MATH 151
Mrs. Bonny Tighe

FINAL EXAM
200 points

NAME Answers

SECTION______ Fri 5/19/06

There are 20 problems worth 10 points each.

1. Find the following limits.

a) $\lim_{x\to 0}\frac{\sin x}{2x} = \underline{1/2}$

$\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}$

b) $\lim_{x\to 1}\frac{\sqrt{x}-1}{1-x} = \underline{-1/2}$

$\frac{\sqrt{x}-1\,(\sqrt{x}+1)}{(1-x)(\sqrt{x}+1)} = \frac{\cancel{x-1}(-1)}{\cancel{1-x}\,(\sqrt{x}+1)}$

$\lim_{x\to 1}\frac{-1}{\sqrt{x}+1} = -1/2$

c) $\lim_{x\to\infty}\frac{2x-1}{3-x} = \underline{-2}$

2. Use the **definition of the derivative of a function** at a point as the limit of the slope of the secant as h approaches 0 to evaluate f '(x) for f(x) = $\frac{1}{x+1} - 2$

$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(\frac{\frac{1}{x+h+1}-2-\left(\frac{1}{x+1}-2\right)}{h}\right)$

$\lim_{h\to 0}\left(\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h}\right) = \lim_{h\to 0}\frac{x+1-(x+h+1)}{h(x+h+1)(x+1)} =$

$\lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}(x+h+1)(x+1)} = \lim_{h\to 0}\frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+1)^2}$

3. Differentiate: $f(x)=\dfrac{\sin^3 x}{1-x^2}$

$$f'(x) = \frac{(1-x^2)(3\sin^2 x\cos x) - \sin^3 x(-2x)}{(1-x^2)^2}$$

$$= \frac{\sin^2 x\left[3\cos x(1-x^2) + 2x\sin x\right]}{(1-x^2)^2}$$

4. Find dy/dx. $2x - 3\sec y = x^2 y^3$

$$2 - 3\sec y\tan y\, dy/dx = x^2(3y^2\, dy/dx) + y^3(2x)$$

$$2 - 2xy^3 = 3y^2x^2\, dy/dx + 3\sec y\tan y\, dy/dx$$

$$dy/dx = \frac{2-2xy^3}{3y^2x^2 + 3\sec y\tan y}$$

5. Find the equation of the tangent line to $y=1+\cos^2 x$ at $(\frac{\pi}{6}, \frac{7}{4})$

$$\boxed{y - 7/4 = -\sqrt{3}/2\,(x - \pi/6)}$$

$m = dy/dx\ @\ x = \pi/6$

$dy/dx = 2\cos^1 x(-\sin x)$

$= 2(\cos \pi/6)(-\sin \pi/6)$

$2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right) =$

$m = -\sqrt{3}/2$

6. Air is being pumped into a spherical balloon and the radius is expanding at the rate of 10 cm / min. Find how fast the volume of the balloon is increasing when the radius is 20 cm. $V = \frac{4}{3}\pi r^3$

Given: $dr/dt = 10\text{cm/min}$

Find dV/dt

when $r = 20\text{cm}$

$V = \frac{4}{3}\pi r^3$

$dV/dt = \frac{4}{3}\pi 3r^2\, dr/dt$

$dV/dt = 4\pi(20)^2(10)$

$dV/dt = 16000\pi \text{ cm}^3/\text{min}$

7. Find the asymptotes and intercepts. $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 4x}$

$\dfrac{\cancel{(x-4)}(x+1)}{x\cancel{(x-4)}} = \dfrac{x+1}{x}$

VA : $x = 0$

HA : $y = 1$

x-int: $(-1, 0)$

y-int: none

8. A box is to be constructed from 900 cm^2 of material with an open top and a square base. Find the maximum volume that can be contained in this box.

Max Volume

$V = lwh$

$V = x^2\left(\frac{225}{x} - \frac{1}{4}x\right)$

$V = 225x - \frac{1}{4}x^3$

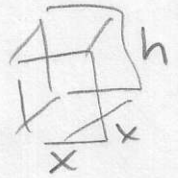
$dV/dx = 225 - \frac{3}{4}x^2 = 0$

$225 = \frac{3}{4}x^2$

$\frac{4}{3}(225) = x^2$

$300 = x^2$

$x = 10\sqrt{3}$ cm



$x^2 + 4xh = 900$

$h = \dfrac{900 - x^2}{4x}$

$h = \dfrac{225}{x} - \frac{1}{4}x$

$f(\pm 2) = 16 - 8(4) - 20$

$16 - 32 - 20 = -36$

9. Find the intercepts, intervals of increasing, decreasing, concave up and concave down and sketch. $f(x) = x^4 - 8x^2 - 20$

x-int: $(\pm\sqrt{10}, 0)$

y-int: $(0, -20)$

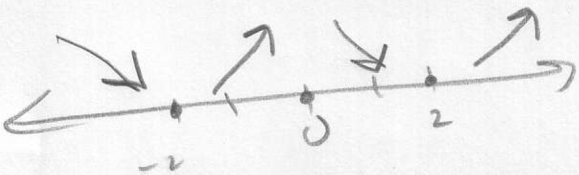
$(x^2 - 10)(x^2 + 2) = 0$

$x = \pm\sqrt{10}$

$f'(x) = 4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

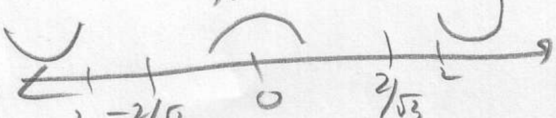
$x = 0, \pm 2$

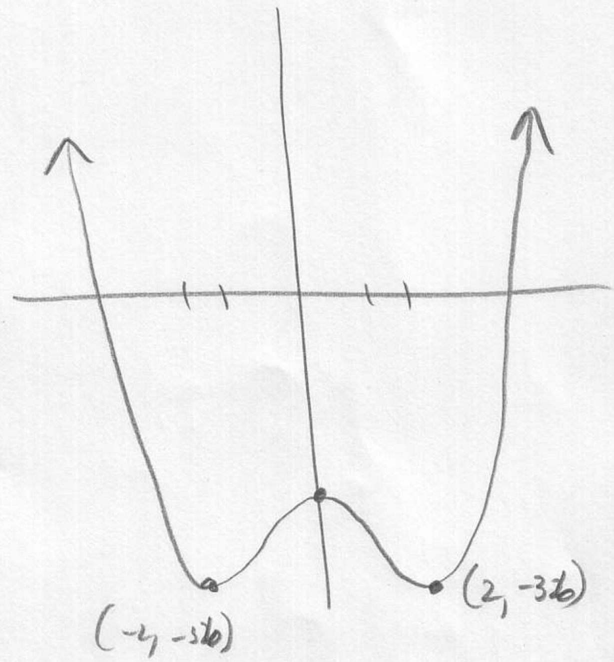


$f''(x) = 12x^2 - 16 = 0$

$4(3x^2 - 4) = 0$

$x = \pm\sqrt{4/3} = \pm 2/\sqrt{3}$





10. Use the Fundamental Theorem of Calculus to evaluate the area under the curve $f(x) = \csc^2 x$ on the interval $[\pi/6, \pi/4]$.

$\int_{\pi/6}^{\pi/4} \csc^2 x \, dx = -\cot x \Big|_{\pi/6}^{\pi/4} = -\cot \pi/4 - (-\cot \pi/6)$

$-1 + \sqrt{3}$

11. A particle is moving with velocity, $v(t) = 4t - 4\ {}^{m}/_{s}$. Find the displacement of the particle and then find the distance traveled on the interval [0,3].

a) displacement - $\int_0^3 (4t-4)\,dt = 2t^2 - 4t \Big|_0^3 =$

$(2(9) - 12) - (0) = 18 - 12 = 6m$

b) total distance : $v = 0$, $4t - 4 = 0$ $t = 1$ sec

$\left| \int_0^1 (4t-4)\,dt \right| + \left| \int_1^3 (4t-4)\,dt \right|$

$2t^2 - 4t \Big|_0^1 + 2t^2 - 4t \Big|_1^3$

$(2-4) = |-2| = 2$

$2(9) - 4(3) - (2-4)$

$18 - 12 + 2$

8

10m

12. Evaluate the definite integral, if it exists. $\int_1^2 \frac{\left(\frac{1}{x}+2\right)^3}{x^2}\,dx =$ _______

$-\int u^3\,du$ $u = \frac{1}{x} + 2$ $du = -\frac{1}{x^2}$

$-\frac{1}{4}u^4 \Big|_1^2 = -\frac{1}{4}\left(\frac{1}{x}+2\right)^4 \Big|_1^2 =$

$-\frac{1}{4}\left(\frac{1}{2}+2\right)^4 - \left(-\frac{1}{4}\left(\frac{1}{1}+2\right)^4\right) =$

$-\frac{1}{4}\left(\frac{5}{2}\right)^4 + \frac{1}{4}(3)^4$

$\int_0^2 (2x^2+1)\,dx \qquad \Delta x = \frac{2-0}{n} = \frac{2}{n}$

13. Evaluate the area under the curve $f(x) = 2x^2 + 1$ on the interval [0,2], using the definition, as a limit of the summation as n approaches infinity and evaluate using the following:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2\left(\frac{2i}{n}\right)^2 + 1\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{8i^2}{n^2} + 1\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{16i^2}{n^3} + \frac{2}{n}\right) = \lim_{n\to\infty} \left(\frac{16}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + \frac{2}{n}(2)\right)$$

$$\lim_{n\to\infty} \left(\frac{16}{3} + \frac{8}{n} + \frac{16}{6n^2} + 2\right) = \frac{16}{3} + 2 = 7\tfrac{1}{3} \text{ or } 22/3$$

Check $\int_0^2 (2x^2+1) = \frac{2}{3}x^3 + x \Big|_0^2 =$

$\left(\frac{2}{3}(8) + 2\right) - 0 = \frac{16}{3} + 2 = 7\tfrac{1}{2}$ or $22/3$ ✓

14. Evaluate the indefinite integral using substitution.

$\int \sec^2(\sin 3x)\cos 3x \; dx =$ __________

$\frac{1}{3}\int \sec^2 u \, du$

$u = \sin 3x$

$du = \cos 3x \,(3)\,dx$

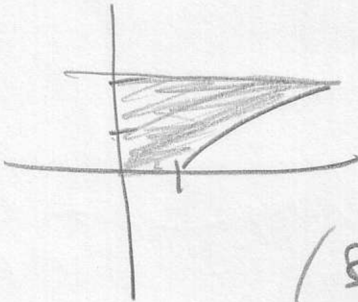
$\frac{1}{3}\tan u + C$

$\frac{1}{3}\tan(\sin 3x) + C$

15. Find the area bounded by the y-axis, the function $y=\sqrt{x-1}$, y=0 and y=2 Sketch the graph.

$y^2 = x - 1$

$y^2 + 1 = x$



$$\int_0^2 (y^2+1)\,dy = \frac{1}{3}y^3 + y \Big|_0^2 =$$

$$\left(\frac{8}{3}+2\right) - 0 = 4\tfrac{2}{3}$$

16. Find the points, if any, on the curve $g(x) = (x^2-1)^3$ where the tangent line is horizontal.

$g'(x) = 0$

$$g'(x) = 3(x^2-1)^2(2x) = 0$$

$$x = \pm 1, 0$$

$(0,-1)$ $(1,0)$ $(-1,0)$

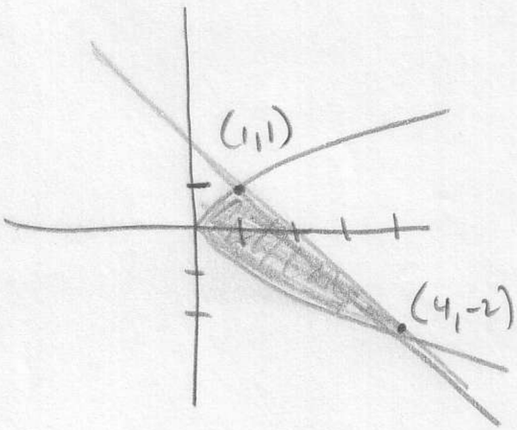
17. Find the second derivative for the function $h(x) = \tan 3x + 2x$

$h'(x) = \sec^2 3x(3) + 2$

$h''(x) = 2\sec 3x(3)(\sec 3x \tan 3x)(3)$

$18\sec^2 3x \tan 3x$

18. Sketch the region enclosed by the given curves, decide whether to integrate with respect to x or y. Find the area that is enclosed. $x = y^2$ and $y = 2 - x$



$x = 2-y$ graphs (Cross)

$2-y = y^2$

$0 = y^2 + y - 2$

$(y+2)(y-1)$

$y = 1, -2$

$\int_{-2}^{1} (2-y) - (y^2)\, dy$

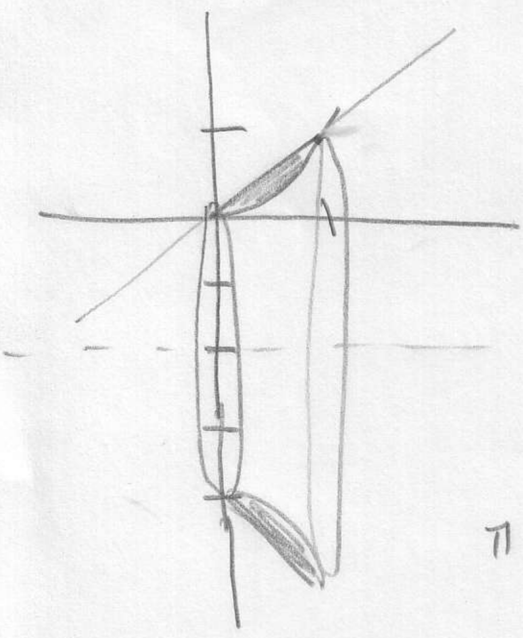
$2y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_{-2}^{1} =$

$\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{1}{2}(4) + \frac{8}{3}\right)$

$2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - 8/3$

$8 - \frac{9}{3} - 1/2 = 5 - 1/2$ $4\frac{1}{2}$

19. Rotate the region bounded by $y = x$ and $y = x^2$ about the line $y = -2$. Find the volume of the solid that has been generated using the washer or slicing method.



$\pi \int_0^1 (x+2)^2 - (x^2+2)^2\, dx$

$\pi \int_0^1 x^2 + 4x + \cancel{4} - x^4 - 4x^2 - \cancel{4}\, dx$

$\pi \int_0^1 (4x - x^4 - 3x^2)\, dx$

$\pi \left(2x^2 - \frac{1}{5}x^5 - x^3\right)\Big|_0^1 = \left(2 - \frac{1}{5} - 1\right)\pi$

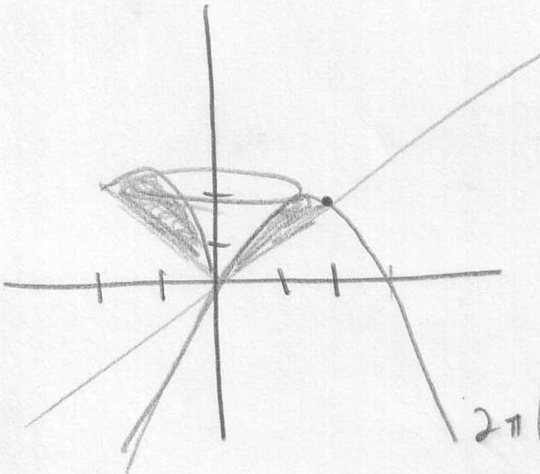
$4/5\,\pi$

$\left(\text{graphs cross}\right)$ $3x - x^2 = x$

$0 = x^2 - 2x$ $\quad x(x-2)$

$x = 0, 2$

20. Use the method of cylindrical shells to find the volume generated by the rotation of the region bounded by $y = 3x - x^2$ and $y = x$ about the y-axis.



$$2\pi \int_0^2 x(3x - x^2) - x(x)\, dx$$

$$2\pi \int_0^2 (3x^2 - x^3 - x^2)\, dx = 2\pi \int_0^2 (2x^2 - x^3)\, dx$$

$$2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4 \right) \Big|_0^2 = 2\pi \left(\frac{2}{3}(8) - \frac{1}{4}(16) \right) - 0$$

$$2\pi \left(\frac{16}{3} - 4 \right)$$

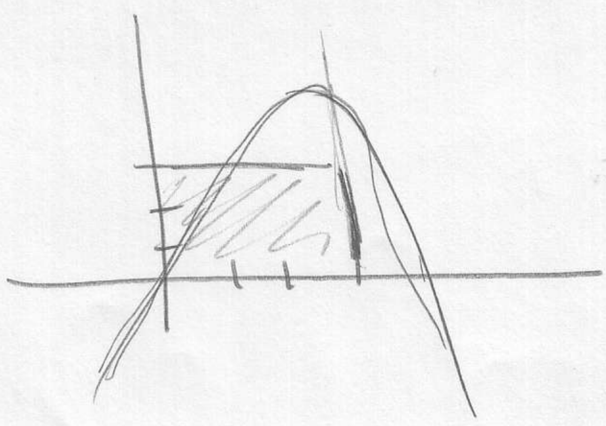
$$\left(5\tfrac{1}{3} - 4 \right)$$

$$2\pi \left(\frac{4}{3} \right) = \frac{8\pi}{3}$$

21. Find the average value of f(x)= $4x - x^2$ on the interval [0,3]. Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

$$\frac{1}{b-a} \int_a^b f(x)\, dx = \frac{1}{3-0} \int_0^3 (4x - x^2)\, dx =$$

$$\frac{1}{3} \left(2x^2 - \frac{1}{3}x^3 \right) \Big|_0^3 = \frac{1}{3} \left(2(9) - 9 \right) = 3$$



22. Use Newton's Method to approximate $\sqrt{35}$ to the third iteration, x_3.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x = \sqrt{35}$

$x^2 - 35 = 0 = f$

$2x = f'$

Choose $x_1 = 6$

$$x_2 = 6 - \frac{36-35}{2(6)} = 6 - \frac{1}{12} = \frac{71}{12}$$

$$x_3 = \frac{71}{12} - \frac{(71/12)^2 - 35}{2(71/12)}$$