Finite Element Formulation of Three-dimensional Nonlinear Elasticity Problem

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Abstract

The purpose of the present work is to give a brief description of the finite elasticity and of its approximation via finite element method. We formulate the problem for the case of compressible elasticity. Weak formulation allows to use any isotropic hyperelastic material model that satisfies polyconvexity assumptions. Discretization using FEM leads to systems of non-linear equations. Finally we also show the strategy of solving such systems of equations by modification of Newton's method that can be used under some restrictions.

1 Introduction

The main object of the finite three-dimensional elasticity is to predict changes in the geometry of solid bodies. The starting point of the classical theory of linear elasticity is the concept of small strains: the deformation of structures under working loads are not detectable by human eye. In contrast, many modern situations involve large deformations. The nonlinear behavior of polymers and synthetic rubbers are such examples. Applications in biomechanics are even more critical because the most of vital organs such as eye, heart trachea or vocal apparatus fulfill their function only because of their large deformations. In this framework the concept of finite elasticity covers the simplest case where internal forces (stresses) depend only on the present deformation of the body and not on the history. In the paper we show the finite element approximation and strategy of solution of this nonlinear and 'visible' stress-strain relationship.

2 Formulation of elasticity problem

Let us consider body Ω before deformation and Ω^φ after the deformation φ

$$\varphi : \Omega \rightarrow \Omega^\varphi \tag{1}$$

and

$$\bar{\Omega}^\varphi = \Omega^\varphi \cup \Gamma^\varphi \qquad (\text{i.e. } \Gamma^\varphi = \partial\Omega^\varphi). \tag{2}$$

We can write the classic formulation of equilibrium equation in component form as

$$\mathrm{div}^\varphi T_i^\varphi + f_i^\varphi = 0, \tag{3}$$

resp.

$$\frac{\partial T_{ij}^\varphi}{\partial x_j^\varphi} + f_i^\varphi = 0. \tag{4}$$

2.1 Weak formulation

Multiplication by test function v_i^φ and integration over the whole domain $\bar{\Omega}^\varphi$ gives

$$\int_{\Omega^\varphi} \frac{\partial T_{ij}^\varphi}{\partial x_j^\varphi} v_i^\varphi dx^\varphi + \int_{\Omega^\varphi} f_i^\varphi v_i^\varphi dx^\varphi = 0. \tag{5}$$

After applying Green's theorem on the first term

$$\int_{\partial\Omega^\varphi} T_{ij}^\varphi n_j^\varphi v_i^\varphi da^\varphi - \int_{\Omega^\varphi} T_{ij}^\varphi \frac{\partial v_i^\varphi}{\partial x_j^\varphi} dx^\varphi + \int_{\Omega^\varphi} f_i^\varphi v^\varphi dx^\varphi = 0. \tag{6}$$

On the parts of boundary $\Gamma^\varphi = \Gamma_v^\varphi \cup \Gamma_\tau^\varphi$ we prescribe following boundary conditions

$$\begin{aligned} T_{ij}^\varphi n_j^\varphi &= g_i^\varphi \qquad \text{on } \Gamma_\tau^\varphi && (7)\\ v_i^\varphi &= 0 \qquad \text{on } \Gamma_v^\varphi. && (8) \end{aligned}$$

Finally we obtain after small rearrangement the weak formulation of equilibrium equations of a body after deformation

$$\int_{\Omega^\varphi} T^\varphi_{ij} \frac{\partial v^\varphi_i}{\partial x^\varphi_j} dx^\varphi = \int_{\Omega^\varphi} f^\varphi_i v^\varphi_i dx^\varphi + \int_{\Gamma^\varphi_\tau} g^\varphi_i v^\varphi_i da^\varphi. \tag{9}$$

Unfortunatelly, in finite elasticity, Ω^φ is unknown and may be very different from the known reference configuration Ω. Therefore, it is more convenient to rewrite the equilibrium equations on Ω, using the formula for changes of variables in multiple integrals. Doing this, we get

$$\int_{\Omega} T^\varphi_{ij} \frac{\partial v^\varphi_i}{\partial x_k} \frac{\partial x_k}{\partial x^\varphi_j} \det\nabla\varphi \, dx = \int_{\Omega} f^\varphi_i v^\varphi_i \det\nabla\varphi \, dx + \int_{\Gamma_\tau} g^\varphi_i v^\varphi_i \frac{da^\varphi}{da} \, da. \tag{10}$$

Rearranging the first term (considering that $x = \varphi^{-1}(x^\varphi)$) in the following way

$$\int_{\Omega} \left[T^\varphi_{ij} \left(\frac{\partial x_k}{\partial x^\varphi_j} \right)^T \det\nabla\varphi \right] \frac{\partial v^\varphi_i}{\partial x_k} \, dx = \int_{\Omega} \left[T^\varphi_{ij} \nabla\varphi^{-T}_{kj} \det\nabla\varphi \right] \frac{\partial v^\varphi_i}{\partial x_k} \, dx, \tag{11}$$

we obtain

$$\int_{\Omega} \left[T^\varphi_{ij} \left(\frac{\partial x_k}{\partial x^\varphi_j} \right)^T \det\nabla\varphi \right] \frac{\partial v^\varphi_i}{\partial x_k} \, dx = \int_{\Omega} f^\varphi_i \det\nabla\varphi v^\varphi_i \, dx + \int_{\Gamma_\tau} g^\varphi_i \frac{da^\varphi}{da} v^\varphi_i \, da. \tag{12}$$

Using Piola transform

$$T(x) = T^\varphi(x^\varphi(x))(\nabla\varphi)^{-T} \det\nabla\varphi \tag{13}$$

we may write

$$T_{ik} = T^\varphi_{ij} \left(\frac{\partial x_k}{\partial x^\varphi_j} \right)^T \det\nabla\varphi \tag{14}$$

and by setting

$$\begin{aligned} v(x) &= v^\varphi(x^\varphi(x)) \\ f(x) &= f^\varphi(x^\varphi(x)) \det\nabla\varphi \\ g(x) &= g^\varphi(x^\varphi(x)) \frac{da^\varphi}{da}, \end{aligned} \tag{15}$$

where $f(x)$ means density of body forces and $g(x)$ density of surface tractions (both in reference configuration), we finally obtain the following weak form of equilibrium equation in the reference configuration

$$\int_\Omega T_{ik} \frac{\partial v_i}{\partial x_k}\, dx = \int_\Omega f_i v_i \, dx + \int_{\Gamma_\tau} g_i v_i \, da \qquad \forall\, v_i \in \mathbf{V}, \tag{16}$$

Considering the following constitutive relation for compressible material

$$T_{ij}(u) = \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u)), \tag{17}$$

we get

$$\int_\Omega \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u)) \frac{\partial v_i}{\partial x_j}\, dx = \int_\Omega f_i v_i \, dx + \int_{\Gamma_\tau} g_i v_i \, da \qquad \forall\, v \ \in \mathbf{V}. \tag{18}$$

3 Constitutive relations

The simplest law uses a quadratic isotropic function of the Green strain tensor $E = \frac{1}{2}(C - I)$. So called St. Venant material is characterized by the stored energy function

$$\hat{W} = \frac{\lambda}{2}(\mathrm{tr}E)^2 + \mu\, \mathrm{tr}(E^2), \tag{19}$$

where λ and μ are the Lamé coefficient introduced in linear elasticity. Unfortunately, such materials can reach infinite compression rates with finite energy and do not satisfy the polyconvexity assumptions used in the existence theory. For these reason we do not use this material model here and suggest to use polyconvex functions given in terms of invariants.

The simplest example of such materials is the neo-Hookean material

$$W = C_{10}(I_1 - 3). \tag{20}$$

If we add a linear term, we get well-known Mooney-Rivlin material

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3). \tag{21}$$

This energy function was further generalized into the third order polynomial in invariants I_1, I_2 which fits well to numerous experimental data

$$W = C_{10}(I_1-3) + C_{01}(I_2-3) + C_{20}(I_1-3)^2 + C_{02}(I_2-3)^2 + C_{11}(I_1-3)(I_2-3) + C_{30}(I_1-3)^3. \tag{22}$$

Theorem (Rivlin-Eriksen representation theorem): *For any isotropic hyperelastic material, the elastic potential W satisfies:*

$$W(x,F) = W(x, I_1(E), I_2(E), I_3(E)). \tag{23}$$

Invariants and their derivatives are given in the following table:

$$\begin{array}{ll} I_1 = \mathrm{tr}\mathbf{E} & \frac{\partial I_1}{\partial E_{ij}} = \delta_{ij} \\ I_2 = \frac{1}{2}\mathrm{tr}\mathbf{E}^2 = \frac{1}{2}E_{ij}E_{ji} & \frac{\partial I_2}{\partial E_{ij}} = E_{ij} \\ I_3 = \frac{1}{3}\mathrm{tr}\mathbf{E}^3 = \frac{1}{3}E_{ij}E_{jk}E_{ki} & \frac{\partial I_3}{\partial E_{ij}} = E_{ik}E_{kj} \end{array} \tag{24}$$

$$\frac{\partial W}{\partial E_{kl}} = \frac{\partial W}{\partial I_1}\delta_{kl} + \frac{\partial W}{\partial I_2}E_{kl} + \frac{\partial W}{\partial I_3}E_{km}E_{ml} \tag{25}$$

$$\begin{aligned} \frac{\partial}{\partial E_{ij}}\left(\frac{\partial W}{\partial E_{kl}}\right) = {} & \frac{\partial^2 W}{\partial I_1^2}\delta_{ij}\delta_{kl} + \frac{\partial^2 W}{\partial I_1 \partial I_2}\left(E_{ij}\delta_{kl} + \delta_{ij}E_{kl}\right) + \\ & + \frac{\partial^2 W}{\partial I_1 \partial I_3}\left(E_{im}E_{mj}\delta_{kl} + \delta_{ij}E_{km}E_{ml}\right) + \frac{\partial W}{\partial I_2}\delta_{ik}\delta_{jl} + \frac{\partial^2 W}{\partial I_2^2}E_{ij}E_{kl} + \\ & + \frac{\partial^2 W}{\partial I_2 \partial I_3}\left(E_{im}E_{mj}E_{kl} + E_{ij}E_{km}E_{ml}\right) + \frac{\partial W}{\partial I_3}\left(\delta_{ik}E_{jl} + \delta_{jl}E_{ik}\right) + \\ & + \frac{\partial^2 W}{\partial I_3^2}\left(E_{in}E_{nj}E_{km}E_{ml}\right) \end{aligned} \tag{26}$$

4 Numerical solution technique

We recall the equilibrium equation in the form

$$\int_\Omega \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u))\frac{\partial w_i}{\partial x_j}\, dx = \int_\Omega f w\, dx + \int_{\Gamma_\tau} g w\, da \qquad \forall\, w \ \in \mathbf{V}. \tag{27}$$

Such equation is generally nonlinear in the displacements u. We would like to find the field of displacements u so that $\mathcal{F}(u) = 0$ using Newton's method in the following way: in the (k+1)-th iteration of Newton's method we are looking for the field of displacements u_{k+1}. The field of displacements from k-th iteration u_k is known and we must find increment $h \in \mathbf{R}^n$ satisfying the equation

$$D\mathcal{F}(u_k) \cdot h = -\mathcal{F}(u_k), \tag{28}$$

then we add the increment h to the previous iteration so that

$$u_{k+1} = u_k + h. \tag{29}$$

Naturally,

$$\mathcal{F}(u) = \int_\Omega \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u)) \frac{\partial w_i}{\partial x_j} \, dx - \int_\Omega f w \, dx - \int_{\Gamma_\tau} g w \, da \tag{30}$$

Now we need to compute the Fréchet derivative of $\mathcal{F}(u)$ as

$$D_v\mathcal{F}(u) = \int_\Omega \frac{\partial^2 \hat{W}}{\partial F_{ij} \partial F_{kl}} \underbrace{Dx}_{=0} \frac{\partial w}{\partial x_j} \, dx + \int_\Omega \frac{\partial^2 \hat{W}}{\partial F_{ij} \partial F_{kl}} D_v F_{kl}(u) \frac{\partial w_i}{\partial x_k} \, dx. \tag{31}$$

Based on definition $F_{kl}(u) = \delta_{kl} + \frac{\partial u_k}{\partial x_l}$, we compute

$$D_v F_{kl}(u) = \lim_{t \to 0} \frac{1}{t} \left(\frac{\delta_{kl} + \partial u_k + t \partial v_k}{\partial x_l} - \frac{\delta_{kl} + \partial u_k}{\partial x_l} \right) = \frac{\partial v_k}{\partial x_l}. \tag{32}$$

After substituting back we get

$$D_v\mathcal{F}(u) = \int_\Omega \left(\frac{\partial^2 \hat{W}(x, F(u))}{\partial F_{ij} \partial F_{kl}} \frac{\partial v_k}{\partial x_l} \right) \frac{\partial w_i}{\partial x_j} \, dx, \tag{33}$$

5 Finite Element Formulation

First we use the equivalence relation $W = \hat{W}(F) = \tilde{W}(C)$ following from definition

$$C_{ij} = F_{ki} F_{kj} = \delta_{ij} + 2E_{ij} \tag{34}$$

to transform $\mathcal{F}(u)$ and $D\mathcal{F}(u)$ in order to use internal energy functions in terms of invariants and strain tensors E_{ij}. For the derivatives then holds

$$\frac{\partial W}{\partial E_{ij}} = \frac{\partial \tilde{W}}{\partial C_{kl}} \frac{\partial C_{kl}}{\partial E_{ij}} = 2 \frac{\partial \tilde{W}}{\partial C_{ij}} \tag{35}$$

Using previous relations we derive

$$T_{ij} = \frac{\partial \hat{W}}{\partial F_{ij}} = \frac{\partial \tilde{W}}{\partial C_{kl}} \frac{\partial C_{kl}}{\partial F_{ij}} = 2 \frac{\partial \tilde{W}}{\partial C_{jk}} F_{ik} = \frac{\partial W}{\partial E_{jk}} F_{ik} \tag{36}$$

and after substitution into $\mathcal{F}(u)$ we obtain

$$\mathcal{F}(u) = \int_\Omega \frac{\partial W}{\partial E_{jk}} F_{ik} \frac{\partial w_i}{\partial x_j} dx - \int_\Omega f w \, dx - \int_{\Gamma_\tau} g w \, da. \tag{37}$$

Exchanging indeces ik and using symmetry of E_{ij} we get

$$\mathcal{F}(u) = \int_\Omega \frac{\partial W}{\partial E_{ij}} F_{ki} \frac{\partial w_i}{\partial x_j} dx - \int_\Omega f w \, dx - \int_{\Gamma_\tau} g w \, da \tag{38}$$

resp.

$$\mathcal{F}(u) = \int_\Omega \frac{\partial W}{\partial E_{ij}} \left[\left((\delta_{ki} + \frac{\partial u_k}{\partial x_i} \right) \right] \frac{\partial w_k}{\partial x_j} \, dx - \int_\Omega f w \, dx - \int_{\Gamma_\tau} g w \, da. \tag{39}$$

Considering $w = (0, N_t, 0)$ and writing b instead of $\mathcal{F}(u)$ we get the component form and righ-hand side vectors into for the Newton's method

$$\begin{aligned}
b_x &= \int_\Omega \frac{\partial W}{\partial E_{ij}} \left[\left((\delta_{xi} + \frac{\partial u_x}{\partial x_i} \right) \right] \frac{\partial N_t}{\partial x_j} \, dx - \int_\Omega f_x N_t \, dx - \int_{\Gamma_\tau} g_x N_t \, da \\
b_y &= \int_\Omega \frac{\partial W}{\partial E_{ij}} \left[\left((\delta_{yi} + \frac{\partial u_y}{\partial x_i} \right) \right] \frac{\partial N_t}{\partial x_j} \, dx - \int_\Omega f_y N_t \, dx - \int_{\Gamma_\tau} g_y N_t \, da \\
b_z &= \int_\Omega \frac{\partial W}{\partial E_{ij}} \left[\left((\delta_{zi} + \frac{\partial u_z}{\partial x_i} \right) \right] \frac{\partial N_t}{\partial x_j} \, dx - \int_\Omega f_z N_t \, dx - \int_{\Gamma_\tau} g_z N_t \, da
\end{aligned} \tag{40}$$

Now, using the relation $C_{ij} = \delta_{ij} + 2E_{ij}$ we can derive

$$\frac{\partial^2 W}{\partial E_{ij} \partial E_{kl}} = 4 \frac{\partial^2 W}{\partial C_{ij} \partial C_{kl}} \tag{41}$$

and together with the definition of the second Piola-Kirchhoff stress tensor

$$S_{ij} = 2 \frac{\partial W}{\partial C_{ij}} \tag{42}$$

we get

$$\frac{\partial}{\partial F_{mn}}\left(\frac{\partial \hat{W}}{\partial F_{ij}}\right) = 4\frac{\partial^2 \tilde{W}}{\partial C_{jk}\partial C_{ns}} F_{ms}F_{ik} + S_{jn}\delta_{im} = \frac{\partial^2 W}{\partial E_{jk}\partial E_{ns}} F_{ms}F_{ik} + S_{jn}\delta_{im} \quad (43)$$

Substituting into

$$D_v\mathcal{F}(u) = \int_\Omega \frac{\partial^2 W}{\partial F_{ij}\partial F_{mn}} \frac{\partial v_i}{\partial x_j}\frac{\partial w_m}{\partial x_n}\, dx \quad (44)$$

leads after some rearrangements to

$$\begin{aligned} D_v\mathcal{F}(u) &= \int_\Omega \left[\frac{\partial^2 W}{\partial E_{jk}\partial E_{ns}} F_{ik}\frac{\partial v_i}{\partial x_j} F_{ms}\frac{\partial w_m}{\partial x_n} + S_{jn}\frac{\partial v_i}{\partial x_j}\delta_{im}\frac{\partial w_m}{\partial x_n}\right] dx = \quad (45)\\ &= \int_\Omega \left\{\frac{\partial^2 W}{\partial E_{jk}\partial E_{ns}}\left[\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\frac{\partial v_i}{\partial x_j}\right]\left[\left(\delta_{ms} + \frac{\partial u_m}{\partial x_s}\right)\frac{\partial v_m}{\partial x_n}\right] + S_{jn}\frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_n}\right\} dx \end{aligned}$$

Exchanging indeces *ki,sk,nl,ms,ir* in the first term, *ji,nj,ik* in the second term and writing a_u instead of $D_v\mathcal{F}(u)$ we have

$$a_u = \int_\Omega \left\{\frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}}\left[\left(\delta_{ri} + \frac{\partial u_r}{\partial x_i}\right)\frac{\partial v_r}{\partial x_j}\right]\left[\left(\delta_{sk} + \frac{\partial u_s}{\partial x_k}\right)\frac{\partial v_s}{\partial x_l}\right] + S_{ij}\frac{\partial v_k}{\partial x_i}\frac{\partial w_k}{\partial x_j}\right\} dx \quad (46)$$

Taking $w = (0, N_t, 0)$ and $v = h = (h_x, h_y, h_z)$ where

$$h_x = \textstyle\sum_{u=1}^{N_h} h_{xu}N_u \quad h_y = \sum_{u=1}^{N_h} h_{yu}N_u \quad h_z = \sum_{u=1}^{N_h} h_{zu}N_u \quad (47)$$

we get the components of the stiffness matrix as

$$
\begin{aligned}
a_{xx} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{xi} + \frac{\partial u_x}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{xk} + \frac{\partial u_x}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] + S_{ij} \frac{\partial N_u}{\partial x_i} \frac{\partial N_t}{\partial x_j} \right\} dx \\
a_{xy} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{yi} + \frac{\partial u_y}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{xk} + \frac{\partial u_x}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] \right\} dx \\
a_{xz} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{zi} + \frac{\partial u_z}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{xk} + \frac{\partial u_x}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] \right\} dx \\
a_{yx} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{xi} + \frac{\partial u_x}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{yk} + \frac{\partial u_y}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] \right\} dx \\
a_{yy} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{yi} + \frac{\partial u_y}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{yk} + \frac{\partial u_y}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] + S_{ij} \frac{\partial N_u}{\partial x_i} \frac{\partial N_t}{\partial x_j} \right\} dx \\
a_{yz} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{zi} + \frac{\partial u_z}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{yk} + \frac{\partial u_y}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] \right\} dx \\
a_{zx} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{xi} + \frac{\partial u_x}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{zk} + \frac{\partial u_z}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] \right\} dx \\
a_{zy} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{yi} + \frac{\partial u_y}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{zk} + \frac{\partial u_z}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] \right\} dx \\
a_{zz} &= \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}} \left[\left(\delta_{zi} + \frac{\partial u_z}{\partial x_i} \right) \frac{\partial N_u}{\partial x_j} \right] \left[\left(\delta_{zk} + \frac{\partial u_z}{\partial x_k} \right) \frac{\partial N_t}{\partial x_l} \right] + S_{ij} \frac{\partial N_u}{\partial x_i} \frac{\partial N_t}{\partial x_j} \right\} dx
\end{aligned}
$$

Finally, the (k+1)-th iteration of Newton's method consists of two steps:

(i) First, we solve the following system of equations

$$A^k \cdot h = b^k, \tag{48}$$

where the components of the matrix A and vectors h, b are defined as

$$A = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \tag{49}$$

$$b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \quad h = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \tag{50}$$

(ii) Next, we update the field of displacements

$$u^{k+1} = u^k + h \tag{51}$$

6 Conclusion

In the paper we showed finite element approximation of the non-linear three-dimensional finite elasticity problem for compressible material model. We also described the strategy of solution of the non-linear system of equations arised.

Currently we are developing and testing finite element code in programming language FORTRAN. The linearized system will be solved by frontal (direct) solver. We will also study possibility of application of iterative solvers, esp. conjugate gradient solver with BDDC preconditioning described in [3], [6], which is also very promising from the parallelisation point of view.

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