Finite Element Solution Algorithm for Nonlinear Elasticity Problems by Domain Decomposition Method

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Abstract

The purpose of the present work is to give a brief description of the finite elasticity and of its approximation via finite element method. We formulate the problem for the case of compressible elasticity. Weak formulation allows to use any isotropic hyperelastic material model that satisfies polyconvexity assumptions. Discretization using FEM leads to systems of non-linear equations. Such systems of equations can be solved by Newton's method and its modifications. In each iteration of Newton's method a linearized system of equations has to be solved. We propose to use BDDC preconditioning iterative method for the solution of linear system of equations.

1 Introduction

The main object of the finite three-dimensional elasticity is to predict changes in the geometry of solid bodies. The starting point of the classical theory of linear elasticity is the concept of small strains: the deformation of structures under working loads are not detectable by human eye. In contrast, many modern situations involve large deformations. The nonlinear behavior of polymers and synthetic rubbers are such examples. Applications in biomechanics are even more critical because the most of vital organs such as eye, heart trachea or vocal apparatus fulfill their function only because of their large deformations. In this framework the concept of finite elasticity covers the simplest case where internal forces (stresses) depend only on the present deformation of the body and not on the history. In the paper we show the finite element approximation and strategy of solution of this nonlinear and 'visible' stress-strain relationship.

2 Formulation of elasticity problem

Let us describe the deformation of a body by $\varphi : \Omega \to \Omega^\varphi$, and the occupied domain we decompose in an interior and boundary as $\bar{\Omega}^\varphi = \Omega^\varphi \cup \Gamma^\varphi$, (i.e. $\Gamma^\varphi = \partial\Omega^\varphi$).

The classic formulation of equilibrium equation we write as

$$\frac{\partial T_{ij}^\varphi}{\partial x_j^\varphi} + f_i^\varphi = 0. \tag{1}$$

2.1 Weak formulation

Multiplication by test function v_i^φ, integration over the whole domain $\bar{\Omega}^\varphi$ and application of Green's theorem gives

$$\int_{\partial\Omega^\varphi} T_{ij}^\varphi n_j^\varphi v_i^\varphi \, da^\varphi - \int_{\Omega^\varphi} T_{ij}^\varphi \frac{\partial v_i^\varphi}{\partial x_j^\varphi} dx^\varphi + \int_{\Omega^\varphi} f_i^\varphi v^\varphi dx^\varphi = 0. \tag{2}$$

On the parts of boundary $\Gamma^\varphi = \Gamma_v^\varphi \cup \Gamma_\tau^\varphi$ we prescribe following boundary conditions

$$\begin{aligned} T_{ij}^\varphi n_j^\varphi &= g_i^\varphi \qquad \text{on } \Gamma_\tau^\varphi & (3)\\ v_i^\varphi &= 0 \qquad \text{on } \Gamma_v^\varphi. & (4) \end{aligned}$$

After some rearrangements we obtain the weak formulation of equilibrium equations of a body after deformation

$$\int_{\Omega^\varphi} T_{ij}^\varphi \frac{\partial v_i^\varphi}{\partial x_j^\varphi} dx^\varphi = \int_{\Omega^\varphi} f_i^\varphi v_i^\varphi dx^\varphi + \int_{\Gamma_\tau^\varphi} g_i^\varphi v_i^\varphi da^\varphi. \tag{5}$$

Unfortunatelly, in finite elasticity, Ω^φ is unknown and may be very different from the known reference configuration Ω. Therefore, it is more convenient to rewrite the equilibrium equations on Ω, using the formula for changes of variables in multiple integrals. Doing this, using Piola transform and considering that $x = \varphi^{-1}(x^\varphi)$ we finally obtain the following weak form of equilibrium equation in the reference configuration

$$\int_\Omega T_{ik} \frac{\partial v_i}{\partial x_k} \, dx = \int_\Omega f_i v_i \, dx + \int_{\Gamma_\tau} g_i v_i \, da \qquad \forall \, v_i \in \mathbf{V}, \tag{6}$$

Considering the following constitutive relation for compressible material

$$T_{ij}(u) = \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u)), \tag{7}$$

we get

$$\int_\Omega \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u)) \frac{\partial v_i}{\partial x_j}\, dx = \int_\Omega f_i v_i\, dx + \int_{\Gamma_\tau} g_i v_i\, da \qquad \forall\, v\ \in \mathbf{V}. \tag{8}$$

3 Constitutive relations

The simplest law uses a quadratic isotropic function of the Green strain tensor $E = \frac{1}{2}(C - I)$. So called St. Venant material unfortunately can reach infinite compression rates with finite energy and do not satisfy the polyconvexity assumptions used in the existence theory. For these reason we do not use this material model here and suggest to use polyconvex functions given in terms of invariants.

The simplest example of such materials is the neo-Hookean material

$$W = C_{10}(I_1 - 3). \tag{9}$$

If we add a linear term, we get well-known Mooney-Rivlin material

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3). \tag{10}$$

This energy function was further generalized into the third order polynomial in invariants I_1, I_2 which fits well to numerous experimental data

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{02}(I_2 - 3)^2 + C_{11}(I_1 - 3)(I_2 - 3) + C_{30}(I_1 - 3)^3. \tag{11}$$

Theorem (Rivlin-Eriksen representation theorem): *For any isotropic hyperelastic material, the elastic potential W satisfies:*

$$W(x, F) = W(x, I_1(E), I_2(E), I_3(E)). \tag{12}$$

Invariants and their derivatives are given in the following table:

$$\begin{array}{ll} I_1 = \mathrm{tr}\mathbf{E} & \frac{\partial I_1}{\partial E_{ij}} = \delta_{ij} \\ I_2 = \frac{1}{2}\mathrm{tr}\mathbf{E}^2 = \frac{1}{2}E_{ij}E_{ji} & \frac{\partial I_2}{\partial E_{ij}} = E_{ij} \\ I_3 = \frac{1}{3}\mathrm{tr}\mathbf{E}^3 = \frac{1}{3}E_{ij}E_{jk}E_{ki} & \frac{\partial I_3}{\partial E_{ij}} = E_{ik}E_{kj} \end{array} \tag{13}$$

4 Numerical solution technique

Equilibrium equations are generally nonlinear in the displacements u. We would like to find the field of displacements u so that $\mathcal{F}(u) = 0$ using Newton's method in the following way: in the (k+1)-th iteration of Newton's method we are looking for the field of displacements u_{k+1}. The field of displacements from k-th iteration u_k is known and we must find increment $h \in \mathbf{R}^n$ satisfying the equation

$$D\mathcal{F}(u_k) \cdot h = -\mathcal{F}(u_k), \tag{14}$$

then we add the increment h to the previous iteration so that

$$u_{k+1} = u_k + h. \tag{15}$$

Naturally, in our case,

$$\mathcal{F}(u) = \int_\Omega \frac{\partial \hat{W}}{\partial F_{ij}}(x, F(u)) \frac{\partial w_i}{\partial x_j}\, dx - \int_\Omega f w\, dx - \int_{\Gamma_\tau} g w\, da \tag{16}$$

and the Fréchet derivative of $\mathcal{F}(u)$ is

$$D_v\mathcal{F}(u) = \int_\Omega \left(\frac{\partial^2 \hat{W}(x, F(u))}{\partial F_{ij} \partial F_{kl}} \frac{\partial v_k}{\partial x_l} \right) \frac{\partial w_i}{\partial x_j}\, dx, \tag{17}$$

5 Finite Element Formulation

First we use the equivalence relation $W = \hat{W}(F) = \tilde{W}(C)$ following from definition $C_{ij} = F_{ki}F_{kj} = \delta_{ij} + 2E_{ij}$ to transform $\mathcal{F}(u)$ and $D\mathcal{F}(u)$ in order to use internal energy functions in terms of invariants and strain tensors E_{ij}.

$$\mathcal{F}(u) = \int_\Omega \frac{\partial W}{\partial E_{ij}} \left[\left((\delta_{ki} + \frac{\partial u_k}{\partial x_i} \right) \right] \frac{\partial w_k}{\partial x_j}\, dx - \int_\Omega f w\, dx - \int_{\Gamma_\tau} g w\, da. \tag{18}$$

and using the second Piola-Kirchhoff stress tensor $S_{ij} = 2\frac{\partial W}{\partial C_{ij}}$ we get

$$D_v\mathcal{F}(u) = \int_\Omega \left\{ \frac{\partial^2 W}{\partial E_{ij} \partial E_{kl}} \left[\left(\delta_{ri} + \frac{\partial u_r}{\partial x_i} \right) \frac{\partial v_r}{\partial x_j} \right] \left[\left(\delta_{sk} + \frac{\partial u_s}{\partial x_k} \right) \frac{\partial v_s}{\partial x_l} \right] + S_{ij} \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial x_j} \right\} dx \tag{19}$$

Taking $w = (0, N_t, 0)$ and $v = h = (h_x, h_y, h_z)$ where

$$h_x = \sum_{u=1}^{N_h} h_{xu} N_u \quad h_y = \sum_{u=1}^{N_h} h_{yu} N_u \quad h_z = \sum_{u=1}^{N_h} h_{zu} N_u \tag{20}$$

we get the components of the stiffness matrix and right hand side vectors into Newton's method. Finally, the (k+1)-th iteration of Newton's method consists of two steps:

(i) First, we solve the following system of equations

$$A^k \cdot h = b^k, \tag{21}$$

where the components of the matrix A and vectors h, b are defined as

$$A = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \tag{22}$$

$$b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \quad h = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \tag{23}$$

(ii) Next, we update the field of displacements

$$u^{k+1} = u^k + h \tag{24}$$

6 Formulation of BDDC method

Clearly, in each step of Newton's iterations we solve a linear system of equation. Here, we describe the BDDC preconditioning method. This method is closely related to FETI-DP method that is also briefly described. Let us decompose the domain Ω into the nonoverlapping set of substructures (subdomains) Ω_i, where $i = 1, ...N$ and N means the number of subistructures. Let K_i be the stiffness matrix and v_i the vector of degrees of freedom (dofs) for substructure i. We want to solve the problem in decomposed form

$$\frac{1}{2} v^T K v - v^T f \to \min \tag{25}$$

where

$$K = \begin{bmatrix} K_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & K_N \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

subject to continuity dofs between substructures. Partitioning the dofs in each subdomain i into internal and interface (boundary)

$$K_i = \begin{bmatrix} K_i^{ii} & K_i^{ib} \\ K_i^{ib\,T} & K_i^{bb} \end{bmatrix}, \quad v_i = \begin{bmatrix} v_i^i \\ v_i^b \end{bmatrix}, \quad f_i = \begin{bmatrix} f_i^i \\ f_i^b \end{bmatrix},$$

and eliminating the interior dofs we obtain the problem reduced to interfaces

$$\frac{1}{2} w^T S w - w^T g \to \min, \quad S = \operatorname{diag}(S_i), \quad S_i = K_i^{bb} - K_i^{ib^T} K_i^{ii-1} K_i^{ib},$$

again subject to continuity of dofs between substructures

In BDD type methods, the continuity of dofs between substructures is enforced by imposing common values on substructures interfaces: $w = Ru$ for some u, where

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix}$$

and R_i is the operator of restriction of global dofs on the interfaces to substructure i. In FETI type methods, continuity of dofs between substructures is enforced by the constraint $Bw = 0$, where the entries of B are typically $0, \pm 1$. By construction, we have

$$R_i R_i^T = I, \quad \operatorname{range} R = \operatorname{null} B.$$

A BDDC or FETI-DP method is specified by the choice of coarse dofs and the choice of weights for intersubdomain averaging. To define the coarse problem for BDDC, choose a matrix Q_P^T that selects coarse dofs u_c from global interface dofs u, e.g. as values at corners or averages on sides:

$$u_c = Q_P^T u.$$

We define $\widetilde{W}$ as the space of all vectors of substructure interface dofs that are continuous between substructures,

$$\widetilde{W} = \{w \in W : \exists u_c : Cw = R_c u_c\} = \{w \in W : Cw \in \operatorname{range} R_c\} \qquad (26)$$

where

$$C_i = R_{ci} Q_P^T R_i^T, \quad C = \begin{bmatrix} C_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_N \end{bmatrix}, \quad R_c = \begin{bmatrix} R_{c1} \\ \vdots \\ R_{cN} \end{bmatrix}, \qquad (27)$$

and the matrix R_{ci} restricts the vector all coarse dof values into a vector of coarse dof values that can be nonzero on substructure i . The dual approach in FETI-DP is to construct Q_D such that

$$\widetilde{W} = \left\{ w \in W : Q_D^T B w = 0 \right\} \qquad (28)$$

In BDDC, the intersubdomain averaging is defined by the matrices

$$D_P = \mathrm{diag}\left(D_{Pi}\right)$$

that form a decomposition of unity,

$$R^T D_P R = I.$$

The corresponding dual matrices in FETI-DP are are

$$B_D = \left[D_{D1} B_1, \dots D_{DN} B_N \right],$$

where the dual weights D_{Di} are defined so that

$$B_D^T B + R R^T D_P = I. \qquad (29)$$

The BDDC method is then the method of conjugate gradients for the assembled system $Au = R^T g$ with the system matrix

$$A = R^T S R$$

and the preconditioner P defined by

$$Pr = R^T D_P \left(\Psi u_c + z \right),$$

where u_c is the solution of the coarse problem

$$\Psi^T S \Psi u_c = \Psi^T D_P^T R r$$

and z is the solution of

$$\begin{array}{ccccc} Sz & + & C^T \mu & = & D_P^T R r \\ Cz & & & = & 0 \end{array},$$

which is a collection of independent substructure problems. The coarse basis functions Ψ are defined by energy minimization,

$$\begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ R_c \end{bmatrix}.$$

7 Conclusion

In the paper we showed finite element approximation of the non-linear three-dimensional finite elasticity problem for compressible material model. We also described the strategy of solution of the non-linear system of equations arised.

The linearized system of equations will be solved by an iterative solver of conjugate gradient type with BDDC preconditioning proposed and studied in [1], [3]. Currently we are developing and testing this finite element code in programming language FORTRAN. We will also continue in developing of adaptive strategies for the selection of constraints studied in the paper [4]. The whole code will be finally parallelized using MPI library.

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