MATH 152
Mrs. Bonny Tighe

# FINAL EXAM
200 points

NAME Answers

Section ________ Fri 5/19/06

1. Find the derivative using logarithmic differentiation.

a) $f(x) = \left(\dfrac{x+1}{x+2}\right)^x$

$\ln y = x(\ln(x+1) - \ln(x+2))$

$\frac{1}{y}\,dy/dx = x\left[\frac{1}{x+1} - \frac{1}{x+2}\right] + \ln\left[\frac{x+1}{x+2}\right](1)$

$dy/dx = \left[\left(\frac{x}{x+1} - \frac{x}{x+2}\right) + \ln\left(\frac{x+1}{x+2}\right)\right]\left(\frac{x+1}{x+2}\right)^x.$

(10)

2. Evaluate: a) $\int \csc^3 \alpha \, d\alpha$ = _______

$\int \underbrace{\csc^2\alpha}_{dv} \underbrace{\csc\alpha}_{u} \, d\alpha$

$v = -\cot\alpha$ $\quad du = -\csc\alpha\cot\alpha \, d\alpha$

(5)

$\int \csc^3\alpha = -\csc\alpha\cot\alpha - \int -\cot\alpha(-\csc\alpha\cot\alpha)\,d\alpha$

$= -\csc\alpha\cot\alpha - \int \csc\alpha \underbrace{\cot^2\alpha}_{(\csc^2\alpha - 1)}\,d\alpha$

$= -\csc\alpha\cot\alpha - \int \csc^3\alpha\,d\alpha + \int \csc\alpha\,d\alpha$

$2\int \csc^3\alpha\,d\alpha = -\csc\alpha\cot\alpha + \ln|\csc\alpha - \cot\alpha| + C$

$\int \csc^3\alpha\,d\alpha = \frac{1}{2}\left[-\csc\alpha\cot\alpha + \ln|\csc\alpha - \cot\alpha| + C\right]$

b) $\int e^{-x} \sin 2x \, dx = \left(\frac{-1}{2}e^{-x}\cos 2x + \frac{1}{4}e^{-x}\sin 2x\right)\frac{4}{5}$

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$u \quad dv$

$du = -e^{-x} \quad v = \frac{-1}{2}\cos 2x \, dx$

$e^{-x}\left(-\frac{1}{2}\cos 2x\right) - \int -\frac{1}{2}\cos 2x\,(-e^{-x})\,dx$

$-\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int \cos 2x \, e^{-x}\, dx$

$dv \quad u$

$du = -e^{-x}$

$v = \frac{1}{2}\sin 2x$

$\int e^{-x}\sin 2x\,dx = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\left[-e^{-x}\frac{1}{2}(\sin 2x) - \int \frac{1}{2}\sin 2x(-e^{-x})\right]$

$-\frac{1}{2}e^{-x}\cos 2x + \frac{1}{4}e^{-x}\sin 2x + \frac{1}{4}\int e^{-x}\sin 2x\,dx$

$\frac{5}{4}\int e^{-x}\sin 2x\,dx = -\frac{1}{2}e^{-x}\cos 2x + \frac{1}{4}e^{-x}\sin 2x$

3. Find the following limits. a) $\lim_{x\to\infty}\left(\frac{e^{2x}}{x^3}\right) = \infty$

$\frac{\infty}{\infty}$ L'H $\lim_{x\to\infty}\frac{2e^{2x}}{3x^2} = \frac{\infty}{\infty} =$ L'H $\lim_{x\to\infty}\frac{4e^{2x}}{6x} = \frac{\infty}{\infty}$

L'H $= \lim_{x\to\infty}\frac{8e^{x}}{6} = \infty$

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b) $\lim_{x\to 0^+}(\cos x)^{1/x^2} = e^{-1/2}$

$\ln y = \lim_{x\to 0^+}\frac{1}{x^2}\ln(\cos x) = \frac{0}{0}$

L'H $= \lim_{x\to 0^+}\frac{\frac{1}{\cos x}(-\sin x)}{2x} = \frac{0}{0}$

L'H $= \lim_{x\to 0^+} = \frac{-\sec^2 x}{2} = \frac{-1}{2}$

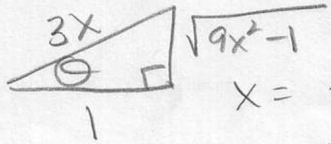
$\ln y = -\frac{1}{2}$

$y = e^{-1/2}$

c) $\lim_{x\to 0^+} x^2 \ln x = 0$

$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}} = \frac{\infty}{\infty} =$ L'H

$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0^+} -\frac{x^3}{2}\cdot\frac{1}{x} = 0$




$x = \frac{1}{3}\sec\theta$, $\sec\theta = 3x$

$dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$

4. Evaluate: $\int x^3\sqrt{9x^2-1}\,dx$

$\int \frac{1}{27}\sec^3\theta\sqrt{\sec^2\theta-1}\left(\frac{1}{3}\sec\theta\tan\theta\, d\theta\right)$

$\frac{1}{81}\int \sec^4\theta\tan^2\theta\, d\theta$

$\frac{1}{81}\int \underset{(1+\tan^2\theta)\tan^2\theta}{\sec^2\theta\tan^2\theta}\ \underset{\sec^2\theta\, d\theta}{\sec^2\theta\, d\theta}$

$\frac{1}{81}\int \sec^2\theta + \tan^4\theta\sec^2\theta\, d\theta$

$\frac{1}{81}\left(\tan\theta + \frac{1}{5}\tan^5\theta + C\right) =$

$\frac{1}{81}\left(\frac{\sqrt{9x^2-1}}{1}\right) + \frac{1}{5}\left(\sqrt{9x^2-1}\right)^5\Big) + C$

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5. Evaluate the integral using partial fractions: $\int \frac{x^3+3}{x^2-2x-3}dx =$ ________

$$\begin{array}{r} x+2 \\ x^2-2x-3\ \overline{)\ x^3 \qquad\qquad +3} \\ -(x^3-2x^2-3x) \\ \hline 2x^2+3x+3 \\ -(2x^2-4x-6) \\ \hline 7x+9 \end{array}$$

$\int x+2 + \frac{7x+9}{(x-3)(x+1)}\,dx$

$\int (x+2)\,dx + \frac{15}{2}\int\left(\frac{1}{x-3}\right)dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$

$A(x+1) + B(x-3) = 7x+9$

$(x=-1)$ $\quad -4B = 2$, $B = -1/2$

$(x=3)$ $\quad 4A = 30$, $A = 15/2$

$\frac{1}{2}x^2 + 2x + \frac{15}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C$

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6. Find the Macluarin series for the given function and find the radius and interval of convergence. $f(x) = \dfrac{1}{\sqrt{x+4}}$

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \frac{f''''(0)x^4}{4!} + \cdots$$

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$$f(x) = (x+4)^{-1/2}$$
$$f'(x) = -\tfrac{1}{2}(x+4)^{-3/2}$$
$$f''(x) = \tfrac{3}{4}(x+4)^{-5/2}$$
$$f'''(x) = \frac{-5\cdot 3}{2^3}(x+4)^{-7/2}$$

@ $x=0$

$$4^{-1/2} = 1/2 = \frac{1}{2}$$
$$-\tfrac{1}{2}\left(\tfrac{1}{2^3}\right) = -\frac{1}{2^4}$$
$$\tfrac{3}{4}\left(\tfrac{1}{2^5}\right) = \frac{3}{2^7}$$
$$\frac{-15}{2^3}\left(\frac{1}{2^7}\right) = \frac{-5\cdot 3}{2^{10}}$$

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n (1\cdot 3\cdot 5\cdots(2n-1))}{n!\, 2^{3n+1}}$$

$n=0$, $n=1$, $n=2$, $n=3$

$$\frac{1}{2} + \frac{1}{2^4} + \frac{3}{2^7} + \frac{5\cdot 3}{2^{10}}$$

$$\lim_{n\to\infty}\left| \frac{x^{n+1}(1\cdot 3\cdot 5\cdots(2n-1)(2n+1))}{2^{3n+1}\cdot 2^3\, n!\,(n+1)} \cdot \frac{2^{3n+1}\, n!}{x^n (1\cdot 3\cdot 5\cdots(2n-1))}\right|$$

$$\lim_{n\to\infty}\left| \frac{x\,(2n+1)}{2^3(n+1)}\right| = \left|\frac{x}{4}\right| < 1$$

$R = 4$

$I = (-4, 4]$

$$(x=-4)\quad \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n(1\cdot 3\cdot 5\cdots(2n-1))}{n!\, 2^{3n+1}} = \sum_{n=1}^{\infty}\frac{(1\cdot 3\cdot 5\cdots(2n-1))}{n!\,2^{n+1}}$$

$$(x=4)\quad \sum_{n=1}^{\infty} \frac{(-1)^n(1\cdot 3\cdot 5\cdots(2n-1))}{n!\,2^{n+1}}$$ Convergent AST

7. Determine whether the series are convergent or divergent using the given test.

Comparison Test

a) $\sum_{n=0}^{\infty} \frac{\sin n}{n^2+1}$

Sin n ≤ 1 so

$\frac{1}{n^2+1} \leq \frac{\sin n}{n^2+1}$

choose $\frac{1}{n^2}$ which is convergent by p-series

so $\frac{1}{n^2+1} \leq \frac{1}{n^2}$ so both

are convergent

Integral Test

b) $\sum_{n=2}^{\infty} \frac{2}{n \ln n}$

$\lim_{t\to\infty} 2\int_2^t \frac{1}{\ln n}\left(\frac{1}{n}\right) dn$ 10

$\lim_{t\to\infty} \ln(\ln n)\Big|_2^t$

$\lim_{t\to\infty} \left(\ln(\ln t) - \ln(\ln 2)\right)$

$\infty$

So Diverget

8. Find the radius of convergence and the interval of convergence if it is convergent.

$$\sum_{n=2}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-5)(x+1)^{2n-1}}{(\ln n)n!}(-1)^{n-1}$$

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$$\lim_{n\to\infty} \left| \frac{\cancel{1\cdot4\cdot7\cdots(3n-5)}(3n-2)(x+1)^{2n+1}}{\ln(n+1)(n+1)!} \cdot \frac{\ln n\,(n!)}{\cancel{1\cdot4\cdot7\cdots(3n-5)}(x+1)^{2n-1}} \right|$$

$$\lim_{n\to\infty} \left| (x+1)^2 \underbrace{\frac{3n-2}{n+1}}_{(3)} \underbrace{\left(\frac{\ln n}{\ln n+1}\right)}_{(1)} \right| = 3\left|(x+1)^2\right| < 1$$

R = 1/3

$-\frac{1}{3} < (x+1)^2 < \frac{1}{3}$

$(x+1) < \pm\sqrt{1/3}$

$x = -1 \pm \sqrt{1/3}$

$(1+x)$

9. Expand $\dfrac{1}{(1-2x)^3}$ as a power series using the binomial series. State the radius of convergence.

$(1+x)^k$

$1 + kx + \dfrac{k(k-1)x^2}{2!} + \dfrac{k(k-1)(k-2)x^3}{3!} + \cdots$

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$(1+(-2x))^{-3} = 1 + \dfrac{2(-3)}{2} + \dfrac{2-3(-4)}{2} + \dfrac{2-3(-4)(-5)}{2} + \dfrac{2(-3)(-4)(-5)(-6}{2}$

$1 + \sum_{n=1}^{\infty} \dfrac{(-2x)^n}{n!} = 1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n 2^n x^n}{n!} \dfrac{(-1)^n (n+2)!}{2}$

$\boxed{1 + \sum_{n=1}^{\infty} 2^{n-1} x^n (n+1)(n+2)}$

$\lim_{n\to\infty} \left| \dfrac{2^n x^{n+1} (n+2)(n+3)}{2^{n-1} x^n (n+1)(n+2)} \right| = |2x| < 1$

$R = 1/2$

10. Set up but do not evaluate the following:

$\frac{dx}{dt} = \sec t \tan t$

a) Find the length of the curve, $x = \sec t$, $y = \ln(1+t)$, $0 \le t \le 2$

$dy/dt = \frac{1}{1+t}$

$L = \int_0^2 \sqrt{\sec^2 t \tan^2 t + \left(\frac{1}{1+t}\right)^2}\, dt$

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b) Find the surface area generated by revolving the given curve about the y-axis.
$y = \ln(\cos x)$, $0 \le y \le 3$

$dy/dx = \frac{1}{\cos x}(-\sin x)$

$S = 2\pi \int_0^3 y \sqrt{1 + (dy/dx)^2}\, dx$

$2\pi \int_0^3 \ln(\cos x) \sqrt{1 + (-\tan x)^2}\, dx$

11. Sketch the curve and find the area that it encloses. $r = 3\cos 2\theta$

$\frac{1}{2}\int_0^{\pi/2} (3\cos 2\theta)^2\, d\theta \quad \times 4$

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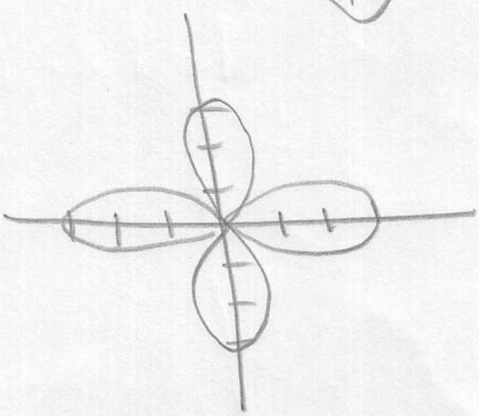

$\frac{9}{2}\int_0^{\pi/4} (\cos^2 2\theta)\, d\theta$

$4\left(\frac{9}{2}\right)\int_0^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$

$18\left(\frac{1}{2}\theta\right)\Big|_0^{\pi/2} + \frac{18}{2}\cdot\frac{1}{4}\sin 4\theta\, \Big|_0^{\pi/4}$

$9\theta\Big|_0^{\pi/2} + \frac{9}{4}[\sin\pi - \sin 0]$

$9\left(\frac{\pi}{2}\right) - 0 + 0$

$\frac{9\pi}{2}$

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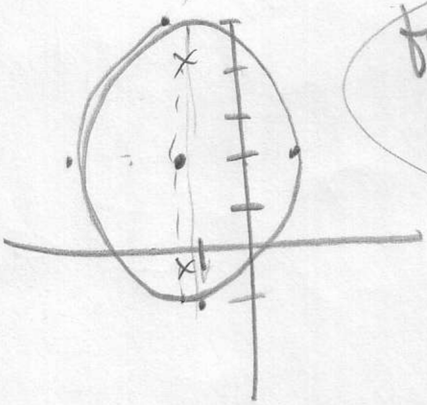
12. a) Find the foci, vertices and center of the ellipse $9x^2+4y^2+18x-16y=11$ and then sketch the graph.

$9(x^2+2x+1)+4(y^2-4y+4)=11+9+16$

$9(x+1)^2+4(y-2)^2=36 \rightarrow \frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}=1$

$c=\sqrt{9-4}=\sqrt{5}$



foci $(-1, 2\pm\sqrt{5})$

vertices $(-1, 2\pm3)$

Center $(-1,2)$

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b) Find the vertices, foci and asymptotes of the hyperbola and sketch its graph.

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

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foci $c=\sqrt{16+9}=5$

$(\pm5,0)$

vertices $(\pm3,0)$

asymptotes : $y=\pm\frac{4}{3}x$