MATH 152 | **EXAM III** | NAME Answers
Mrs. Bonny Tighe | 100 points | Wednesday 11/30/05
12.4-11.2

There are 11 problems worth 10 points each.

1. Test the series for convergence or divergence. **State the test** you use and show all work. If the series is an Alternating Series, find if it is Absolutely or Conditionally convergent.

a) Use the Comparison Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+n}$$

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms

i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n then $\sum a_n$ is also

ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n then $\sum a_n$ is also

$b_n = \frac{1}{n^3}$ which is convergent $p > 1$, p-Series

$a_n = \frac{1}{n^3+n}$

$\frac{1}{n^3+n} \le \frac{1}{n^3}$ for all n

So a_n is also convergent

So absolutely convergent

b) Use the Ratio Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

$L < 1$ convergent
$L > 1$ divergent
$L = 1$ inconclusive

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{2n+2}(n+1)!} \cdot \frac{(n!)^2 \, 2^{2n}}{x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{4(n+1)(n+1)} \right| = 0 < 1$$ always

So Convergent

So Absolutely Convergent

2. Test the series for convergence or divergence. State the test used.

a) Use the Integral test

$$\sum_{n=2}^{\infty} \frac{\cancel{dx}}{\cancel{x\sqrt{\ln x}}} \quad \frac{1}{n\sqrt{\ln n}}$$

$a_n = f(n)$ & f is a continuous positive decreasing function on $[2,\infty)$ then if $\int_2^\infty f(x)dx$ is cvg then $\sum a_n$ is also and if $\int_2^\infty f(x)dx$ is divergent the $\sum a_n$ is also

$$\lim_{t\to\infty} \int_2^t (\ln x)^{-1/2} \cdot \frac{1}{x}\,dx$$

$$\lim_{t\to\infty} \frac{1}{1/2}(\ln x)^{1/2}\Big|_2^t =$$

$$\lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right)$$

$$\infty - 2\sqrt{\ln 2}$$

$$\infty$$

Diverges so $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ also diverges

b) Use the Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$$

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ a finite number >0 then either both converge or both diverge

choose $b_n = \frac{1}{n\sqrt{n}}$ is convergent by p-series $p<1$,

$$\lim_{n\to\infty} \left|\frac{\frac{\sin(\frac{1}{n})}{\sqrt{n}}}{\frac{1}{n\sqrt{n}}}\right| = \lim_{n\to\infty} \left[\frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right].$$

$1\ (p) > 0$

So Convergent by Limit Comparison Test

3. Express the function $f(x)=\coth^{-1}(x^2)$ as a power series by first finding it's derivative, $f'(x)=\dfrac{1}{1-x^4}$, as a power series and then integrating. Find the interval of convergence.

$$f'(x)=\sum_{n=0}^{\infty} x^{4n} \quad \text{so} \quad \boxed{f(x)=C+\sum_{n=0}^{\infty}\frac{x^{4n+1}}{4n+1}=\coth^{-1}(x^2)}$$

$$\lim_{n\to\infty}\left|\frac{x^{4n+5}}{4n+5}\cdot\frac{4n+1}{x^{4n+1}}\right|=|x^4|<1 \quad \text{so } I=[0,1)$$

$R=1$

4. Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence. $f(x)=\dfrac{3}{x^2+x-2}$ $\quad (x+2)(x-1)$

$$\frac{A}{x+2}+\frac{B}{x-1} \qquad \frac{1}{x-1}-\frac{1}{x+2}$$

$A(x-1)+B(x+2)=3$

$B=1$

$A=-1$

$$\frac{-1}{1-x}+\frac{1}{2-x}=\frac{-1}{1-(x)}+\frac{+1/2}{1-\left(\frac{x}{2}\right)}$$

$$-1\sum_{n=0}^{\infty}x^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{+x^n}{2^{n+1}}-x^n \text{ or } \sum_{n=0}^{\infty}x^n\left[\frac{-1}{2^{n+1}}-1\right]$$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+2}}\cdot\frac{2^{n+1}}{x^n}\right|=\left|\frac{x}{2}\right|<1$$

$R=2 \qquad I=-2,2$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|=|x|<1 \quad R=1$$

$I=[-1,1)$ ✓ works for both

5. Evaluate the indefinite integral as an infinite series. $\int \frac{\cos x}{x} dx$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

so $\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!} dx$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n)!} + C}$$

6. Find the sum of the series. a) $\sum_{n=0}^{\infty} \frac{2^n}{3^n n!} =$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$\sum_{n=0}^{\infty} \frac{(2/3)^n}{n!}$$

$$e^{2/3}$$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} =$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} = \cos 3$$

7. Find the Taylor series for f(x) centered at the given value of a and find its radius of convergence. $f(x) = \ln x,\ \textit{centered at}\ a = 2$

	at $a=2$
$f(x) = \ln x$	$= \ln 2$
$f'(x) = \frac{1}{x}$	$= 1/2$
$f''(x) = -\frac{1}{x^2}$	$= -1/4$
$f'''(x) = \frac{2}{x^3}$	$= \frac{2}{8}$
$f''''(x) = \frac{-6}{x^4}$	$= \frac{-6}{2^4}$
$f'''''(x) = \frac{24}{x^5}$	$= \frac{4!}{2^5}$

$$\ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{n!} \left(\frac{(n-1)!}{2^n}\right)$$

$$\boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n\, 2^n}}$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n(2^n)}{(x-2)^n} \right| =$$

$$\lim_{n\to\infty} \left| \frac{(x-2)}{2} \left(\frac{n}{n+1}\right) \right| = \left|\frac{x-2}{2}\right| < 1$$

$R = 2$

$I = (0, 4)$

8. Find the Maclaurin series of f(x) and its radius of convergence. $f(x)=\frac{1}{\sqrt{x+1}}$

	@ $x=0$
$f(x)=(x+1)^{-1/2}$	$=1$
$f'(x)=-\frac{1}{2}(x+1)^{-3/2}$	$-\frac{1}{2}$
$f''(x)=\frac{3}{4}(x+1)^{-5/2}$	$3/4$
$f'''(x)=\frac{-1\cdot3\cdot5}{2^3}(x+1)^{-7/2}$	$-\frac{1\cdot3\cdot5\cdot}{2^3}$

$$1+\sum_{n=1}^{\infty}\frac{x^n(-1)^n}{n!\,2^n}$$

$$1-\frac{1}{2}+\frac{3}{4}-\frac{1\cdot3\cdot5}{2^3}+\frac{1\cdot3\cdot5\cdot7}{2^4} \qquad 1\cdot3\cdot5\cdots(2n-1)$$

$$\boxed{1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n(1\cdot3\cdot5\cdots(2n-1))}{n!\,2^n}}$$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}(1\cdot3\cdot5\cdots(2n-1)(2n+1))}{(n+1)!\,2^{n+1}}\cdot\frac{n!\,2^n}{x^n(1\cdot3\cdot5\cdots(2n-1))}\right|$$

$$\lim_{n\to\infty}\left|\frac{x(2n+1)}{(n+1)(2)}\right|=|x|$$

$$\lim_{n\to\infty}\left|x\left(\frac{2n+1}{2n+2}\right)\right|=|x|<1 \qquad R=1$$

9. Expand $\frac{x}{\sqrt{4-x}}$ as a power series using the binomial series. State the radius of convergence.

$$x\left(\frac{1}{\sqrt{4}}\right)\left(1-\frac{x}{4}\right)^{-1/2}$$

$$\frac{x}{2}\left(1+\left(\frac{-x}{4}\right)\right)^{-1/2}$$

$$\frac{x}{2}\left(1+\sum_{n=1}^{\infty}\frac{\left(\frac{-x}{4}\right)^n(-1)^n(1\cdot3\cdot5\cdots(2n-1))}{n!\,2^n}\right) = \frac{x}{2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n(-1)^n(1\cdot3\cdot5\cdots(2n-1))}{4^n n!\,2^n}\right)$$

$$\frac{x}{2}+\frac{x}{2}\sum_{n=1}^{\infty}$$

$$1+k+k(k-1)+k(k-1)(k-2)+$$

$$1+{-1/2}+{-1/2}(-3/2)+{-1/2}(-3/2)(-5/2)$$

$$\boxed{\frac{x}{2}+\sum_{n=1}^{\infty}\frac{x^{n+1}(1\cdot3\cdot5\cdots(2n-1))}{n!\,2^{3n+1}}}$$

$R=4$

$\left|\frac{-x}{4}\right|<1$

$$\frac{dx}{d\theta} = 2\cos 2\theta \quad \frac{dy}{d\theta} = -2\sin 2\theta$$

10. Find the length of the curve, $x = \sin 2\theta$, $y = \cos 2\theta$, $0 \le \theta \le \pi$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_0^{\pi} \sqrt{(2\cos 2\theta)^2 + (-2\sin 2\theta)^2}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{4\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta =$$

$$\int_0^{\pi} \sqrt{4}\, d\theta = \int_0^{\pi} 2\, d\theta = 2\theta \Big|_0^{\pi} =$$

$$2\pi$$

11. Find the radius and interval of convergence for the power series:

a) $\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{1\cdot 4\cdot 7\cdot 10\cdots(3n-2)}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n 3x^{2n}}{n\ln(n+1)}$

a) $\lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{1\cdot 4\cdot 7\cdots(3n-2)(3n+1)} \cdot \frac{1\cdot 4\cdot 7\cdots(3n-2)}{2^n(x-1)^n} \right|$

$\lim_{n\to\infty} \left| \frac{2(x-1)}{3n+1} \right| = 0 < 1$ always true

so $R = \infty$

$I = (-\infty, \infty)$

b) $\lim_{n\to\infty} \left| \frac{3x^{2n+2}}{(n+1)\ln(n+2)} \cdot \frac{n\ln(n+1)}{3x^{2n}} \right| =$

$\lim_{n\to\infty} \left| x^2 \frac{n}{n+1} \cdot \frac{\ln(n+1)}{\ln(n+2)} \right| = |x^2| < 1$

$R = 1$

$I = [0, 1]$

$\sum_{n=1}^{\infty} \frac{(-1)^n (0)}{} = 0$ cvg

$\sum_{n=1}^{\infty} \frac{(-1)^n 3(1)^{2n}}{n(\ln(n+1))} = $ cvg