MATH 152 **EXAM IIIA** NAME Answers

Mrs. Bonny Tighe 100 points

12.5-11.4 Section _______ Wed. 5/10/06

There are 11 problems worth 10 points each.

1. Test the series for convergence or divergence. State and show the test.

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}\right| =$

$\lim_{n\to\infty} \left|\frac{3}{n+1} \cdot \frac{(n+1)^2}{n^2}\right| = 0 < 1$ so absolutely convergent

2, Test the series for divergence or convergence. State and show the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n \ln n}$$

Comparison Test: $a_n = \frac{1}{n \ln n}$ $b_n = \frac{\sin n}{n \ln n}$

$\frac{1}{n \ln n} \geq \frac{\sin n}{n \ln n}$ because $\sin n \leq 1$

Integral test: $\int_1^{\infty} \frac{1}{x} \frac{1}{\ln x} dx = \ln(\ln x)\Big|_1^{\infty} =$

$\lim_{t\to\infty} \left(\ln(\ln t) - \ln(\ln 1)\right) = \infty$ so divergent

so both are divergent

$-1 < 4x+1 < 1$
$-2 < 4x < 0$
$-1/2 < x < 0$

3. Find the radius of convergence and the interval of convergence of the series.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n (4x+1)^n}{n^2}$

$$\lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x+1)^n} \right| = \lim_{n\to\infty} \left| 4x+1 \cdot \frac{n^2}{(n+1)^2} \right| = |4x+1| < 1$$

$R = 1/4$

$I = [-1/2, 0]$

$x = -1/2$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{n^2} = \frac{1}{n^2}$ Convergent p-series

$x = 0$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n (1)^n}{n^2} = \frac{(-1)^n}{n^2}$ Convergent p-series or AST

b) $\sum_{n=0}^{\infty} n^3 (x-3)^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3 (x-3)^{n+1}}{n^3 (x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} (x-3) \right| = |x-3| < 1$$

$-1 < x-3 < 1$
$2 < x < 4$

$R = 1$
$I = (2, 4)$

$x = 2$ $\quad \sum_{n=0}^{\infty} n^3 (-1)^n$ Divergent, test for divergence

$x = 4$ $\quad \sum_{n=0}^{\infty} (1)^n n^3$ " "

$$\frac{1}{1-x} = \sum_{n=0} x^n$$

4. Find a power series representation for the function and determine the radius and interval of convergence. $f(x) = \dfrac{1}{2-\sqrt{x}} = \dfrac{1/2}{1-\frac{\sqrt{x}}{2}}$

$$\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\sqrt{x}}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{x^{n/2}}{2^{n+1}}\right)$$

$$\lim_{n\to\infty}\left|\frac{x^{\frac{n+1}{2}}}{2^{n+2}}\cdot\frac{2^{n+1}}{x^{n/2}}\right| = \lim_{n\to\infty}\left|\frac{x^{1/2}}{2}\right| < 1$$

$$0 < \sqrt{x} < 2$$

$$0 < x < 4$$

$R = 2$

$I = (0, 4]$

$x = 0$ $\quad \sum \frac{0^{n/2}}{2^{n+1}}$ convergent (all 0)

$x = 4$ $\quad \sum_{n=0}^{\infty}\frac{2}{2^{n+1}} = \frac{1}{2^n}$ Convergent r-series $|R|<1$

5. Evaluate the indefinite integral as an infinite series. $\displaystyle\int \frac{\tan^{-1}x - e^{-x}}{x}\,dx$

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ $\quad \tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ from text

$$\int \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{2n+1} - \frac{(-1)^n x^n}{n!}\right)dx =$$

$$C + \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{(2n+1)^2} - \frac{(-1)^n x^{n+1}}{n!(n+1)}\right]$$

6. Find the Taylor series for f(x) centered at the given value of $f(x) = \frac{1}{x-1}$ at a = 2 and find the radius of convergence.

$$f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

$f = (x-1)^{-1}$ at $a=2$ 1

$f' = -1(x-1)^{-2} = -1$

$f'' = 2(x-1)^{-3} = 2$

$f''' = -6(x-1)^{-4} = -6$

$1, \; -1, \; 2, \; -6, \; 24$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$$1 + \sum_{n=0}^{\infty} \frac{(x-2)^n(-1)^n \, n!}{n!} = \sum_{n=0}^{\infty} (-1)^n (x-2)^n$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(x-2)^n} \right| = |x-2| < 1 \qquad R = 1$$

7. Find the Maclaurin series of f(x) and its radius of convergence. $f(x) = \frac{1}{(2x+1)^3}$

$f(x) = (2x+1)^{-3} = 1$

$f'(x) = -3(2x+1)^{-4}(2) = -3\ (2)$

$f''(x) = 12(2x+1)^{-5}\,2^2 = 12\ (2^2)$

$f'''(x) = 5\cdot 4\cdot 3\cdot 2\cdot 1\,(2x+1)^{-4}\,2^3 = 5!\ (2^3)$

1 3 4! 5!

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n (n+1)!\,2^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n (n+1)\,2^n}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}(n+2)2^{n+1}}{x^n (n+1)2^n} \right| = |2x| < 1 \qquad R = 1/2$$

$$(1+x)^k = 1 + \frac{kx}{1!} + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

8. Expand $\frac{2}{\sqrt{2-x}}$ as a power series using the binomial series. State the radius and interval of convergence.

$$\frac{2}{\sqrt{2}\sqrt{1+(-\frac{x}{2})}} = \sqrt{2}\left(1+\left(-\frac{x}{2}\right)\right)^{-1/2} \qquad \sqrt{2}\sum_{n=0}^{\infty} \frac{\left(-\frac{x}{2}\right)^n (-1)^n}{n!\, 2^n}$$

$$1 + -\tfrac{1}{2} + -\tfrac{1}{2}\left(-\tfrac{3}{2}\right) + -\tfrac{1}{2}\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right) + -\tfrac{1}{2}\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)\left(-\tfrac{7}{2}\right) \cdots$$

$$\sqrt{2} + \sqrt{2}\sum_{n=1}^{\infty} \frac{x^n (1\cdot 3\cdot 5 \cdots (2n-1))}{n!\, 2^{2n}} = \boxed{\sqrt{2} + \sum_{n=1}^{\infty} \frac{x^n (1\cdot 3\cdot 5 \cdots (2n-1))}{n!\, 2^{2n-1/2}}}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}(1\cdot 3\cdot 5\cdots(2n-1)(2n+1))}{(n+1)!\, 2^{2n-1/2}\, 2^2} \cdot \frac{n!\, 2^{2n-1/2}}{x^n(1\cdot 3\cdot 5\cdots(2n-1))} \right| = \lim_{n\to\infty} \left| \frac{x(2n+1)}{(n+1)2^2} \right| = \left|\frac{x}{2}\right|$$

$R = 2$

$I = [-2, 2]$

$x = -2$ $$\sum_{n=1}^{\infty} \frac{(-2)^n (1\cdot 3\cdot 5 \cdots (2n-1))}{n!\, 2^{2n-1/2}} = \frac{(-1)^n (1\cdot 3\cdot 5 \cdots (2n-1))\sqrt{2}}{n!\, 2^{n}}$$ Converges

$x = 2$ $$\sum_{n=1}^{\infty} \frac{2^n (1\cdot 3\cdot 5 \cdots (2n-1))}{n!\, 2^{2n-1/2}} = \frac{(1\cdot 3\cdot 5 \cdots (2n-1))\sqrt{2}}{n!}$$ converges

$\frac{1}{\cos^3\theta} \quad \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\cos\theta} \cdot \frac{1}{\sin^2\theta}$

$= (1+t)^{-1}$

9. Find the length of the curve $x = \dfrac{1}{1+t}$ *and* $y = \ln(1+t) \quad 0 \le t \le 2$

$dy/dt = \frac{1}{1+t}$

$dx/dt = -1(1+t)^{-2}$

$\int_0^2 \sqrt{\left(\frac{1}{(1+t)}\right)^2 + \left(\frac{-1}{(1+t)^2}\right)^2}\, dt$

$\int_0^2 \sqrt{\frac{(1+t)^2+1}{(1+t)^4}}\, dt = \int_0^2 \frac{\sqrt{(1+t)^2+1}}{(1+t)^2}\, dt$

$1+t = \tan\theta$
$dt = \sec^2\theta\, d\theta$

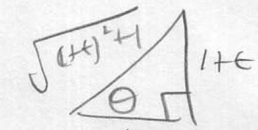
$\int_0^2 \frac{\sqrt{\tan^2\theta+1}}{\tan^2\theta} \sec^2\theta\, d\theta = \int_0^2 \frac{\sec^3\theta}{\tan^2\theta}\, d\theta = \int_0^2 \csc^2\theta \cdot \sec\theta\, d\theta$

$dv \quad u$

$v = -\cot\theta \quad du = \sec\theta\tan\theta$

$-\sec\theta\cot\theta - \int -\cot\theta \sec\theta\tan\theta\, d\theta =$

$-\sec\theta\cot\theta + \ln|\sec\theta + \tan\theta| \Big|_0^2 =$



$-\sqrt{(1+t)^2+1}\left(\frac{1}{1+t}\right) + \ln\left|\sqrt{(1+t)^2+1} + 1+t\right| \Big|_0^2 = \left(-\sqrt{10}\left(\frac{1}{3}\right) + \ln|\sqrt{10}+3|\right) - \left(-\sqrt{2}(1) + \ln(\sqrt{2}+1)\right)$

$-\frac{\sqrt{10}}{3} + \ln|\sqrt{10}+3| + \sqrt{2} - \ln|\sqrt{2}+1|$

10. a) Find a Cartesian equation for the curve described by the polar equation
$r = \tan\theta\sec\theta$

$r^2 = r\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$

$r^2\cos^2\theta = r\sin\theta$

$x^2 = y$

b) Convert the Cartesian coordinates (2, -3) to polar coordinates.

$r^2 = x^2+y^2 = 4+9 \quad r = \sqrt{13}$

$\tan\theta = -3/2$

$(\sqrt{13}, \tan^{-1}(-3/2))$

c) Convert the polar coordinates $(4, \frac{7\pi}{6})$ to Cartesian coordinates.

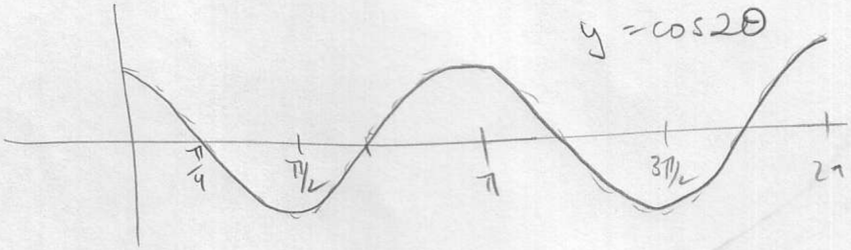
$x = r\cos\theta \qquad y = r\sin\theta$

$x = 4\cos 7\pi/6 \qquad y = 4\sin 7\pi/6$

$4(-\sqrt{3}/2) \qquad 4(-1/2)$

$(-2\sqrt{3}, -2)$

11. Sketch the curve of the polar equation $r = 1 + \cos 2\theta$



θ	
0	2
$\pi/6$	$\frac{1}{2}+1$
$\pi/4$	$0+1$
$\pi/3$	$-1/2+1 = 1/2$
$\pi/2$	0
$2\pi/3$	$-\frac{1}{2}+1 = 1/2$.
$3\pi/4$	1
$5\pi/6$	$1/2+1 = 3/2$
π	2
$7\pi/6$	$-1/2+1 = 3/2$
$5\pi/4$	$0+1$
$\frac{4\pi}{3}$	$-1/2+1 = 1/2$
$3\pi/2$	0
$5\pi/3$	
$7\pi/4$	
$11\pi/6$