi}(0) - \delta$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$,

$$\overline{F}_{t_n}(\overline{\chi}(0) - \delta) > [x(t_n) - 1]^+ \wedge \overline{F}_{t_n}(\overline{\chi}(0) + t_n).$$

By using the fact that $[x(0) - 1]^+ = \overline{F}_0(\overline{\chi}(0))$ and using the continuity of $x(\cdot)$, the continuity of $\overline{F}_t(x)$ as a function of t for each $x \in \mathbb{R}_+$ and the continuity of $\overline{F}_t(\overline{\chi}(0) + t)$ as a function of t, we have that

$$\overline{F}_0(\overline{\chi}(0) - \delta) \geq [x(0) - 1]^+ \wedge \overline{F}_0(\overline{\chi}(0)) = [x(0) - 1]^+.$$

It follows that

$$\overline{\chi}(0) - \delta \geq (\overline{F}_0)^{-1}\left(\overline{F}_0(\overline{\chi}(0) - \delta)\right) \geq (\overline{F}_0)^{-1}\left([x(0) - 1]^+\right) = \overline{\chi}(0),$$

which is a contradiction. Thus, we proved that $\liminf_{t\to 0} \tilde{\chi}(t) \geq \overline{\chi}(0)$ and then $\lim_{t\to 0} \tilde{\chi}(t) = \overline{\chi}(0)$. By (3.52), $\overline{\eta}_0^k(\{\overline{\chi}(0)\}) = 0$ and the right continuity of h_k^r (since g_k^r is assumed to be right continuous) for each $k \in \mathcal{K}$, the local boundedness of h_k^r, $k \in \mathcal{K}$ in Assumption 1 and an application of the dominated convergence theorem, we get that for each $k \in \mathcal{K}$,

$$\lim_{t\downarrow 0} B^k(t, x(t)) = \int_{[0,\overline{\chi}(0)]} \frac{g_k^r(x)}{\bar{G}_k^r(x)}\, \overline{\eta}_0^k(dx)$$

and then

$$\sum_{k\in\mathcal{K}} \left(B^k(t, x(t)) - \int_{[0,\overline{\chi}(0)]} \frac{g_k^r(x+t)}{\bar{G}_k^r(x)}\, \overline{\eta}_0^k(dx) - \int_0^t g_k^r(t-u)\overline{\lambda}_k(u)du \right) \to 0 \text{ as } t \downarrow 0.$$

By using (3.53), it follows that there exists $T > 0$ such that $h'(t) > 0$ for all $t \in [0, T]$. Thus, in both cases, we yield that there exists $T > 0$ such that $h'(t) > 0$ for all $t \in [0, T]$, which in turn implies that for all $t \in [0, T]$,

$$\overline{F}_t(\overline{\chi}(0) + t) + 1 - \tilde{x}(t) > \overline{F}_0(\overline{\chi}(0)) + 1 - \tilde{x}(0) = [x(0) - 1]^+ + 1 - x(0) \geq 0,$$

then $\overline{F}_t(\overline{\chi}(0) + t) > \tilde{x}(t) - 1 \geq x(t) - 1$ for all $t \in [0, T]$. □

It follows from Lemma 6 that $\overline{F}_t(\overline{\chi}(0) + t) \geq [x(t) - 1]^+$ for all $t \in [0, T]$. By (3.48), for each $t \in [0, T]$,

$$\tilde{\chi}(t) = (\overline{F}_t)^{-1}\left([x(t) - 1]^+\right) \text{ and then } \tilde{\chi}(t) \leq \overline{\chi}(0) + t. \tag{3.54}$$

Also it follows from (3.29) and Lemma 6 that for each $k \in \mathcal{K}$ and $t \in [0,T]$,

$$\begin{cases} A^k(t,x(t)) = \overline{F}^k_t\left((\overline{F}_t)^{-1}([x(t)-1]^+)\right) = \overline{F}^k_t(\tilde{\chi}(t)), \\ B^k(t,x(t)) = \int_0^{[x(t)-1]^+} h^r_k((\overline{F}_t)^{-1}(u))d\overline{F}^k_t((\overline{F}_t)^{-1}(u)) = \int_{[0,\tilde{\chi}(t)]} h^r_k(x)d\overline{F}^k_t(x). \end{cases} \tag{3.55}$$

We now define a time shifted version of the functional map of Λ along with the time shifted auxiliary processes $x,\ \overline{E},\ \overline{F},\ A^k,\ B^k,\ \overline{F}^k,\ k \in \mathcal{K}$. For each $t \in [0,T]$, let

$$x^{[t]} \doteq x(t+\cdot),\ \overline{E}^{[t]} \doteq \overline{E}(t+\cdot) - \overline{E}(t) \text{ and } \overline{\eta}^{[t]} \doteq \overline{\eta}_{t+\cdot}.$$

By Lemma 1, for each $k \in \mathcal{K}$, $\overline{\eta}^{[t],k}$ satisfies (2.5) with $\overline{\eta}^k_t$ and $\overline{E}^{[t]}_k$ in place of $\overline{\eta}^k_0$ and $\overline{E}_k$. For each $k \in \mathcal{K}$ and $u,\ x \in \mathbb{R}_+$, let

$$\overline{F}^{[t],k}_u(x) \doteq \overline{\eta}^{[t],k}_u[0,x] = \overline{\eta}^k_{t+u}[0,x] \text{ and } \overline{F}^{[t]}_u(x) \doteq \sum_{k\in\mathcal{K}} \overline{F}^{[t],k}_u(x),$$

and then

$$\overline{F}^{[t],k}_u(x) = \overline{F}^k_{t+u}(x) \text{ and } \overline{F}^{[t]}_u(x) = \overline{F}_{t+u}(x).$$

For each $k \in \mathcal{K}$, $t \in [0,T]$ and $u \in \mathbb{R}_+$, let

$$A^{[t],k}(u,x) \doteq A^k(t+u,x) \text{ and } B^{[t],k}(u,x) \doteq B^k(t+u,x). \tag{3.56}$$

It follows from (3.55) that for each $t \in [0,T]$ and $u \in \mathbb{R}_+$ such that $t+u \in [0,T]$,

$$A^{[t],k}(u,x^{[t]}(u)) = \overline{F}^{[t],k}_u\left((\overline{F}^{[t]}_u)^{-1}\left([x^{[t]}(u)-1]^+\right)\right) \tag{3.57}$$

and

$$B^{[t],k}(u,x^{[t]}(u)) = \int_0^{[x^{[t]}(u)-1]^+} h^r_k((\overline{F}^{[t]}_u)^{-1}(w))d\overline{F}^{[t],k}_u((\overline{F}^{[t]}_u)^{-1}(w)). \tag{3.58}$$

For each $t \in [0,T]$, let $\xi^{[t]}$ be the function on $\mathbb{R}_+$ defined as

$$\begin{aligned} \xi^{[t]}(u) \doteq{}& x(t) - \sum_{k\in\mathcal{K}} \overline{Q}_k(t)G^s_k(u) + \sum_{k\in\mathcal{K}} \int_0^u \bar{G}^s_k(u-s)d\overline{E}^{[t]}_k(s) \\ &- \sum_{k\in\mathcal{K}} \int_{[0,H^s_k)} \frac{G^s_k(x+t+u) - G^s_k(x+t)}{\bar{G}^s_k(x)} \overline{\nu}^k_0(dx) \\ &- \sum_{k\in\mathcal{K}} \overline{L}_k(t)G^s_k(u) - \sum_{k\in\mathcal{K}} \int_0^t (g^s_k(t+u-s) - g^s_k(t-s))\overline{L}_k(s)ds, \end{aligned} \tag{3.59}$$

and let $\Lambda^{[t]}$ be the time-shifted functional map of Λ by t defined on $C(\mathbb{R}_+)$ as

$$\Lambda^{[t]}(y)(u) \doteq \xi^{[t]}(u) - \sum_{k\in\mathcal{K}} \int_0^u \bar{G}_k^s(u-v)B^{[t],k}(v,y(v))dv + \sum_{k\in\mathcal{K}} \int_0^u g_k^s(u-v)A^{[t],k}(v,y(v))dv. \tag{3.60}$$

Lemma 7. *For each $t \in [0,T]$, $x^{[t]}(\cdot)$ is the unique fixed point of $\Lambda^{[t]}$ in (3.60).*

Proof Fix $t \in [0,T]$. Since $x(\cdot)$ be the unique continuous solution to the equation (3.30) associated with $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)$, we have from (3.31), (3.56) and an application of change of variables that for each $h \in \mathbb{R}_+$,

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$$\begin{aligned} x(t+h) &= \xi(t+h) - \sum_{k\in\mathcal{K}} \int_0^{t+h} \bar{G}_k^s(t+h-u)B^k(u,x(u))du \\ &\quad + \sum_{k\in\mathcal{K}} \int_0^{t+h} g_k^s(t+h-u)A^k(u,x(u))du \\ &= \zeta^{[t]}(h) - \sum_{k\in\mathcal{K}} \int_0^h \bar{G}_k^s(h-u)B^{[t],k}(u,x^{[t]}(u))du + \sum_{k\in\mathcal{K}} \int_0^h g_k^s(h-u)A^{[t],k}(u,x^{[t]}(u))du, \end{aligned} \tag{3.61}$$

where

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$$\zeta^{[t]}(h) \doteq \xi(t+h) - \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t+h-u)B^k(u,x(u))du + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t+h-u)A^k(u,x(u))du.$$

Using the fact that $x(\cdot)$ is the unique continuous solution to the equation (3.30), we have that for each $h \in \mathbb{R}_+$,

$$\begin{aligned}
\xi^{[t]}(h) &= \xi(t) - \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u) B^k(u,x(u))du + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) A^k(u,x(u))du \\
&\quad + (\xi(t+h)-\xi(t)) - \sum_{k\in\mathcal{K}} \int_0^t (\bar{G}_k^s(t+h-u) - \bar{G}_k^s(t-u)) B^k(u,x(u))du \\
&\quad + \sum_{k\in\mathcal{K}} \int_0^t (g_k^s(t+h-u) - g_k^s(t-u)) A^k(u,x(u))du \\
&= x(t) + (\xi(t+h)-\xi(t)) + \sum_{k\in\mathcal{K}} \int_0^t (g_k^s(t+h-u) - g_k^s(t-u)) A^k(u,x(u))du \\
&\quad - \sum_{k\in\mathcal{K}} \int_0^t (\bar{G}_k^s(t+h-u) - \bar{G}_k^s(t-u)) B^k(u,x(u))du.
\end{aligned}$$

Using the definition ξ in (3.27) and applications of change of variables and integration by parts, we can see that

$$\begin{aligned}
\xi(t+h)-\xi(t) &= -\sum_{k\in\mathcal{K}} \overline{Q}_k(0)(G_k^s(t+h) - G_k^s(t)) + \sum_{k\in\mathcal{K}} \int_0^h \bar{G}_k^s(h-u) d\overline{E}_k^{[t]}(u) \\
&\quad + \sum_{k\in\mathcal{K}} \int_0^t (\bar{G}_k^s(t+h-u) - \bar{G}_k^s(t-u)) d\overline{E}_k(u) \\
&\quad - \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t+h) - G_k^s(x+t)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) \\
&= -\sum_{k\in\mathcal{K}} \overline{Q}_k(0)(G_k^s(t+h) - G_k^s(t)) + \sum_{k\in\mathcal{K}} \int_0^h \bar{G}_k^s(h-u) d\overline{E}_k^{[t]}(u) \\
&\quad - \sum_{k\in\mathcal{K}} \overline{E}_k(t) G_k^s(h) - \sum_{k\in\mathcal{K}} \int_0^t (g_k^s(t+h-u) - g_k^s(t-u)) \overline{E}_k(u)du \\
&\quad - \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{G_k^s(x+t+h) - G_k^s(x+t)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx).
\end{aligned}$$

By another application of integration by parts and (3.41), we have that

$$
\begin{aligned}
&\sum_{k\in\mathcal{K}}\int_0^t (\bar{G}_k^s(t+h-u)-\bar{G}_k^s(t-u))B^k(u,x(u))du \\
&= -\sum_{k\in\mathcal{K}}\overline{R}_k(t)G_k^s(h)-\sum_{k\in\mathcal{K}}\int_0^t (g_k^s(t+h-u)-g_k^s(t-u))\overline{R}_k(u)du.
\end{aligned}
$$

Combining the above three displays and the definition of $\overline{L}_k$ in (3.42), we have that for each $h\in\mathbb{R}_+$,

$$
\begin{aligned}
\zeta^{[t]}(h) &= x(t)-\sum_{k\in\mathcal{K}}\overline{Q}_k(0)(G_k^s(t+h)-G_k^s(t))+\sum_{k\in\mathcal{K}}\int_0^h \bar{G}_k^s(h-u)d\overline{E}_k^{[t]}(u) \\
&\quad -\sum_{k\in\mathcal{K}}\overline{E}_k(t)G_k^s(h)-\sum_{k\in\mathcal{K}}\int_0^t (g_k^s(t+h-u)-g_k^s(t-u))\overline{E}_k(u)du \\
&\quad -\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{G_k^s(x+t+h)-G_k^s(x+t)}{\bar{G}_k^s(x)}\overline{\nu}_0^k(dx) \\
&\quad +\sum_{k\in\mathcal{K}}\int_0^t (g_k^s(t+h-u)-g_k^s(t-u))A^k(u,x(u))du \\
&\qquad +\sum_{k\in\mathcal{K}}\overline{R}_k(t)G_k^s(h)+\sum_{k\in\mathcal{K}}\int_0^t (g_k^s(t+h-u)-g_k^s(t-u))\overline{R}_k(u)du \\
&= x(t)+\sum_{k\in\mathcal{K}}(-\overline{Q}_k(0)-\overline{E}_k(t)+\overline{R}_k(t))G_k^s(h)+\sum_{k\in\mathcal{K}}\int_0^h \bar{G}_k^s(h-u)d\overline{E}_k^{[t]}(u) \\
&\quad -\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{G_k^s(x+t+h)-G_k^s(x+t)}{\bar{G}_k^s(x)}\overline{\nu}_0^k(dx) \\
&\quad -\sum_{k\in\mathcal{K}}\int_0^t (g_k^s(t+h-u)-g_k^s(t-u))(\overline{Q}_k(0)+\overline{E}_k(u)-\overline{Q}_k(u)-\overline{R}_k(u))du \\
&= x(t)-\sum_{k\in\mathcal{K}}\overline{Q}_k(t)G_k^s(h)+\sum_{k\in\mathcal{K}}\int_0^h \bar{G}_k^s(h-u)d\overline{E}_k^{[t]}(u) \\
&\quad -\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{G_k^s(x+t+h)-G_k^s(x+t)}{\bar{G}_k^s(x)}\overline{\nu}_0^k(dx) \\
&\quad -\sum_{k\in\mathcal{K}}\overline{L}_k(t)G_k^s(h)-\sum_{k\in\mathcal{K}}\int_0^t (g_k^s(t+h-u)-g_k^s(t-u))\overline{L}_k(u)du.
\end{aligned}
$$

Comparing this with (3.59), we see that $\xi^{[t]}=\zeta^{[t]}$. This and (3.61) together imply that for each $h\in\mathbb{R}_+$,

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$$x^{[t]}(h) = \xi^{[t]}(h) - \sum_{k\in\mathcal{K}} \int_0^h \bar{G}_k^s(h-u) B^{[t],k}(u, x^{[t]}(u)) du + \sum_{k\in\mathcal{K}} \int_0^h g_k^s(h-u) A^{[t],k}(u, x^{[t]}(u)) du.$$

Thus, $x^{[t]}(\cdot)$ is a fixed point of $\Lambda^{[t]}$. The uniqueness of fixed points of $\Lambda^{[t]}$ follows exactly the same argument as for the uniqueness of fixed points of Λ in Proposition 2. □

COROLLARY 1. *Suppose that Assumption 1 holds, the service time densities $\{g_k^s,\ k\in\mathcal{K}\}$ are right continuous on their supports and are absolutely continuous on $[0,\delta]$ for some $\delta > 0$, and suppose that $\sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} h_k^s(x)\, \overline{\nu}_0^k(dx) > 0$. Then there exists $\tilde{T} \in (0,T)$ such that for each $0 \le t < t+u \le \tilde{T}$, $\tilde{\chi}(t+u) \le \tilde{\chi}(t) + u$.*

Proof Note that g_k^s, $k\in\mathcal{K}$, are absolutely continuous on $[0,\delta]$ for some $\delta > 0$, without loss of generality, we may assume that g_k^s, $k\in\mathcal{K}$, are absolutely continuous on $[0,T]$. For each $t\in[0,T]$, by Lemma 7, $x^{[t]}(\cdot) = x(t+\cdot)$ is a fixed point of $\Lambda^{[t]}$ restricted on $[0, T-t]$.

For each $t\in[0,T]$, $k\in\mathcal{K}$ and $u\in\mathbb{R}_+$, Define

$$\tilde{A}^{[t],k}(u,y) \doteq \overline{F}_u^{[t],k}\left((\overline{F}_u^{[t]})^{-1}\left([y-1]^+ \wedge \overline{F}_u^{[t]}(\tilde{\chi}(t)+u) \right) \right) \tag{3.62}$$

and

$$\tilde{B}^{[t],k}(u,y) \doteq \int_0^{[y-1]^+\wedge\overline{F}_u^{[t]}(\tilde{\chi}(t)+u)} h_k^r((\overline{F}_u^{[t]})^{-1}(w)) d\overline{F}_u^{[t],k}((\overline{F}_u^{[t]})^{-1}(w)). \tag{3.63}$$

Since, by Lemma 1, for each $k\in\mathcal{K}$, $\overline{\eta}^{[t],k}$ satisfies (2.5) with $\overline{\eta}_t^k$ and $\overline{E}_k^{[t]}$ in place of $\overline{\eta}_0^k$ and $\overline{E}_k$, then it can be checked readily that $\tilde{A}^{[t],k}$ and $\tilde{B}^{[t],k}$, $k\in\mathcal{K}$, also satisfy the three properties of Lemma 4 with $\tilde{C}_u^{r,k} \doteq \sup_{0\le v\le\tilde{\chi}(t)+u} h_k^r(v)$ in place of $C_t^{r,k}$. Consider the following functional map $\tilde{\Lambda}^{[t]}$ analog to Λ in (3.31):

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$$\begin{aligned}\tilde{\Lambda}^{[t]}(y)(u) &\doteq \xi^{[t]}(u) - \sum_{k\in\mathcal{K}}\int_0^u \bar{G}_k^s(u-s)\tilde{B}^{[t],k}(s,y(s))ds + \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\tilde{A}^{[t],k}(s,y(s))ds \\ &= \xi^{[t]}(u) - \sum_{k\in\mathcal{K}}\int_0^u \tilde{B}^{[t],k}(s,y(s))ds \\ &+ \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\left(\tilde{A}^{[t],k}(s,y(s)) + \int_0^s \tilde{B}^{[t],k}(w,y(w))dw\right)ds,\end{aligned}$$

where $\xi^{[t]}$ is given in (3.59). Note that for each $k\in\mathcal{K}$ and $x\in\mathbb{R}_+$, $\overline{\eta}_t^k(\{x\})=0$ due to the fact that $\overline{\eta}_0^k(\{x\})=0$ for each $k\in\mathcal{K}$ and $x\in\mathbb{R}_+$, the absolute continuity of $\overline{E}_k$ for each $k\in\mathcal{K}$, the definition of $\overline{\eta}_t$ and Remark 1. Also note that for each $k\in\mathcal{K}$, $\overline{E}_k^{[t]}$ is absolutely continuous with a.e. derivative $\overline{\lambda}^{[t]}(\cdot)\doteq\overline{\lambda}(t+\cdot)$ and $\tilde{\chi}(t)\le\overline{\chi}(0)+t$. Thus, $\overline{\eta}_t$, $\overline{E}^{[t]}$ and $\tilde{\chi}(t)$ also satisfies Assumption 1. It follows from Proposition 2 that $\tilde{\Lambda}^{[t]}$ has a unique continuous fixed point $\tilde{y}^{[t]}$ on $\mathbb{R}_+$. Consider the process $z^{[t]}(\cdot)$ defined as

$$z^{[t]}(u)\doteq\xi^{[t]}(u)-\sum_{k\in\mathcal{K}}\int_0^u\tilde{B}^{[t],k}(s,\tilde{y}^{[t]}(s))ds+g^{[t]}(u),$$

where

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$$g^{[t]}(u)\doteq\sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\left(\tilde{A}^{[t],k}(s,1+\overline{F}_s^{[t]}(\tilde{\chi}(t)+s))+\int_0^s\tilde{B}^{[t],k}(w,1+\overline{F}_w^{[t]}(\tilde{\chi}(t)+w))dw\right)ds.$$

Then $z^{[t]}(u)\ge\tilde{y}^{[t]}(u)$ for each $u\in\mathbb{R}_+$. Consider the continuous function $h^{[t]}$ on $\mathbb{R}_+$ defined by

$$h^{[t]}(u)\doteq\overline{F}_u^{[t]}(\tilde{\chi}(t)+u)+1-z^{[t]}(u),\qquad u\in\mathbb{R}_+.$$

By using exactly the same argument as in the proof of Lemma 6, we have that $\overline{F}_u^{[t]}(\tilde{\chi}(t)+u)$, as a function of u, is absolutely continuous with a.e derivative

$$\Big(\overline{F}_u^{[t]}(\tilde{\chi}(t)+u)\Big)' = -\sum_{k\in\mathcal{K}}\int_{[0,\tilde{\chi}(t)]}\frac{g_k^r(x+u)}{\bar{G}_k^r(x)}\,\overline{\eta}_0^{[t],k}(dx) + \sum_{k\in\mathcal{K}}\overline{\lambda}_k^{[t]}(u) - \sum_{k\in\mathcal{K}}\int_0^u g_k^r(u-s)\overline{\lambda}_k^{[t]}(s)\,ds, \tag{3.64}$$

and that $g^{[t]}$ is absolutely continuous with a.e. derivative $(g^{[t]})'(u)$ given by

$$\begin{aligned}(g^{[t]})'(u) &= \sum_{k\in\mathcal{K}} g_k^s(u)\tilde{A}^{[t],k}(0,1+\overline{F}_0^{[t]}(\tilde{\chi}(t))) + \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\overline{\lambda}_k^{[t]}(s)ds\\ &= \sum_{k\in\mathcal{K}} g_k^s(u)\overline{F}_0^{[t],k}\left((\overline{F}_0^{[t]})^{-1}\left(\overline{F}_0^{[t]}(\tilde{\chi}(t))\right)\right) + \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\overline{\lambda}_k^{[t]}(s)ds\\ &= \sum_{k\in\mathcal{K}} g_k^s(u)\overline{F}_t^k(\tilde{\chi}(t)) + \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\overline{\lambda}_k^{[t]}(s)ds\\ &= \sum_{k\in\mathcal{K}} g_k^s(u)\overline{Q}_k(t) + \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\overline{\lambda}_k^{[t]}(s)ds,\end{aligned}$$

where the last equality follows from the definition of $\overline{Q}$ in (3.40) and the expression of $\tilde{\chi}(t)$ in (3.54). For each $k\in\mathcal{K}$, since g_k^s is absolutely continuous on $[0,T]$, g_k^s has a.e. derivative $(g_k^s)'$ that is Lebesgue integrable. From the definition $\xi^{[t]}$ in (3.59), we see that $\xi^{[t]}$ is absolute continuous with a.e. derivative given by

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$$\begin{aligned}(\xi^{[t]})'(u) = &-\sum_{k\in\mathcal{K}}\overline{Q}_k(t)g_k^s(u) + \sum_{k\in\mathcal{K}}\overline{\lambda}_k^{[t]}(u) - \sum_{k\in\mathcal{K}}\int_0^u g_k^s(u-s)\overline{\lambda}_k^{[t]}(s)ds\\ &-\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{g_k^s(x+t+u)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx) - \sum_{k\in\mathcal{K}}\overline{L}_k(t)g_k^s(u) - \sum_{k\in\mathcal{K}}\int_0^t (g_k^s)'(t+u-s)\overline{L}_k(u)ds.\end{aligned}$$

Combining the above three displays, we see that $h^{[t]}$ is also absolute continuous with a.e. derivative

$$
\begin{aligned}
(h^{[t]})'(u) &= \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{g_k^s(x+t+u)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx) + \sum_{k\in\mathcal{K}} \overline{L}_k(t) g_k^s(u) \\
&\quad + \sum_{k\in\mathcal{K}} \int_0^t (g_k^s)'(t+u-s)\overline{L}_k(s)ds \\
&\quad + \sum_{k\in\mathcal{K}} \left(\tilde{B}^{[t],k}(u,\tilde{y}^{[t]}(u)) - \int_{[0,\tilde{\chi}(t)]} \frac{g_k^r(x+u)}{\bar{G}_k^r(x)} \overline{\eta}_0^{[t],k}(dx) - \int_0^u g_k^r(u-s)\overline{\lambda}_k^{[t]}(s)\, ds \right).
\end{aligned}
$$

Moreover, for each $k \in \mathcal{K}$ and $u \in \mathbb{R}_+$,

$$
\begin{aligned}
&\tilde{B}^{[t],k}(u,\tilde{y}^{[t]}(u)) \\
&= \int_0^{[\tilde{y}^{[t]}(u)-1]^+ \wedge \overline{F}_u^{[t]}(\tilde{\chi}(t)+u)} h_k^r((\overline{F}_u^{[t]})^{-1}(w)) d\overline{F}_u^{[t],k}((\overline{F}_u^{[t]})^{-1}(w)) \\
&= \int_{\mathbb{R}_+} \mathbf{1}_{[0,\tilde{\chi}^{[t]}(u)]}(x) h_k^r(x) d\overline{F}_u^{[t],k}(x) \\
&= \mathbf{1}_{\{\tilde{\chi}^{[t]}(u)>u\}} \left(\int_{\mathbb{R}_+} \mathbf{1}_{[0,\tilde{\chi}^{[t]}(u)-u]}(x) \frac{g_k^r(x+u)}{\bar{G}_k^r(x)} \overline{\eta}_0^{[t],k}(dx) + \int_0^u g_k^r(u-s)\overline{\lambda}_k^{[t]}(s)ds \right) \\
&\quad + \mathbf{1}_{\{\tilde{\chi}^{[t]}(u)\le u\}} \int_{u-\tilde{\chi}^{[t]}(u)}^u g_k^r(u-s)\overline{\lambda}_k^{[t]}(s)ds,
\end{aligned}
$$

where

$$\tilde{\chi}^{[t]}(u) \doteq (\overline{F}_u^{[t]})^{-1}([\tilde{y}^{[t]}(u)-1]^+ \wedge \overline{F}_u^{[t]}(\tilde{\chi}(t)+u)) \le \tilde{\chi}(t)+u.$$

By considering the two mutually exclusive cases of $\tilde{\chi}(t)=0$ and $\tilde{\chi}(t)>0$ using exactly the same argument as the one after (3.52) in the proof of Lemma 6, we have that as $u \downarrow 0$,

$$\sum_{k\in\mathcal{K}} \left(\tilde{B}^{[t],k}(u,\tilde{y}^{[t]}(u)) - \int_{[0,\tilde{\chi}(t)]} \frac{g_k^r(x+u)}{\bar{G}_k^r(x)} \overline{\eta}_0^{[t],k}(dx) - \int_0^u g_k^r(u-s)\overline{\lambda}_k^{[t]}(s)\, ds \right) \to 0.$$

By the assumed right continuity of g_k^s, $k \in \mathcal{K}$ and an application of Fatou's Lemma, we have that

$$0 < \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} h_k^s(x)\, \overline{\nu}_0^k(dx) \le \liminf_{t,u\downarrow 0} \sum_{k\in\mathcal{K}} \int_{[0,H_k^s)} \frac{g_k^s(x+t+u)}{\bar{G}_k^s(x)} \overline{\nu}_0^k(dx).$$

By the definition $\overline{L}_k$, $k \in \mathcal{K}$, in (3.42), $\overline{L}_k$ is continuous with $\overline{L}_k(0) = 0$ for each $k \in \mathcal{K}$. This and the fact that for each $k \in \mathcal{K}$, g_k^s is absolutely continuous on $[0,T]$ with a.e. Lebesgue integrable derivative $(g_k^s)'$ imply that

$$\sum_{k\in\mathcal{K}} \overline{L}_k(t) g_k^s(u) + \sum_{k\in\mathcal{K}} \int_0^t (g_k^s)'(t+u-s)\overline{L}_k(s)ds \to 0 \text{ as } t,u \downarrow 0.$$

Thus, there exists $\tilde{T} \in (0,T]$ such that $\left(h^{[t]}\right)'(u) > 0$ for all $0 \le t < t+u \le \tilde{T}$. Since by (3.54) and (3.59),

$$h^{[t]}(0) = \overline{F}_0^{[t]}(\tilde{\chi}(t)) + 1 - z^{[t]}(0) = \overline{F}_t(\tilde{\chi}(t)) + 1 - \xi^{[t]}(0) = [x(t)-1]^+ + 1 - x(t) \ge 0,$$

it follows that for each $t \in [0,\tilde{T})$ and each $u \in [0, \tilde{T}-t]$,

$$\overline{F}_u^{[t]}(\tilde{\chi}(t)+u) \ge z^{[t]}(u) - 1 \ge \tilde{y}^{[t]}(u) - 1 \text{ and hence } \overline{F}_u^{[t]}(\tilde{\chi}(t)+u) \ge [\tilde{y}^{[t]}(u)-1]^+.$$

Note that for $t \in [0,\tilde{T})$, by (3.54), $\tilde{\chi}(t) \le \overline{\chi}(0) + t$, then it follows that

$$[\tilde{y}^{[t]}(u)-1]^+ \le \overline{F}_u^{[t]}(\tilde{\chi}(t)+u) \le \overline{F}_u^{[t]}(\overline{\chi}(0)+t+u). \tag{3.65}$$

Then by the definitions of $\tilde{A}^{[t],k}$ and $\tilde{B}^{[t],k}$ in (3.62) and (3.63), respectively and the definitions of $A^{[t],k}$ and $B^{[t],k}$ in (3.56), we have that for each $t \in [0,\tilde{T})$ and each $u \in [0,\tilde{T}-t]$,

$$\tilde{A}^{[t],k}(u,\tilde{y}^{[t]}(u)) = \overline{F}_u^{[t],k}\left((\overline{F}_u^{[t]})^{-1}\left([\tilde{y}^{[t]}(u)-1]^+ \wedge \overline{F}_u^{[t]}(\overline{\chi}(0)+t+u)\right)\right) = A^{[t],k}(u,\tilde{y}^{[t]}(u))$$

and

$$\begin{aligned}\tilde{B}^{[t],k}(u,\tilde{y}^{[t]}(u)) &= \int_0^{[\tilde{y}^{[t]}(u)-1]^+ \wedge \overline{F}_u^{[t]}(\overline{\chi}(0)+t+u)} h_k^r((\overline{F}_u^{[t]})^{-1}(w))d\overline{F}_u^{[t],k}((\overline{F}_u^{[t]})^{-1}(w)) \\ &= B^{[t],k}(u,\tilde{y}^{[t]}(u)).\end{aligned}$$

For each $t \in [0,\tilde{T})$, combining the above two displays with the fact that $\tilde{y}^{[t]}$ is the unique fixed point of $\tilde{\Lambda}^{[t]}$, we have that for each $u \in [0,\tilde{T}-t]$,

$$\begin{aligned}\tilde{y}^{[t]}(u) = \tilde{\Lambda}^{[t]}(\tilde{y}^{[t]})(u) &= \xi^{[t]}(u) - \sum_{k\in\mathcal{K}} \int_0^u \bar{G}_k^s(u-s)\tilde{B}^{[t],k}(s,\tilde{y}^{[t]}(s))ds \\ &\quad + \sum_{k\in\mathcal{K}} \int_0^u g_k^s(u-s)\tilde{A}^{[t],k}(s,\tilde{y}^{[t]}(s))ds \\ &= \xi^{[t]}(u) - \sum_{k\in\mathcal{K}} \int_0^u \bar{G}_k^s(u-s)B^{[t],k}(s,\tilde{y}^{[t]}(s))ds \\ &\quad + \sum_{k\in\mathcal{K}} \int_0^u g_k^s(u-s)A^{[t],k}(s,\tilde{y}^{[t]}(s))ds.\end{aligned}$$

Then $\tilde{y}^{[t]}$ is a fixed point of $\Lambda^{[t]}$ on $[0,\tilde{T}-t]$. Since $\Lambda^{[t]}$ admits a unique fixed point by Lemma 7, then $x^{[t]} = \tilde{y}^{[t]}$ on $[0,\tilde{T}-t]$. Thus, by (3.65), for each $t \in [0,\tilde{T})$ and each $u \in [0,\tilde{T}-t]$, $\overline{F}_u^{[t]}(\tilde{\chi}(t)+u) \geq [x^{[t]}(u)-1]^+$ and then

$$\tilde{\chi}(t+u) = (\overline{F}_{t+u})^{-1}\left([x(t+u)-1]^+\right) = (\overline{F}_u^{[t]})^{-1}\left([x^{[t]}(u)-1]^+\right) \leq \tilde{\chi}(t)+u.$$

This completes the proof of this lemma. □

COROLLARY 2. *Suppose that Assumptions 1 holds, the service time densities $\{g_k^s,\ k\in\mathcal{K}\}$ are right continuous on their supports and are absolutely continuous on $[0,\delta]$ for some $\delta>0$, and suppose that $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_0^k(dx) > 0$. Then for $\tilde{T}$ given in Corollary 1 and $k\in\mathcal{K}$, $\overline{L}_k$ is non-decreasing on $[0,\tilde{T}]$.*

Proof By Corollary 1, for each $0 \leq t < t+u \leq \tilde{T}$, $\tilde{\chi}(t+u) \leq \tilde{\chi}(t)+u$. By following the same argument as in Lemma 3, for each $k\in\mathcal{K}$, we can show that $\overline{L}_k(t+u) \geq \overline{L}_k(t)$ for each $0 \leq t < t+u \leq \tilde{T}$. This implies that $\overline{L}_k$ is non-decreasing on $[0,\tilde{T}]$ for each $k\in\mathcal{K}$. □

Now we extend the result of Corollary 2 from $[0,\tilde{T}]$ to $\mathbb{R}_+$.

PROPOSITION 4. *Suppose that Assumptions 1 and 2 hold. Then $\overline{L}_k$ is non-decreasing on $\mathbb{R}_+$ for each $k\in\mathcal{K}$ and $x(t)-1 \leq \overline{F}_t(\overline{\chi}(0)+t)$ for all $t\in\mathbb{R}_+$.*

Proof First suppose that Condition A in Assumption 2 holds. Then $H_k^s = \infty$ since otherwise, for all $t \geq H_k^s$, $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} \frac{g_k^s(x+H_k^s)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx) = 0$, which contradicts Condition A. It follows from Condition A that $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_0^k(dx) > 0$. It follows from Corollary 2, Lemma 6 and

Proposition 3 that the triple $(\overline{X},\overline{\nu},\overline{\eta})$ constructed from $x(\cdot)$ is a continuous solution to the fluid model equations associated with $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)$ restricted on $[0,\tilde{T}]$. In particular, $\overline{\nu}_t$ satisfies (2.3) for each $t\in[0,\tilde{T}]$. By Remark 1, we have that for each $t\in[0,\tilde{T}]$,

$$\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_t^k(dx)=\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{g_k^s(x+t)}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)+\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-s)\,d\overline{L}_k(s).$$

Then Condition A and the fact that $\overline{L}_k$ is non-decreasing on $[0,\tilde{T}]$ for each $k\in\mathcal{K}$ together imply that $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_{\tilde{T}}^k(dx)>0$. By Lemma 7, $x^{[\tilde{T}]}$ is the unique fixed point of $\Lambda^{[\tilde{T}]}$, where

$$\begin{aligned}
\xi^{[\tilde{T}]}(u)&=x(\tilde{T})-\sum_{k\in\mathcal{K}}\overline{Q}_k(\tilde{T})G_k^s(u)+\sum_{k\in\mathcal{K}}\int_0^u\bar{G}_k^s(u-s)d\overline{E}_k^{[\tilde{T}]}(s)\\
&\quad-\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{G_k^s(x+\tilde{T}+u)-G_k^s(x+\tilde{T})}{\bar{G}_k^s(x)}\,\overline{\nu}_0^k(dx)\\
&\quad-\sum_{k\in\mathcal{K}}\overline{L}_k(\tilde{T})G_k^s(u)-\sum_{k\in\mathcal{K}}\int_0^{\tilde{T}}(g_k^s(\tilde{T}+u-s)-g_k^s(\tilde{T}-s))\overline{L}_k(s)ds\\
&=x(\tilde{T})-\sum_{k\in\mathcal{K}}\overline{Q}_k(\tilde{T})G_k^s(u)+\sum_{k\in\mathcal{K}}\int_0^u\bar{G}_k^s(u-s)d\overline{E}_k^{[\tilde{T}]}(s)\\
&\quad-\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)}\frac{G_k^s(x+u)-G_k^s(x)}{\bar{G}_k^s(x)}\,\overline{\nu}_{\tilde{T}}^k(dx).
\end{aligned}$$

Note that $\Lambda^{[\tilde{T}]}$ can be viewed as a functional map defined from the initial data $(\overline{E}^{[\tilde{T}]},\overline{X}(\tilde{T}),\overline{\nu}_{\tilde{T}},\overline{\eta}_{\tilde{T}})$ in the same way as the functional map Λ in 3.31 defined from $(\overline{E},\overline{X}(0),\overline{\nu}_0,\overline{\eta}_0)$. Then by applying Lemma 6 and Corollary 2 to $\Lambda^{[\tilde{T}]}$ with initial data $(\overline{E}^{[\tilde{T}]},\overline{X}(\tilde{T}),\overline{\nu}_{\tilde{T}},\overline{\eta}_{\tilde{T}})$ and using (3.54), there exists $\tilde{T}'>0$ such that for each $k\in\mathcal{K}$, $\overline{L}_k^{[\tilde{T}]}$ defined by (3.42) with $x^{[\tilde{T}]}(\cdot)$ in place of $x(\cdot)$ is non-decreasing on $[0,\tilde{T}']$ and for each $t\in[0,\tilde{T}']$, $x^{[\tilde{T}]}(t)-1<\overline{F}_t^{[\tilde{T}]}(\overline{\chi}^{[\tilde{T}]}(0)+t)$, where

$$\overline{\chi}^{[\tilde{T}]}(0)\doteq(\overline{F}_t^{[\tilde{T}]})^{-1}\left(\sum_{k\in\mathcal{K}}\overline{Q}_k(\tilde{T})\right)=(\overline{F}_t^{[\tilde{T}]})^{-1}\left([x(\tilde{T})-1]^+\right)=\tilde{\chi}(\tilde{T})\le\overline{\chi}(0)+\tilde{T}.$$

Note that for each $k\in\mathcal{K}$ and $t\in[0,\tilde{T}]$, $\overline{L}_k(\tilde{T}+t)=\overline{L}_k^{[\tilde{T}]}(t)$ and

$$x(\tilde{T}+t)-1=x^{[\tilde{T}]}(t)-1\le\overline{F}_t^{[\tilde{T}]}(\overline{\chi}^{[\tilde{T}]}(0)+t)\le\overline{F}_{\tilde{T}+t}(\overline{\chi}(0)+\tilde{T}+t).$$

Then $\overline{L}_k$ is non-decreasing on $[0,\tilde{T}+\tilde{T}']$ and $x(t)-1\le \overline{F}_t(\overline{\chi}(0)+t)$ for all $t\in[0,\tilde{T}+\tilde{T}']$. By applying a simple contradiction argument, it is clear that the maximal interval on which $\overline{L}_k$ is non-decreasing for each $k\in\mathcal{K}$ and $x(t)-1\le \overline{F}_t(\overline{\chi}(0)+t)$ has to be $[0,\infty)$.

Suppose Condition B in Assumption 2 holds. If $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_0^k(dx)>0$. By Corollary 2, there exists $\tilde{T}>0$ such that $\overline{L}_k$ is non-decreasing on $[0,\tilde{T}]$ for each $k\in\mathcal{K}$ and $x(t)-1\le \overline{F}_t(\overline{\chi}(0)+t)$ for all $t\in[0,\tilde{T}]$. Otherwise, if $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_0^k(dx)=0$, by Condition B, $\sum_{k\in\mathcal{K}}\langle \mathbf{1},\overline{\nu}_0^k\rangle=0$, that is, $\overline{\nu}_0^k=\mathbf{0}$ for all $k\in\mathcal{K}$. By the definition of $\mathcal{S}_0$ in (2.1), $\sum_{k\in\mathcal{K}}\overline{X}_k(0)=0$. Since $x(\cdot)$ is the continuous solution to the equation (3.30), then

$$x(t)=\sum_{k\in\mathcal{K}}\int_0^t \bar{G}_k^s(t-u)d\overline{E}_k(u)-\sum_{k\in\mathcal{K}}\int_0^t \bar{G}_k^s(t-u)B^k(u,x(u))du$$
$$+\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-u)A^k(u,x(u))du.$$

Since $x(0)=\sum_{k\in\mathcal{K}}\overline{X}_k(0)=0$, let $\tilde{T}=\inf\{t\ge 0:\ x(t)=1\}$. By the continuity of $x(\cdot)$, $\tilde{T}>0$ and $x(t)<1$ for all $t\in[0,\tilde{T}]$. It follows that $A^k(t,x(t))=B^k(t,x(t))=0$ for all $t\in[0,\tilde{T}]$ and $k\in\mathcal{K}$, then $x(t)=\sum_{k\in\mathcal{K}}\int_0^t \bar{G}_k^s(t-u)d\overline{E}_k(u)$ and for each $k\in\mathcal{K}$, $\overline{L}_k(t)=\overline{E}_k(t)$ for all $t\in[0,\tilde{T}]$. Thus $\overline{L}_k$ is non-decreasing on $[0,\tilde{T}]$ and it is clear that $x(t)-1\le 0\le \overline{F}_t(\overline{\chi}(0)+t)$ for all $t\in[0,\tilde{T}]$. Let $[0,S)$ be the maximal interval on which $\overline{L}_k$ is non-decreasing for all $k\in\mathcal{K}$ and $x(t)-1\le \overline{F}_t(\overline{\chi}(0)+t)$. If $S<\infty$, it follows that $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_S^k(dx)=0$, since, otherwise, if $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_S^k(dx)>0$, then there exists $S'>0$ such that $\overline{L}_k$ is non-decreasing on $[0,S+S']$ for all $k\in\mathcal{K}$ and $x(t)-1\le \overline{F}_t(\overline{\chi}(0)+t)$ for all $t\in[0,S+S']$. This contradicts the maximality of $[0,S]$. Since $\sum_{k\in\mathcal{K}}\int_{[0,H_k^s)} h_k^s(x)\,\overline{\nu}_S^k(dx)=0$, then there exists $S^*>0$ such that $\overline{L}_k^{[S]}(t)=\overline{E}_k^{[S]}(t)$ for all $t\in[0,S^*]$ and $x^{[S]}(t)\le 1$ for all $t\in[0,S^*]$. Thus, $\overline{L}_k$ is non-decreasing on $[0,S+S']$ for all $k\in\mathcal{K}$ and $x(t)-1\le \overline{F}_t(\overline{\chi}(0)+t)$ for all $t\in[0,S+S']$, which again contradicts the maximality of $[0,S]$. This shows that $S=\infty$. □

Proof of Theorem 3.2. The statement in Theorem 3.2 follows directly from Proposition 3, Lemmas 6 and 7, Corollaries 1 and 2 and Proposition 4. □

4. Sensitivity Analysis of the Fluid Model Equations This section is devoted to the sensitivity analysis of the fluid model equations on perturbations of the input data. To be specific, for each $n\ge 0$, let $(\overline{E}^n,\overline{X}^n(0),\overline{\nu}_0^n,\overline{\eta}_0^n)$ be a sequence of input data in $\mathcal{S}_0$ and let $(\overline{X}^n,\overline{\nu}^n,\overline{\eta}^n)$ be a solution to the fluid model equations associated with the input data $(\overline{E}^n,\overline{X}^n(0),\overline{\nu}_0^n,\overline{\eta}_0^n)$.

ASSUMPTION 3. *We make the following set of assumptions on the sequence of input data* $(\overline{E}^n, \overline{X}^n(0), \overline{\nu}^n_0, \overline{\eta}^n_0)$, $n \geq 0$. *For each* $k \in \mathcal{K}$, *as* $n \to \infty$,

1. *for each* $T > 0$, $\sup_{t\in[0,T]} |\overline{E}^n_k(t) - \overline{E}^0_k(t)| \to 0$,
2. $\overline{X}^n_k(0) \to \overline{X}^0_k(0)$,
3. $\overline{\nu}^{n,k}_0 \to \overline{\nu}^{0,k}_0$ *in total variation,*
4. $\overline{\eta}^{n,k}_0 \to \overline{\eta}^{0,k}_0$ *in total variation.*

In addition, the patience time densities $\{g^r_k,\ k \in \mathcal{K}\}$ *are absolutely continuous on* $[0,\delta)$ *for some* $\delta \in \mathbb{R}_+ \cup \{\infty\}$ *and their hazard rate functions* $\{h^r_k,\ k \in \mathcal{K}\}$ *are assumed to be locally bounded and it is assumed that* $\left[\sum_{k\in\mathcal{K}} \overline{X}^0_k(0) - 1\right]^+ < \sum_{k\in\mathcal{K}} \left\langle \mathbf{1}, \overline{\eta}^{0,k}_0 \right\rangle$ *if* h^r_k *is not bounded for some* $k \in \mathcal{K}$.

In this section, to ease the notation, for any two functions f^1 and f^2 defined on $\mathbb{R}_+$, let $\Delta f(\cdot)$ denote $f^1(\cdot) - f^2(\cdot)$ and for any two measures η^1 and η^2 on $\mathbb{R}_+$, let $\Delta\eta$ denote $\eta^1 - \eta^2$ and let $|\Delta\eta|$ denote the total variation measure on $\mathbb{R}_+$ of $\Delta\eta$. The main result of this section is the following theorem.

THEOREM 4.3. *Suppose that Assumption 3 holds. Then for each* $k \in \mathcal{K}$ *and* $T \in [0,\delta)$, *as* $n \to \infty$, $\sup_{t\in[0,T]} \left|\overline{X}^n_k(t) - \overline{X}^0_k(t)\right| \to 0$, $\overline{\nu}^{n,k} \to \overline{\nu}^{0,k}$ *and* $\overline{\eta}^{n,k} \to \overline{\eta}^{0,k}$ *weakly on* $[0,T]$.

We first establish a local Lipschitz property on the solutions to the fluid model equations relative to the input data in Section 4.1, then the proof of Theorem 4.3 is given in Section 4.2.

4.1. A Local Lipschitz Property

PROPOSITION 5. *For* $i = 1,2$, *let* $(\overline{X}^i, \overline{\nu}^i, \overline{\eta}^i)$ *be a solution to the fluid model equations associated with* $(\overline{E}^i, \overline{X}^i(0), \overline{\nu}^i_0, \overline{\eta}^i_0) \in \mathcal{S}_0$. *Suppose that the patience time densities* $\{g^r_k,\ k \in \mathcal{K}\}$ *are absolutely continuous on* $[0,\delta)$ *for some* $\delta \in \mathbb{R}_+ \cup \{\infty\}$ *and for each* $k \in \mathcal{K}$ *and* $t \in \mathbb{R}_+$,

$$K^{r,k}_t \doteq \sup_{0\leq u\leq \overline{\chi}^1(0)\vee\overline{\chi}^2(0)+t} h^r_k(u) < \infty, \tag{4.66}$$

where $\overline{\chi}^i(0)$, $i = 1,2$, *is defined as in (2.15) from* $(\overline{E}^i, \overline{X}^i(0), \overline{\nu}^i_0, \overline{\eta}^i_0)$ *with* $t = 0$. *Then for each* $T \in [0,\delta)$, *there exists a constant* $C_T \in (0,\infty)$ *such that*

$$\sup_{t\in[0,T]} |\Delta x(t)| \leq C_T \zeta_T, \tag{4.67}$$

where

$$\zeta_T \doteq \sum_{k\in\mathcal{K}} \left(|\Delta\overline{X}_k(0)| + \langle \mathbf{1}, |\Delta\overline{\nu}^k_0| \rangle + \sup_{w\in[0,T]} \left|\Delta\overline{E}_k(w)\right| + \langle \mathbf{1}, |\Delta\overline{\eta}^k_0| \rangle \right). \tag{4.68}$$

Proof Fix $T \in [0,\delta)$. For each $i = 1,2$, let $\Lambda^i(x)$ be the functional map in (3.31) associated with $(\overline{E}^i, \overline{X}^i(0), \overline{\nu}_0^i, \overline{\eta}_0^i) \in \mathcal{S}_0$, where ξ^i is defined in (3.27) from $(\overline{E}^i, \overline{X}^i(0), \overline{\nu}_0^i, \overline{\eta}_0^i)$. It follows that $x^i(\cdot) \doteq \sum_{k\in\mathcal{K}} \overline{X}^i(\cdot)$ is a fixed point of Λ^i. Then, for each $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\left|x^1(t) - x^2(t)\right| \\
&\le |\xi^1(t) - \xi^2(t)| + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) \left|A^{k,1}(u, x^1(u)) - A^{k,2}(u, x^2(u))\right| du \\
&+ \sum_{k\in\mathcal{K}} \int_0^t \bar{G}_k^s(t-u) \left|B^{k,1}(u, x^1(u)) - B^{k,2}(u, x^2(u))\right| du.
\end{aligned} \tag{4.69}$$

It follows from the definition of $\xi(\cdot)$ in (3.27) and the definition of ζ_T in (4.68) that for each $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\left|\xi^1(t) - \xi^2(t)\right| \\
&\le 2\sum_{k\in\mathcal{K}} |\Delta\overline{X}_k(0)| + 2\sum_{k\in\mathcal{K}} \langle \mathbf{1}, |\Delta\overline{\nu}_0^k|\rangle + \sum_{k\in\mathcal{K}} |\Delta\overline{E}_k(t)| + \sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) \left|\Delta\overline{E}_k(u)\right| du \\
&\le 2\sum_{k\in\mathcal{K}} |\Delta\overline{X}_k(0)| + 2\sum_{k\in\mathcal{K}} \langle \mathbf{1}, |\Delta\overline{\nu}_0^k|\rangle + 2\sum_{k\in\mathcal{K}} \sup_{u\in[0,t]} \left|\Delta\overline{E}_k(u)\right| \le 2\zeta_T.
\end{aligned} \tag{4.70}$$

We first estimate the second term on the right-hand side of (4.69), that is,

$$\sum_{k\in\mathcal{K}} \int_0^t g_k^s(t-u) \left|A^{k,1}(u, x^1(u)) - A^{k,2}(u, x^2(u))\right| du.$$

Note that, by applying the usual triangle inequality and property (2) of Lemma 4, for each $u \in [0,t]$ and $k \in \mathcal{K}$,

$$\begin{aligned}
&\left|A^{k,1}(u, x^1(u)) - A^{k,2}(u, x^2(u))\right| \\
&\le \left|A^{k,1}(u, x^1(u)) - A^{k,1}(u, x^2(u))\right| + \left|A^{k,1}(u, x^2(u)) - A^{k,2}(u, x^2(u))\right| \\
&\le |x^1(u) - x^2(u)| + \left|A^{k,1}(u, x^2(u)) - A^{k,2}(u, x^2(u))\right|.
\end{aligned} \tag{4.71}$$

By applying Lemma 2, we have that for each $t \in \mathbb{R}_+$, $\overline{\chi}^2(t) \le \overline{\chi}^2(0) + t$, which in turn implies that $[x^2(t) - 1]^+ \le \overline{F}_t^2(\overline{\chi}^2(0) + t)$ and then by the definition of $A^k(t,x)$ in (3.29), we have that for each $u \in [0,t]$ and $k \in \mathcal{K}$,

$$A^{k,2}(u, x^2(u)) = \overline{F}_u^{k,2}\left((\overline{F}_u^2)^{-1}\left([x^2(u) - 1]^+\right)\right).$$

From this and an application of triangle inequality, we see that for each $u \in [0,t]$ and $k \in \mathcal{K}$,

$$\begin{aligned}
&\left|A^{k,1}(u,x^2(u)) - A^{k,2}(u,x^2(u))\right| \qquad (4.72)\\
&= \left|\overline{F}^{k,1}_u\left((\overline{F}^1_u)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) - \overline{F}^{k,2}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right|\\
&\le \left|\overline{F}^{k,1}_u\left((\overline{F}^1_u)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) - \overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right|\\
&+ \left|\overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^{k,2}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right|.
\end{aligned}$$

For $i = 1,2$ and $k \in \mathcal{K}$, it follows from (2.5) with $f(x) = \mathbf{1}_{\left[0,(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]}(x)$ (cf. Remark 1) and $t = u$ that

ATTENTION: The following displayed equation, in its current form, exceeds the column width that will be used in the published edition of your article. Please break or rewrite this equation to fit, including the equation number, within a column width of 470 pt / 165.81 mm / 6.53 in (the width of this red box).

$$\begin{aligned}
\overline{F}^{k,i}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) &= \int_{[0,H^r_k)} \mathbf{1}_{\left[0,(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]}(x)\,\overline{\eta}^{k,i}_u(dx)\\
&= \int_{[0,H^r_k)} \mathbf{1}_{\left[0,(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]}(x+u)\frac{\bar{G}^r_k(x+u)}{\bar{G}^r_k(x)}\,\overline{\eta}^{k,i}_0(dx)\\
&\quad + \int_0^u \mathbf{1}_{\left[0,(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]}(u-w)\bar{G}^r_k(u-w)\,d\overline{E}^i_k(w).
\end{aligned}$$

Note that, by an application of integration by parts, the second term on the right-hand side of the above display can be rewritten as

$$\begin{aligned}
&\int_0^u \mathbf{1}_{\left[0,(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]}(u-w)\bar{G}^r_k(u-w)\,d\overline{E}^i_k(w)\\
&= \int_{\left[u-(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]^+}^u \bar{G}^r_k(u-w)\,d\overline{E}^i_k(w)\\
&= \overline{E}^i_k(u) - \bar{G}^r_k\left(u \wedge (\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\overline{E}^i_k\left(\left[u-(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]^+\right)\\
&\quad - \int_{\left[u-(\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right]^+}^u \overline{E}^i_k(w)g^r_k(u-w)\,dw.
\end{aligned}$$

Then the second term on the right-hand side of (4.72) can be estimated as

$$
\begin{aligned}
&\left|\overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)-\overline{F}_u^{k,2}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right| \\
&\leq \left|\int_{[0,H_k^r)} \mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(x+u)\frac{\bar{G}_k^r(x+u)}{\bar{G}_k^r(x)}\,\Delta\overline{\eta}_0^k(dx)\right| \\
&+\left|\Delta\overline{E}_k(u)\right|+\left|\Delta\overline{E}_k\left(\left[u-(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]^+\right)\right|+\int_0^u\left|\overline{E}_k^i(w)\right|g_k^r(u-w)dw \\
&\leq \langle \mathbf{1}, |\Delta\overline{\eta}_0^k|\rangle+3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|.
\end{aligned}
\tag{4.73}
$$

For the first term on the right-hand side of (4.72), that is,

$$
\left|\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)-\overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right|,
$$

we consider the following three mutually exclusive cases:

Case 1: $(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\geq(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)$.

For each $k\in\mathcal{K}$, since $\overline{F}_u^{k,1}(x)$ as a function of x is non-decreasing, then

$$
\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)\geq F_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right).
$$

This, together with the fact that $\overline{F}_u^1=\sum_{k\in\mathcal{K}}\overline{F}_u^{k,1}$, implies that for each $k\in\mathcal{K}$,

$$
\begin{aligned}
&\left|\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)-F_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right| \\
&=\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)-F_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) \\
&\leq\overline{F}_u^1\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)-\overline{F}_u^1\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\overline{F}_u^1\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)&=[x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u) \\
&=\overline{F}_u^2\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right) \\
&\leq\overline{F}_u^2\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right).
\end{aligned}
\tag{4.74}
$$

From this and (4.73), we have that for each $k \in \mathcal{K}$,

$$
\begin{aligned}
&\left|\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right) - F_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right| \\
&\le \overline{F}_u^2\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}_u^1\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right).
\end{aligned}
$$

Case 2: $(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right) < (\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)$ and $[x^2(u)-1]^+ \le \overline{F}_u^1(\overline{\chi}^1(0)+u)$.

In this case, we have that for each $k \in \mathcal{K}$,

$$
\begin{aligned}
&\left|\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right) - \overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right| \\
&= \overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right) \\
&= \overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\right)\right) \\
&\le \overline{F}_u^1\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}_u^1\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\right)\right) \\
&= \overline{F}_u^1\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}_u^2\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) \\
&\le \left|\overline{F}_u^1\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}_u^2\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right|,
\end{aligned}
$$

where the first equality is due to the first case condition, the second equality is due to the second case condition and the third equality is due to (4.74).

Case 3: $(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right) < (\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)$ and $[x^2(u)-1]^+ > \overline{F}_u^1(\overline{\chi}^1(0)+u)$.

The second case condition and Lemma 2 together imply that

$$
[x^2(u)-1]^+ > \overline{F}_u^1(\overline{\chi}^1(0)+u) \ge [x^1(u)-1]^+.
$$

Then for each $k \in \mathcal{K}$,

$$
\begin{aligned}
&\left|\overline{F}^{k,1}_u\left((\overline{F}^1_u)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) - \overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right| \\
&= \overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^{k,1}_u\left((\overline{F}^1_u)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) \\
&= \overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^{k,1}_u\left((\overline{F}^1_u)^{-1}\left(\overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) \\
&\le \overline{F}^1_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^1_u\left((\overline{F}^1_u)^{-1}\left(\overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) \\
&= \overline{F}^1_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^2_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) \\
&\qquad\qquad + \overline{F}^2_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^1_u(\overline{\chi}^1(0)+u) \\
&\le \overline{F}^1_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^2_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) + [x^2(u)-1]^+ - [x^1(u)-1]^+ \\
&\le \overline{F}^1_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^2_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) + |\Delta x(u)|.
\end{aligned}
$$

Combining all the three cases, we have that the first term on the right-hand side of (4.72) can be estimated as

$$
\begin{aligned}
&\left|\overline{F}^{k,1}_u\left((\overline{F}^1_u)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}^1_u(\overline{\chi}^1(0)+u)\right)\right) - \overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right| \qquad (4.75)\\
&\le \left|\overline{F}^1_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^2_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right| + |\Delta x(u)| \\
&\le \sum_{k\in\mathcal{K}} \left|\overline{F}^{k,1}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right) - \overline{F}^{k,2}_u\left((\overline{F}^2_u)^{-1}\left([x^2(u)-1]^+\right)\right)\right| + |\Delta x(u)| \\
&\le \sum_{k\in\mathcal{K}} \left(\langle \mathbf{1}, |\Delta\overline{\eta}^k_0|\rangle + 3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right) + |\Delta x(u)|.
\end{aligned}
$$

Then it follows from (4.72), (4.73) and (4.75) that for each $u \in [0,t]$ and $k \in \mathcal{K}$,

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$$
\begin{aligned}
&\left|A^{k,1}(u,x^1(u)) - A^{k,2}(u,x^2(u))\right| \qquad (4.76)\\
&\le |\Delta x(u)| + \left(\sum_{k\in\mathcal{K}}\left(\langle \mathbf{1}, |\Delta\overline{\eta}^k_0|\rangle + 3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right) + |\Delta x(u)|\right) + \left(\langle \mathbf{1}, |\Delta\overline{\eta}^k_0|\rangle + 3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right).
\end{aligned}
$$

Thus, the second term on the right-hand side of (4.69) can be estimated by using (4.76) as follows.

$$\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-u)\left|A^{k,1}(u,x^1(u))-A^{k,2}(u,x^2(u))\right|du \tag{4.77}$$
$$\le 2\sum_{k\in\mathcal{K}}\int_0^t g_k^s(t-u)\,|\Delta x(u)|\,du+(3K+3)\sum_{k\in\mathcal{K}}\left(\langle\mathbf{1},|\Delta\bar{\eta}_0^k|\rangle+\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right).$$

We next turn to estimate the third term on the right-hand side of (4.69), that is,

$$\sum_{k\in\mathcal{K}}\int_0^t \bar{G}_k^s(t-u)\left|B^{k,1}(u,x^1(u))-B^{k,2}(u,x^2(u))\right|du.$$

For each $u\in[0,t]$, by an application of the triangle inequality and property (3) of Lemma 4, we have that

$$\begin{aligned}&\left|B^{k,1}(u,x^1(u))-B^{k,2}(u,x^2(u))\right|\\&\le\left|B^{k,1}(u,x^1(u))-B^{k,1}(u,x^2(u))\right|+\left|B^{k,1}(u,x^2(u))-B^{k,2}(u,x^2(u))\right|\\&\le K_t^{r,k}|\Delta x(u)|+\left|B^{k,1}(u,x^2(u))-B^{k,2}(u,x^2(u))\right|,\end{aligned}$$

where $K_t^{r,k}$ is given by (4.66). Also it follows from the definition of $B^k(t,x)$ in (3.29) and Lemma 2 that for each $u \in [0,t]$,

$$
\begin{aligned}
&\left|B^{k,1}(u,x^2(u)) - B^{k,2}(u,x^2(u))\right| \qquad (4.78)\\
&= \Bigg| \int_0^{[x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)} h_k^r((\overline{F}_u^1)^{-1}(w)) d\overline{F}_u^{k,1}((\overline{F}_u^1)^{-1}(w)) \\
&\qquad\qquad - \int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w)) d\overline{F}_u^{k,2}((\overline{F}_u^2)^{-1}(w)) \Bigg| \\
&\le \Bigg| \int_0^{[x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)} h_k^r((\overline{F}_u^1)^{-1}(w)) d\overline{F}_u^{k,1}((\overline{F}_u^1)^{-1}(w)) \\
&\qquad\qquad - \int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w)) d\overline{F}_u^{k,1}((\overline{F}_u^2)^{-1}(w)) \Bigg| \\
&+ \Bigg| \int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w)) d\overline{F}_u^{k,1}((\overline{F}_u^2)^{-1}(w)) \\
&\qquad\qquad - \int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w)) d\overline{F}_u^{k,2}((\overline{F}_u^2)^{-1}(w)) \Bigg|,
\end{aligned}
$$

where the equality uses the fact that $[x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^2(0)+u) = [x^2(u)-1]^+$ by Lemma 2 and the inequality is an application of the triangle inequality. For the first term on the right-hand side of (4.78), note that by an application of change of variables, this term can be rewritten as

$$
\begin{aligned}
&\Bigg| \int_0^{[x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)} h_k^r((\overline{F}_u^1)^{-1}(w)) d\overline{F}_u^{k,1}((\overline{F}_u^1)^{-1}(w)) \qquad (4.79)\\
&\qquad\qquad - \int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w)) d\overline{F}_u^{k,1}((\overline{F}_u^2)^{-1}(w)) \Bigg| \\
&= \Bigg| \int_{\mathbb{R}_+} \mathbf{1}_{\left[0,(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right]}(w) h_k^r(w) d\overline{F}_u^{k,1}(w) \\
&\qquad\qquad - \int_{\mathbb{R}_+} \mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w) h_k^r(w) d\overline{F}_u^{k,1}(w) \Bigg| \\
&\le \int_{\mathbb{R}_+} \Bigg| \mathbf{1}_{\left[0,(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+ \wedge \overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right]}(w) \\
&\qquad\qquad - \mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w) \Bigg| h_k^r(w) d\overline{F}_u^{k,1}(w).
\end{aligned}
$$

Note that the term with absolute value in the right-hand side of (4.79), that is,

$$\left|\mathbf{1}_{\left[0,(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right]}(w)-\mathbf{1}_{[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)]}(w)\right|$$

is in fact an indicator function of an interval with the two boundary points that are determined by $(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)$ and $(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)$. Since for each $u\in[0,t]$,

$$(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\le(\overline{F}_u^1)^{-1}\left(\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\le\overline{\chi}^1(0)+t$$

and as a consequence of Lemma 2,

$$(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\le\overline{\chi}^2(0)+u\le\overline{\chi}^2(0)+t.$$

Thus, the intergrand of the integral in the right-hand side of (4.79) is bounded above as

$$\begin{aligned}&\left|\mathbf{1}_{\left[0,(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right]}(w)-\mathbf{1}_{[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)]}(w)\right|h_k^r(w)\\&\le K_t^{r,k}\left|\mathbf{1}_{\left[0,(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right]}(w)-\mathbf{1}_{[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)]}(w)\right|.\end{aligned}$$

Note that by comparing the two terms $(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)$ and $(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)$, we can see that

$$\begin{aligned}&\int\left|\mathbf{1}_{\left[0,(\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right]}(w)-\mathbf{1}_{[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)]}(w)\right|d\overline{F}_u^{k,1}(w)\\&=\left|\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)-\overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right|.\end{aligned}$$

It follows the above two displays, (4.75) and (4.79) that for each $u\in[0,t]$,

$$\begin{aligned}&\left|\int_0^{[x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)}h_k^r((\overline{F}_u^1)^{-1}(w))d\overline{F}_u^{k,1}((\overline{F}_u^1)^{-1}(w))\right.\\&\qquad\left.-\int_0^{[x^2(u)-1]^+}h_k^r((\overline{F}_u^2)^{-1}(w))d\overline{F}_u^{k,1}((\overline{F}_u^2)^{-1}(w))\right|\\&\le K_t^{r,k}\left|\overline{F}_u^{k,1}\left((\overline{F}_u^1)^{-1}\left([x^2(u)-1]^+\wedge\overline{F}_u^1(\overline{\chi}^1(0)+u)\right)\right)-\overline{F}_u^{k,1}\left((\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\right|\\&\le K_t^{r,k}\left(\sum_{k\in\mathcal{K}}\left(\langle\mathbf{1},|\Delta\overline{\eta}_0^k|\rangle+3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right)+|\Delta x(u)|\right).\end{aligned}\tag{4.80}$$

Now for the second term on the right-hand side of (4.78), by an application of change of variables, this term is the same as

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$$\begin{aligned}&\Bigg|\int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w))d\overline{F}_u^{k,1}((\overline{F}_u^2)^{-1}(w)) \\ &\qquad -\int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w))d\overline{F}_u^{k,2}((\overline{F}_u^2)^{-1}(w))\Bigg| \\ &=\left|\int_{\mathbb{R}_+}\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w)h_k^r(w)d\overline{F}_u^{k,1}(w)-\int_{\mathbb{R}_+}\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w)h_k^r(w)d\overline{F}_u^{k,2}(w)\right|\end{aligned} \tag{4.81}$$

Note that for each $i=1,2$, by (2.5) with $f(x)=\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(x)h_k^r(x)$ and $t=u$,

$$\begin{aligned}&\int_{\mathbb{R}_+}\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w)h_k^r(w)d\overline{F}_u^{k,i}(w) \\ &=\int_{\mathbb{R}_+}\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w)h_k^r(w)\overline{\eta}_u^{k,i}(dw) \\ &=\int_{\mathbb{R}_+}\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w+u)\frac{g_k^r(w+u)}{\bar{G}_k^r(w)}\overline{\eta}_0^{k,i}(dw) \\ &\qquad+\int_0^u\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(u-w)g_k^r(u-w))d\overline{E}_k^i(w) \\ &=\int_{\mathbb{R}_+}\mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w+u)\frac{g_k^r(w+u)}{\bar{G}_k^r(w)}\overline{\eta}_0^{k,i}(dw)+\overline{E}_k^i(u)g_k^r(0) \\ &-g_k^r\left(u\wedge(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\overline{E}_k^i\left(\left[u-(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]^+\right) \\ &\qquad-\int_{\left[u-(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]^+}^u\overline{E}_k^i(w)(g_k^r)'(u-w)dw,\end{aligned}$$

where the last equality follows from the fact that the patience time densities $\{g_k^r,\ k\in\mathcal{K}\}$ are absolutely continuous on $[0,\infty)$ and hence a.e. derivatives $\{(g_k^r)',\ k\in\mathcal{K}\}$ exist and locally integrable.

Then it follows that

$$\begin{aligned}
&\left|\int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w))d\overline{F}_u^{k,1}((\overline{F}_u^2)^{-1}(w))\right.\\
&\qquad\qquad\left. - \int_0^{[x^2(u)-1]^+} h_k^r((\overline{F}_u^2)^{-1}(w))d\overline{F}_u^{k,2}((\overline{F}_u^2)^{-1}(w))\right| \qquad (4.82)\\
&\le \int_{\mathbb{R}_+} \mathbf{1}_{\left[0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]}(w+u)\frac{g_k^r(w+u)}{\bar{G}_k^r(w)}\left|\Delta\overline{\eta}_0^k\right|(dw) + \left|\Delta\overline{E}_k(u)\right| g_k^r(0)\\
&+ g_k^r\left(u \wedge (\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right)\left|\Delta\overline{E}_k\left(\left[u-(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]^+\right)\right|\\
&\quad + \int_0^u \left|\Delta\overline{E}_k(w)\right|\left|(g_k^r)'(u-w)\right| dw\\
&\le K_t^{r,k}\int_{\mathbb{R}_+}\frac{\bar{G}_k^r(x+u)}{\bar{G}_k^r(x)}|\Delta\overline{\eta}_0^k|(dx) + K_t^{r,k}\left|\Delta\overline{E}_k\left(\left[u-(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)\right]^+\right)\right|\\
&\quad + K_t^{r,k}\left|\Delta\overline{E}_k(u)\right| + \int_0^u \left|\Delta\overline{E}_k(w)\right|\left|(g_k^r)'(u-w)\right| dw\\
&\le K_t^{r,k}\left(\langle \mathbf{1}, |\Delta\overline{\eta}_0^k|\rangle + 2\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right) + \sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\int_0^T \left|(g_k^r)'(w)\right| dw,
\end{aligned}$$

where the second inequality follows from the fact that $h_k^r(w) \le K_t^{r,k}$ for each $w \in [0,(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right)]$ since $(\overline{F}_u^2)^{-1}\left([x^2(u)-1]^+\right) \le \overline{\chi}^2(0)+u \le \overline{\chi}^2(0)+t$ by Lemma 2. Combining the estimations of the two terms on the right-hand side of (4.78) in (4.80) and (4.82), we have that for each $u\in[0,t]$ and $k\in\mathcal{K}$,

$$\begin{aligned}
&\left|B^{k,1}(u,x^1(u)) - B^{k,2}(u,x^2(u))\right| \qquad (4.83)\\
&\le K_t^{r,k}\left(\sum_{k\in\mathcal{K}}\left(\langle \mathbf{1}, |\Delta\overline{\eta}_0^k|\rangle + 3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right) + |\Delta x(u)|\right)\\
&+ K_t^{r,k}\left(\langle \mathbf{1}, |\Delta\overline{\eta}_0^k|\rangle + 2\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right) + \sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\int_0^T \left|(g_k^r)'(w)\right| dw\\
&\le K_t^{r,k}|\Delta x(u)| + \left(2K_t^{r,k} + \int_0^T \left|(g_k^r)'(w)\right| dw\right)\sum_{k\in\mathcal{K}}\left(\langle \mathbf{1}, |\Delta\overline{\eta}_0^k|\rangle + 3\sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right).
\end{aligned}$$

Then the third term of the right-hand side of (4.69) can be estimated by using (4.83) as follows.

$$\begin{aligned}
&\sum_{k\in\mathcal{K}}\int_0^t \bar{G}_k^s(t-u)\left|B^{k,1}(u,x^1(u)) - B^{k,2}(u,x^2(u))\right| du \\
&\le \sum_{k\in\mathcal{K}} K_t^{r,k}\int_0^t \bar{G}_k^s(t-u)\,|\Delta x(u)|\,du \\
&+3\sum_{k\in\mathcal{K}}\left(2K_t^{r,k} + \int_0^T \left|(g_k^r)'(w)\right| dw\right)\sum_{k\in\mathcal{K}}\left(\langle \mathbf{1}, |\Delta\overline{\eta}_0^k|\rangle + \sup_{w\in[0,T]}\left|\Delta\overline{E}_k(w)\right|\right).
\end{aligned} \tag{4.84}$$

Thus, by combining (4.69), (4.70), (4.77) and (4.84) and the definition of ζ_T in (4.68), we have that for each $t\in[0,T]$,

$$|\Delta x(t)| \le \left(3K+5+6\sum_{k\in\mathcal{K}} K_T^{r,k} + 3\sum_{k\in\mathcal{K}}\int_0^T \left|(g_k^r)'(w)\right| dw\right)\zeta_T + \int_0^t k(t-u)\,|\Delta x(u)|\,du,$$

where

$$k(t) \doteq 2\sum_{k\in\mathcal{K}} g_k^s(t) + \sum_{k\in\mathcal{K}} K_T^{r,k}\bar{G}_k^s(t), \tag{4.85}$$

Let $r(\cdot)$ be the resolvent kernel of $k(\cdot)$ in (4.85). Since $k(\cdot)$ is non-negative and integrable over $\mathbb{R}_+$, then the resolvent $r(\cdot)$ is also non-negative and is locally integrable over $\mathbb{R}_+$. It is well known from the theory of linear Volterra integral equations that for each $t\in[0,T]$, $|\Delta x(t)| \le C_T\zeta_T$, where

$$C_T \doteq \left(1+\int_0^T r(u)du\right)\left(3K+5+6\sum_{k\in\mathcal{K}} K_T^{r,k} + 3\sum_{k\in\mathcal{K}}\int_0^T \left|(g_k^r)'(w)\right| dw\right) \in (0,\infty).$$

□

4.2. Proof of Theorem 4.3. The proof of Theorem 4.3 relies on the local Lipschitz property established in Proposition 5. The following lemma is required to implement the local Lipschitz property.

LEMMA 8. *Suppose that Assumption 3 holds. For each $n \geq 0$, recall that $\overline{\chi}^n(0) = \left(\overline{F}_0^n\right)^{-1}\left(\left[\sum_{k\in\mathcal{K}}\overline{X}_k^n(0)-1\right]^+\right)$. Then either h_k^r is bounded for each $k \in \mathcal{K}$ or h_k^r is not bounded for some $k \in \mathcal{K}$ and there exists $p > 0$ such that*

$$\limsup_{n\to\infty}\overline{\chi}^n(0) < \overline{\chi}^0(0) + p < \infty. \tag{4.86}$$

Proof From Assumption 3, for each $k \in \mathcal{K}$, $\overline{\eta}_0^{n,k} \to \overline{\eta}_0^{0,k}$ in total variation and $\overline{X}_k^n(0) \to \overline{X}_k^0(0)$ as $n \to \infty$, then we have that, as $n \to \infty$,

$$\sup_{y\in\mathbb{R}_+}\left|\overline{F}_0^n(y) - \overline{F}_0^0(y)\right| \leq \sum_{k\in\mathcal{K}}\sup_{y\geq 0}\left|\overline{F}_0^{n,k}(y) - \overline{F}_0^{0,k}(y)\right| \leq \sum_{k\in\mathcal{K}}\left\langle \mathbf{1}, \left|\overline{\eta}_0^{n,k} - \overline{\eta}_0^{0,k}\right|\right\rangle \to 0$$

and

$$\left[\sum_{k\in\mathcal{K}}\overline{X}_k^n(0)-1\right]^+ \to \left[\sum_{k\in\mathcal{K}}\overline{X}_k^0(0)-1\right]^+.$$

Suppose that h_k^r is not bounded for some $k \in \mathcal{K}$, then by Assumption 3, $\overline{\chi}^0(0) < \infty$ and

$$\overline{F}_0^0(\overline{\chi}^0(0)) = \left[\sum_{k\in\mathcal{K}}\overline{X}_k^0(0)-1\right]^+ < \sum_{k\in\mathcal{K}}\left\langle \mathbf{1}, \overline{\eta}_0^{0,k}\right\rangle = \overline{F}_0^0(\infty).$$

Then, there exists $p > 0$ such that $\overline{F}_0^0(\overline{\chi}^0(0)+p) > \overline{F}_0^0(\overline{\chi}^0(0))$. Now we show that (8) holds for the existed p by a contradiction argument. Suppose that $\limsup_{n\to\infty}\overline{\chi}^n(0) \geq \overline{\chi}^0(0)+p$. Then for each $\delta > 0$, there exists a subsequence $\overline{\chi}^{n_k}(0)$ such that $\overline{\chi}^{n_k}(0) > \overline{\chi}^0(0)+p-\delta$ for each $k \in \mathbb{N}$. The definition of $\overline{\chi}^n(0)$ implies that

$$\overline{F}_0^{n_k}(\overline{\chi}^0(0)+p-\delta) < \left[\sum_{k\in\mathcal{K}}\overline{X}_k^{n_k}(0)-1\right]^+ \text{ for each } k \in \mathbb{N}.$$

By taking the limits on both sides of the above inequality as $k \to \infty$ and then letting $\delta \to 0$, it follows that

$$\overline{F}_0^0(\overline{\chi}^0(0)+p) \leq \left[\sum_{k\in\mathcal{K}}\overline{X}_k^0(0)-1\right]^+.$$

Then we have that

$$\left[\sum_{k\in\mathcal{K}}\overline{X}_k^0(0)-1\right]^+ = \overline{F}_0^0(\overline{\chi}^0(0)) < \overline{F}_0^0(\overline{\chi}^0(0)+p) \leq \left[\sum_{k\in\mathcal{K}}\overline{X}_k^0(0)-1\right]^+,$$

which is clearly a contradiction. Thus, we obtained the desired result. ◻

Proof of Theorem 4.3. It follows from Lemma 8 that either h_k^r is bounded for each $k \in \mathcal{K}$ or h_k^r is not bounded for some $k \in \mathcal{K}$ and there exist $p > 0$ and $N > 0$ such that $\overline{\chi}^n(0) < \overline{\chi}^0(0) + p$ for all $n \geq N$. For each $k \in \mathcal{K}$, $n \geq N$ and $t \in \mathbb{R}_+$, consider the quantity in (4.66) for the pair of indexes $n, 0$ in place of $1, 2$, denoted as $K_{n,t}^{r,k}$, and let

$$K_t^r \doteq \begin{cases} \sum_{k\in\mathcal{K}} \sup_{u\in\mathbb{R}_+} h_k^r(u) & \text{if } h_k^r \text{ is bounded for each } k \in \mathcal{K}, \\ \sum_{k\in\mathcal{K}} \sup_{0\leq u\leq \overline{\chi}^0(0)+p+t} h_k^r(u) & \text{if } h_k^r \text{ is not bounded for some } k \in \mathcal{K}. \end{cases}$$

Note that K_t^r is non-decreasing in t and since h_k^r is locally bounded for each $k \in \mathcal{K}$, then for each $k \in \mathcal{K}$ and $t \in \mathbb{R}_+$, $K_{n,t}^{r,k} \leq K_t^r < \infty$.

For each $n \geq N$, since $(\overline{X}^n, \overline{\nu}^n, \overline{\eta}^n)$ and $(\overline{X}^0, \overline{\nu}^0, \overline{\eta}^0)$ are solutions to the fluid model equations associated with $(\overline{E}^n, \overline{X}^n(0), \overline{\nu}_0^n, \overline{\eta}_0^n)$ and $(\overline{E}^0, \overline{X}^0(0), \overline{\nu}_0^0, \overline{\eta}_0^0)$, respectively, and $K_{n,t}^{r,k} < \infty$ for each $t \in [0, \infty)$, then it follows from Proposition 5 that for each $T > 0$, there exists a constant $C_T > 0$ such that

$$\sup_{t\in[0,T]} \left| x^n(t) - x^0(t) \right| \leq C_T \zeta_T^n,$$

where

$$\zeta_T^n = \sum_{k\in\mathcal{K}} |\overline{X}_k^n(0) - \overline{X}^0(0)| + \sum_{k\in\mathcal{K}} \langle \mathbf{1}, \left| \overline{\nu}_0^{k,n} - \overline{\nu}_0^{k,0} \right| \rangle + \sum_{k\in\mathcal{K}} \sup_{w\in[0,T]} \left| \overline{E}_k^n(w) - \overline{E}_k^0(w) \right| \\ + \sum_{k\in\mathcal{K}} \left\langle \mathbf{1}, \left| \overline{\eta}_0^{k,n} - \overline{\eta}_0^{k,0} \right| \right\rangle.$$

The assumed convergences on the input data implies that, as $n \to \infty$,

$$\zeta_T^n \to 0 \text{ and then } \sup_{t\in[0,T]} \left| x^n(t) - x^0(t) \right| \to 0. \tag{4.87}$$

Recall that for $n \geq 0$, $t \in [0, T]$ and $k \in \mathcal{K}$,

$$\begin{aligned} \overline{Q}_k^n(t) &= A^{n,k}(t, x^n(t)), \\ \overline{R}_k^n(t) &= \int_0^t B^{n,k}(w, x^n(w)) dw, \\ \overline{L}_k^n(t) &= \overline{Q}_k^n(0) + \overline{E}_k^n(t) - \overline{Q}_k^n(t) - \overline{R}_k^n(t). \end{aligned}$$

It follows from (4.76) and (4.87) that, for each $k \in \mathcal{K}$, as $n \to \infty$,

$$\sup_{t\in[0,T]} \left|\overline{Q}^n_k(t) - \overline{Q}^0_k(t)\right| = \sup_{t\in[0,T]} \left|A^{n,k}(t, x^n(t)) - A^{0,k}(t, x^0(t))\right| \to 0.$$

It follows from (4.83) and (4.87) that, for each $k \in \mathcal{K}$, as $n \to \infty$,

$$\begin{aligned}\sup_{t\in[0,T]} \left|\overline{R}^n_k(t) - \overline{R}^0_k(t)\right| &= \sup_{t\in[0,T]} \left|\int_0^t \left(B^{n,k}(u, x^n(u)) - B^{0,k}(u, x^0(u))\right) du\right| \\ &\leq T \sup_{t\in[0,T]} \left|B^{n,k}(t, x^n(t)) - B^{0,k}(t, x^0(t))\right| \to 0.\end{aligned}$$

It then follows from (2.12) that for each $k \in \mathcal{K}$, $\sup_{t\in[0,T]} \left|\overline{L}^n_k(t) - \overline{L}^0_k(t)\right| \to 0$ as $n \to \infty$, which, together with the assumed $\sup_{t\in[0,T]} \left|\overline{E}^n_k(t) - \overline{E}^0_k(t)\right| \to 0$ as $n \to \infty$, implies that $\overline{\nu}^n \to \overline{\nu}^0$ weakly on $[0,T]$ and $\overline{\eta}^n \to \overline{\eta}^0$ weakly on $[0,T]$ as $n \to \infty$. Lastly, the convergence of $\overline{X}^n$ to $\overline{X}^0$ uniformly on $[0,T]$ follows from (2.6). This establishes the theorem. □

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