PINK

MATH 152 | **EXAM III** A | NAME Answers
Mrs. Bonny Tighe | 100 points | Wednesday 11/30/05
12.4-11.2

There are 11 problems worth 10 points each.

1. Test the series for convergence or divergence. **State the test** you use and show all work. If the series is an Alternating Series, find if it is Absolutely or Conditionally convergent.

a) Use the a Use the Integral test

$$\sum_{n=2}^{\infty} \cancel{\frac{dx}{x\sqrt{\ln x}}} \quad \frac{1}{n\sqrt{\ln n}}$$

Suppose $f(x)$ is a cont, pos, decreasing function on $[2,\infty)$ and $a_n = f(n)$ then if $\int_2^\infty f(x)\,dx$ is cvg then so is $\sum a_n$ and if $\int_2^\infty f(x)\,dx$ is dvg then so is $\sum a_n$

$$a_n = \frac{1}{n\sqrt{\ln n}}$$

$$\int_2^\infty (\ln x)^{-1/2} \frac{1}{x}\,dx =$$

$$\lim_{t\to\infty} \frac{1}{1/2}(\ln x)^{1/2}\Big|_2^t$$

$$\lim_{t\to\infty} 2\sqrt{\ln x}\Big|_2^t$$

$$\lim_{t\to\infty}\left[2\sqrt{\ln t} - 2\sqrt{\ln 2}\right] = \infty$$

$\ln t - \ln 2$

$(\ln(t/2))$

which is not a finite number

So Divergent

b) Use the Comparison Test

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$$

Suppose $\sum a_n$ + $\sum b_n$ are series with positive terms

i) If $\sum b_n$ is convergent + $a_n \le b_n$ for all n then $\sum a_n$ is also cvg.

ii) If $\sum b_n$ is divergent + $a_n \ge b_n$ for all n then $\sum a_n$ is also dvg.

Choose $b_n = \frac{1}{n\sqrt{n}}$ which is ~~divergent~~ Convergent by the p-series test $p > 1$

$$a_n = \frac{\sin(1/n)}{\sqrt{n}}$$

$$\frac{\sin(\frac{1}{n})}{\sqrt{n}} \le \frac{1}{n\sqrt{n}} \text{ for all } n$$

So $\frac{\sin(\frac{1}{n})}{\sqrt{n}}$ is Convergent

2. Test the series for convergence or divergence

a) Use the Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+n}$$

Suppose $\sum a_n$ and $\sum b_n$ have pos. terms

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, a finite number >0

then either both series diverge or both series converge

Chose $b_n = \frac{1}{n^3}$ which is convergent by p-series $p>1$

$a_n = \frac{1}{n^3+n}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| =$$

$$\lim_{n\to\infty} \left|\frac{\frac{1}{n^3+n}}{\frac{1}{n^3}}\right| =$$

$$\lim_{n\to\infty} \left|\frac{n^3}{n^3+n}\right| = 1 > 0$$

So both are convergent

absolutely convergent

b) Use the Ratio Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

$L<1$ then cvg

$L>1$ then divergent

$L=1$ inconclusive

$$\lim_{n\to\infty} \left|\frac{x^{2n+2}}{2^{2n+2}(n+1)!\,(n+1)!} \cdot \frac{n!\,n!\,2^{2n}}{x^{2n}}\right|$$

$$\lim_{n\to\infty} \left|\frac{x^2}{2^2(n+1)^2}\right| = 0 < 1$$

$\boxed{\text{So convergent}}$

absolutely convergent

3.. Express the function $f(x) = \coth^{-1}(\sqrt{x})$ as a power series by first finding it's derivative, $f'(x) = \dfrac{1}{1-x}$, as a power series and then integrating. Find the interval of convergence.

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n \quad \text{so} \quad \coth^{-1}(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\lim_{n\to\infty} \left| \frac{x^{n+2}}{n+2} \cdot \frac{n+1}{x^{n+1}} \right| = |x| < 1 \qquad I = [-1, 1)$$

-1 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = 0$ cvg AST

1 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ dvg, harmonic series

4. Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence. $f(x) = \dfrac{3}{x^2 + x - 2} = \dfrac{3}{(x+2)(x-1)}$ $\dfrac{A}{x+2} + \dfrac{B}{x-1}$

$A(x-1) + B(x+2) = 3$

$x = 1$ $\quad 3B = 3$ $\quad B = 1$

$x = 2$ $\quad -3A = 3$ $\quad A = -1$

$$-\frac{1}{x+2} + \frac{1}{x-1}$$

$$\frac{-\frac{1}{2}}{1-\left(-\frac{x}{2}\right)} + \frac{-1}{1-x}$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n - \sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1} x^n}{2^{n+1}} - x^n\right]$$

or $\displaystyle\sum_{n=0}^{\infty} x^n\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \qquad \boxed{I = (-1, 1)}$$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{x^n}\right| = \left|\frac{x}{2}\right| < 1$$

$I = (-2, 2)$ so use because it works for both.

5. Evaluate the indefinite integral as an infinite series. $\displaystyle\int \frac{\sin 2x}{x^2}\,dx$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{x^2(2n+1)!}$$

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n-1}}{(2n+1)!} =$$

$$C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n}}{(2n+1)!\, 2n}$$

6. Find the sum of the series. a) $\sum_{n=0}^{\infty} \frac{2^n}{5^{n+1} n!} =$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{2}{5}\right)^n}{5 n!} = \frac{1}{5} e^{2/5}$$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} =$ $\cos 2$

7. Find the Maclaurin series of f(x) and its radius of convergence. $f(x) = \ln(1+x)$ $a=0$

$$f(0) + f'(0) + f''(0) + f'''(0)$$

$$0 + 1 + -1 + 2 + -6 + 24$$

$n=0$ 1 2 3 4

3!

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -1(x+1)^{-2} =$

$f'''(x) = 2(x+1)^{-3}$

$f''''(x) = -6(x+1)^{-4}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| =$$

$|x| < 1$

$R = 1$

$$\sum_{n=1}^{\infty} \frac{x^n (-1)^{n+1} (n-1)!}{n!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

8. Find the Taylor series for f(x) centered at the given value of a and find its radius of convergence. $f(x) = \frac{1}{\sqrt{x}}$, *centered at* $a = 9$

$f(x) = x^{-1/2}$ @ $a = 9$ $= 1/3$

$f'(x) = -\frac{1}{2} x^{-3/2} = -\frac{1}{2}\left(\frac{1}{3^3}\right)$

$f''(x) = +\frac{3}{4} x^{-5/2} = \frac{3}{4}\left(\frac{1}{3^5}\right)$

$f'''(x) = \frac{-15}{8} x^{-7/2} = \frac{-15}{8}\left(\frac{1}{3^7}\right)$

$1 + \sum_{n=1}^{\infty} \frac{(x-9)^n}{n!}\left(\frac{(-1)^n 1\cdot 3\cdot 5 \cdots (2n-1)}{2^n 3^{2n+1}}\right)$

$\frac{1}{3^1} + \frac{+1}{2} \cdot \frac{1}{3^3} + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{3^5} + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{1}{3^7} +$

$1 + 1 + 1\cdot 3 + 1\cdot 3\cdot 5$

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n (x-9)^n (1\cdot 3\cdot 5 \cdots (2n-1))}{n!\, 2^n\, 3^{2n+1}}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x-9)^{n+1} (1\cdot 3\cdot 5 \cdots (2n-1)(2n+1))}{(n+1)!\, 2^{n+1}\, 3^{2n+3}} \cdot \frac{n!\, 2^n\, 3^{2n+1}}{(x-9)^n (1\cdot 3\cdot 5 \cdots (2n-1))} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-9)(2n+1)}{(n+1)\, 2 \cdot 3^2} \right| = \lim_{n\to\infty} \left| \frac{x-9}{18} \cdot \frac{2n+1}{n+1} \right| = \left| \frac{x-9}{9} \right| < 1$$

$R = 9$

9. Find the length of the curve, $x = \sin 2\theta$, $y = \cos 2\theta$, $0 \le \theta \le \pi$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{(2\cos 2\theta)^2 + (-2\sin 2\theta)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{4\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta$$

$$\int_0^{\pi} \sqrt{4}\, d\theta$$

$$L = 2\theta \Big|_0^{\pi}$$

$$2\pi$$

10. Expand $\dfrac{3x^2}{\sqrt{4+2x}}$ as a power series using the binomial series. State the radius of convergence.

$$\frac{3x^2 (}{\sqrt{4}\left(\sqrt{1+\frac{x}{2}}\right)} = \frac{3}{2}x^2\left(1+\frac{x}{2}\right)^{-1/2}$$

$$\frac{3}{2}x^2 \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^n(-1)^n}{n!\,2^n}$$

$$1 + k + k(k-1) + k(k-1)(k-2) +$$

$$1 + -\tfrac{1}{2} + -\tfrac{1}{2}\left(-\tfrac{3}{2}\right) + \tfrac{-1}{2}\left(\tfrac{-3}{2}\right)\left(\tfrac{-5}{2}\right)$$

$$1 + 1 + 1\cdot 3 + 1\cdot 3\cdot 5$$

$$\boxed{\frac{3}{2}x^2 + \sum_{n=1}^{\infty} \frac{(-1)^n 3x^{n+2}(1\cdot 3\cdot 5\cdots(2n-1))}{2^{2n+1}\,n!}}$$

$$\left|\frac{x}{2}\right| < 1 \qquad R = 2$$

11. Find the radius and interval of convergence for the power series:

a) $\sum_{n=1}^{\infty} \frac{2^n (2x+1)^n}{1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n 2x^{3n}}{n \ln(n+1)}$

a) $\lim_{n\to\infty} \left| \frac{2^{n+1}(2x+1)^{n+1}}{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)} \cdot \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{2^n (2x+1)^n} \right|$

$\lim_{n\to\infty} \left| \frac{2(2x+1)}{3n+1} \right| = 0 < 1$ always so $R = \infty$

$(I = -\infty, \infty)$

b) $\lim_{n\to\infty} \left| \frac{2x^{3n+3}}{(n+1)\ln(n+2)} \cdot \frac{n \ln(n+1)}{2x^{3n}} \right|$

$\lim_{n\to\infty} \left| \frac{nx^3}{(n+1)} \frac{\ln(n+1)}{\ln(n+2)} \right|$

$|x^3| < 1$ $R = 1$ $I = [-1, 1]$

$\lim_{n\to\infty} \frac{(-1)^n 2(-1)^{3n}}{n \ln(n+1)} = 0$ cvs

$\lim_{n\to\infty} \frac{(-1)^n 2(1)^{3n}}{n \ln(n+1)} = 0$ cvg