Grad, Div and Curl in Cylindrical and Spherical Coordinates

In applications, we often use coordinates other than Cartesian coordinates. It is important to remember that expressions for the operations of vector analysis are DIFFERENT in different coordinates. Here we give explicit formulae for cylindrical and spherical coordinates.

1 Cylindrical Coordinates

In cylindrical coordinates,

$$x = r\cos\phi\ , \quad y = r\sin\phi\ , \quad z = z\ ,$$

we have

$$\nabla f = \hat{\boldsymbol{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial f}{\partial \phi} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\ ,$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}\ ,$$

$$\nabla\times\boldsymbol{u} = \hat{\boldsymbol{r}}\left(\frac{1}{r}\frac{\partial u_z}{\partial\phi} - \frac{\partial u_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + \hat{\boldsymbol{z}}\frac{1}{r}\left[\frac{\partial(r u_\phi)}{\partial r} - \frac{\partial u_r}{\partial\phi}\right].$$

2 Spherical Coordinates

In spherical coordinates,

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta\ ,$$

we have

$$\nabla f = \hat{\boldsymbol{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\ ,$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\ ,$$

$$\nabla\times\boldsymbol{u} = \hat{\boldsymbol{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r u_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$