MATH 152
Mrs. Bonny Tighe

# EXAM II

100 points
8.4-12.4

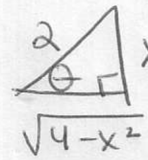
NAME Answers

Section ______ Wed. 4/5/2006

There are 11 problems worth 10 points each

1. a) Evaluate the integrals : a) $\int \frac{\sqrt{4-x^2}}{x}dx$

$x = 2\sin\theta$
$dx = 2\cos\theta\, d\theta$




$$\int \frac{\sqrt{4-4\sin^2\theta}\;\cancel{2}\cos\theta\, d\theta}{\cancel{2}\sin\theta} = 2\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$$2\int \underbrace{\frac{\cos\theta}{\sin\theta}}_{u}\cdot \underbrace{\cos\theta\, d\theta}_{dv} \qquad v = \sin\theta \qquad du = -\csc\theta\cot\theta\, d\theta$$

$$= 2\left[\cot\theta\sin\theta - \int \sin\theta(-\csc\theta\cot\theta)\,d\theta\right]$$

$$2\cot\theta\sin\theta + \int \cot\theta\, d\theta$$

$$2\cot\theta\sin\theta + \ln|\sin\theta| + C$$

$$2\,\frac{\sqrt{4-x^2}}{x}\left(\frac{x}{2}\right) + \ln\left|\frac{x}{2}\right| + C$$

$$\sqrt{4-x^2} + \ln\left|\frac{x}{2}\right| + C$$

or

$$2\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$2\int \csc\theta - \sin\theta\, d\theta$$

$$2\ln|\csc\theta - \cot\theta| + \cos\theta + C$$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \frac{\sqrt{4-x^2}}{2} + C$$

b) $\int \frac{e^x}{e^{2x}-e^x-2}dx$

$u = e^x$
$du = e^x dx$

$$= \int \frac{du}{u^2-u-2} \qquad (u-2)(u+1)$$

$$\frac{A}{u-2} + \frac{B}{u+1}$$

$A(u+1) + B(u-2) = 1$

$u=2 \quad 3A = 1 \quad A = 1/3$

$u=-1 \quad -3B = 1 \quad B = -1/3$

$$\frac{1}{3}\int \frac{1}{u-2}\,du - \frac{1}{3}\int \frac{1}{u+1}\,du$$

$$\frac{1}{3}\ln|u-2| - \frac{1}{3}\ln|u+1| + C$$

$$\frac{1}{3}\ln|e^x-2| - \frac{1}{3}\ln|e^x+1| + C$$

2. Evaluate the integral. $\int e^x \sin 2x \, dx$

$u = e^x \quad dv$

$du = e^x \quad v = -\frac{1}{2}\cos 2x$

$= e^x(-\frac{1}{2}\cos 2x) - \int -\frac{1}{2}\cos 2x \, e^x dx$

$-\frac{1}{2}e^x \cos 2x + \frac{1}{2}\int \cos 2x \, e^x dx$

$dv \quad u$

$v = \frac{1}{2}\sin 2x \quad du = e^x$

$\int e^x \sin 2x dx = -\frac{1}{2}e^x \cos 2x + \frac{1}{2}\left[e^x(\frac{1}{2}\sin 2x) - \int \frac{1}{2}\sin 2x \, e^x dx\right]$

$= -\frac{1}{2}e^x \cos 2x + \frac{1}{4}e^x \sin 2x - \frac{1}{4}\int \sin 2x \, e^x dx$

$\frac{5}{4}\int e^x \sin 2x dx = -\frac{1}{2}e^x \cos 2x + \frac{1}{4}e^x \sin 2x$

$= \frac{4}{5}\left[-\frac{1}{2}e^x \cos 2x + \frac{1}{4}e^x \sin 2x\right] + C$

$-\frac{2}{5}e^x \cos 2x \quad + \frac{1}{5}e^x \sin 2x + C$

3. Determine whether each improper integral is divergent or convergent and evaluate those that are convergent using improper integrals.

a) $\int_0^\infty \frac{1}{3}e^{-3x} \, dx$

$\lim_{t \to \infty} \int_0^t \frac{1}{3}e^{-3x} dx$

$\lim_{t \to \infty} -\frac{1}{9}e^{-3x}\Big|_0^t$

$\lim_{t \to \infty} \left(-\frac{1}{9e^{3t}} + \frac{1}{9e^0}\right) = \frac{1}{9}$

So Convergent

b) $\int_{-1}^1 \frac{1}{(x+1)^3} dx$

$\lim_{t \to -1} \int_t^1 (x+1)^{-3} dx$

$\lim_{t \to -1} \frac{1}{-2}(x+1)^{-2}\Big|_t^1$

$\lim_{t \to -1} \left[\frac{-1}{2(2)^4} + \frac{1}{2(t+1)^2}\right]$ DNE

So Divergent

4. Find the arc length of the curve $y = \ln(\cos x)$ *on the interval* $0 \le x \le \pi/3$ .

$$L = \int_0^{\pi/3} \sqrt{1 + (dy/dx)^2}\, dx \qquad dy/dx = \frac{1}{\cos x} \cdot \sin x$$

$$\int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$\int_0^{\pi/3} \sec x\, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/3} =$$

$$\ln|\sec \pi/3 + \tan \pi/3| - \ln|\sec 0 + \tan 0| =$$

$$\ln|2 + \sqrt{3}| - \ln|1 + 0| = \ln|2+\sqrt{3}|$$

5. Find its sum of the following convergent series.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$

$(n+1)(n+2)$

$$\frac{A}{n+1} + \frac{B}{n+2} = \frac{1}{(n+1)(n+2)}$$

$A(n+2) + B(n+1) = 1$

$(n=-1) \quad A = 1$

$(n=-2) \quad -B = 1, \ B = -1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \frac{1}{6} + \cdots$$

$$\cancel{-\frac{1}{3}} \cancel{-\frac{1}{4}} \cancel{-\frac{1}{5}} - \frac{1}{6} + \cdots$$

Sum $= 1/2$

b) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{7^n}$ , $3\left(\frac{3}{7}\right)^n$

$$\sum_{n=1}^{\infty} \frac{9}{7}\left(\frac{3}{7}\right)^{n-1} \qquad \left(\text{Sum} = \frac{a}{1-r}\right)$$

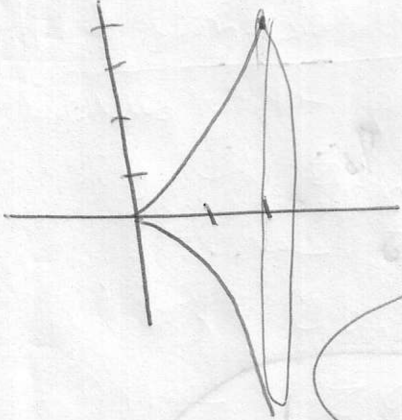
$$\frac{9/7}{1 - 3/7}$$

$$\frac{9}{7-3} = 9/4$$

$y = x^2$

6. SET UP BUT DO NOT EVALUATE the area of the surface generated by revolving the given curve about the x-axis and sketch the figure. $x = \sqrt{y}\ ,\ 0 \le x \le 2$

$0 \le y \le 4$



$$2\pi \int_0^2 x^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$2\pi \int_0^2 x^2 \sqrt{1 + 4x^2}\, dx$$

or

$$2\pi \int_0^4 y \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

7. State and use the Integral Test to determine whether the series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{2}{n^2+4}$ b) $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

Integral Test: Suppose $f$ is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$, then if $\int_1^\infty f(x)dx$ is convergent, then $\sum a_n$ is also and if $\int_1^\infty f(x)dx$ is divergent, then so is $\sum a_n$.

a) $\lim_{t\to\infty} \int_1^t \frac{2}{x^2+4}\, dx$

$\lim_{t\to\infty} 2 \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_1^t$

$\lim_{t\to\infty} \tan^{-1} \frac{t}{2} - \tan^{-1} \frac{1}{2}$

$\pi/2 - \tan^{-1} 1/2$

So both are convergent

b) $\lim_{t\to\infty} \int_1^t \frac{1}{\ln x} \cdot \frac{1}{x}\, dx$

$\lim_{t\to\infty} \ln(\ln x) \Big|_1^t$

$\lim_{t\to\infty} \ln(\ln t) - \ln(\ln 1)$ undefined

$\infty$

So divergent

8. State and use the Limit Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{3}{2n^2+3n+5}$ converges or diverges.

Limit Comparison Test | Suppose $\Sigma a_n$ and $\Sigma b_n$ are series with positive terms, If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, where C is a finite number greater than 0, then either both series converge or both diverge

$a_n = \frac{3}{2n^2+3n+5}$ choose $b_n = \frac{3}{2n^2}$ which is convergent by p-series.

$$\lim_{n\to\infty}\left(\frac{\frac{3}{2n^2+3n+5}}{\frac{3}{2n^2}}\right) = \lim_{n\to\infty}\left(\frac{\cancel{3}}{2n^2+3n+5}\cdot\frac{2n^2}{\cancel{3}}\right) =$$

$$\lim_{n\to\infty}\left(\frac{2n^2}{2n^2+3n+5}\right) = 1$$ which is a finite number $> 0$

So both are convergent

9. State and use The Comparison Test to determine if the following series are convergent or divergent. $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

**The Comparison Test** Suppose $\Sigma a_n$ and $\Sigma b_n$ are series with positive terms.

i) If $\Sigma b_n$ is convergent and $a_n \le b_n$, for all $n$, then $\Sigma a_n$ is also convergent

ii) If $\Sigma b_n$ is divergent and $a_n \ge b_n$, for all $n$, then $\Sigma a_n$ is also convergent.

$a_n = \frac{1}{3^n+2}$ Choose $b_n = \frac{1}{3^n}$ which is convergent geometric series, $|r|<1$

$\frac{1}{3^n+2} \le \frac{1}{3^n}$ for all $n$ so both are convergent.

10. Evaluate: $\int_0^{\pi/8} \tan^3 2x \sec^3 2x \, dx$

$$\int_0^{\pi/8} \underbrace{\tan^2 2x}_{(\sec^2 2x - 1)} \sec^2 2x \sec 2x \tan 2x \, dx$$

$$\frac{1}{2}\int_0^{\pi/8} (\sec^4 2x - \sec^2 2x) \underbrace{\sec 2x \tan 2x \, 2dx}_{du}$$

$$\frac{1}{2}\left[\frac{1}{5}\sec^5 2x - \frac{1}{3}\sec^3 2x\right]_0^{\pi/8} = \frac{1}{10}(\sec \pi/4)^5 - \frac{1}{6}(\sec \pi/4)^3 - \left(\frac{1}{10}\sec^5 0 - \frac{1}{6}\sec^3 0\right)$$

$$\frac{1}{10}(\sqrt{2})^5 - \frac{1}{6}(\sqrt{2})^3 - \frac{1}{10} + \frac{1}{6}$$

11. Determine whether the sequence is convergent or divergent, monotonic or bounded. If it converges, find the limit.

a) $a_n = \dfrac{\sqrt{n}}{2+\sqrt{n}}$ $\quad \frac{1}{3}, \frac{\sqrt{2}}{2+\sqrt{2}}, \frac{\sqrt{3}}{3+\sqrt{3}}, \frac{2}{2+2}, \frac{\sqrt{5}}{2+\sqrt{5}}, \ldots \frac{3}{2+3}, \frac{10}{12}$

Convergent, monotonic + bounded $\quad$ lower bound $= 1/3$, upper bound $= 1$

$\lim_{n\to\infty} a_n = 1$

b) $a_n = \ln(n+2) - \ln(n+1)$

$a_n = \ln\left(\frac{n+2}{n+1}\right)$

Convergent, monotonic, bounded – upper bound $= \ln(3/2)$, lower bound $= 0$

$\lim_{n\to\infty} a_n = \ln(1) = 0$