On a parallel implementation of the BDDC method and its application to the Stokes problem

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keywords: BDDC, domain decomposition, iterative substructuring, Stokes flow

1. Introduction

Numerical solution of linear problems arising from discretization by finite elements is important in many areas of engineering. The matrix of the system is typically large, sparse, and often ill-conditioned. For large problems, iterative methods such as the preconditioned conjugate gradients (PCG) are usually less expensive in terms of memory and computational time. However, their convergence rate deteriorates with growing condition number of the solved linear system and good preconditioning becomes essential. The need of first-rate preconditioners tailored to the solved problem, which can be implemented in parallel, gave rise to the field of domain decomposition methods [12].

The Balancing Domain Decomposition based on Constraints (BDDC) [4, 9] is one of the most advanced preconditioners of this class derived for symmetric positive definite problems.

We have implemented a parallel version of the BDDC method and verified its performance on problems arising from linear elasticity problems. Then the method was successfully applied to the Stokes problem, which is beyond the standard theory of the BDDC method.

2. BDDC domain decomposition method

The Balancing Domain Decomposition by Constraints (BDDC) method can be understood as a preconditioner for large systems arising from finite element analysis. It was introduced by Dohrmann [4] in 2003 and the theory was developed by Mandel and Dohrmann in [8]. The preconditioner was reformulated by Li and Widlund in [7].

Let Ω be a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, let U be a finite element space of piecewise polynomial functions v continuous on Ω and U' its dual space. Let $a(\cdot,\cdot)$ be a bilinear form on $U \times U$ and $f \in U'$, and let $\langle \cdot,\cdot \rangle$ denote the duality pairing of U' and U. Consider an abstract variational problem: *Find* $u \in U$ *such that*

$$a(u,v) = \langle f, v \rangle \quad \forall\, v \in U\,. \tag{1}$$

For the case of linear elasticity,

$$a(u,v) = \int_\Omega (\lambda(\nabla\cdot\mathbf{u}_h)(\nabla\cdot\mathbf{v}_h) + \frac{1}{2}\mu(\nabla\mathbf{u}_h + \nabla^T\mathbf{u}_h) : (\nabla\mathbf{v}_h + \nabla^T\mathbf{v}_h))\mathrm{d}\Omega, \tag{2}$$

$$\langle f, v \rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h \mathrm{d}\Omega. \tag{3}$$

Here solution $u = \mathbf{u}_h$ represents the discretized vector field of displacement, λ and μ represent the first and the second Lammé's constant, respectively, and **f** represents the external load.

For the case of steady Stokes flow we adopt the following slightly unusual notation

$$a(u,v) = \nu \int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \mathrm{d}\Omega - \int_\Omega p_h \nabla \cdot \mathbf{v}_h \mathrm{d}\Omega + \int_\Omega \psi_h \nabla \cdot \mathbf{u}_h \mathrm{d}\Omega, \tag{4}$$

$$\langle f, v\rangle = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \mathrm{d}\Omega. \tag{5}$$

Solution $u = (\mathbf{u}_h, p_h)$ consists of the discretized vector field of velocity and the discretized scalar field of pressure, ν represents the kinematic viscosity of the fluid, **f** represents the external load, and $v = (\mathbf{v}_h, \psi_h)$.

For the case of linear elasticity, $a(u,v)$ is a symmetric positive definite bilinear form on $U \times U$, while for the Stokes problem, it is symmetric indefinite [2, 5].

Write the matrix problem corresponding to (1) as

$$Au = f. \tag{6}$$

The domain Ω is decomposed into N nonoverlapping subdomains Ω_i, $i = 1, ..., N$, with characteristic size H, which form a conforming triangulation of the domain Ω. Each subdomain is a union of several finite elements of the underlying mesh with characteristic mesh size h, i.e. nodes of the finite elements between subdomains coincide.

Unknowns common to at least two subdomains are called *boundary unknowns* and the union of all boundary unknowns is called the *interface* Γ.

Let W_i be the space of finite element functions on subdomain Ω_i and put

$$W = W_1 \times \cdots \times W_N. \tag{7}$$

It is the space where subdomains are completely disconnected, and functions on them independent of each other. Clearly, $U \subset W$.

The main idea of the BDDC preconditioner in the abstract form [10] is to construct an auxiliary finite dimensional space $\widetilde{W}$ such that

$$U \subset \widetilde{W} \subset W, \tag{8}$$

and extend the bilinear form $a(\cdot,\cdot)$ to a form $\widetilde{a}(\cdot,\cdot)$ defined on $\widetilde{W} \times \widetilde{W}$, such that solving the variational problem (1) with $\widetilde{a}(\cdot,\cdot)$ in place of $a(\cdot,\cdot)$ is cheaper and can be split into independent computations performed in parallel. Then the solution restricted to U is used for the preconditioning of (6). Space $\widetilde{W}$ contains functions continuous at selected coarse degrees of freedom such as values at selected nodes called *corners*. This space corresponds to a fictious mesh with connections limited to corners, as illustrated in Figure 1.

In computation, the corresponding matrix denoted $\widetilde{A}$ is used. It is larger than the original matrix of the problem A, but it possesses a simpler structure suitable for direct solution methods. This is the reason why it can be used as a preconditioner.

The projection $E : \widetilde{W} \to U$ is realized as a weighted average of values from different subdomains at unknowns on the interface Γ, thus resulting in functions continuous across the interface.

Let $r \in U'$ be the residual in an iteration of an iterative method. The BDDC preconditioner $M_{BDDC} : U' \to U$ in the abstract form (see [10]) produces the preconditioned residual $v \in U$ as

$$M_{BDDC} : r \to v = Ew,$$

where $w \in \widetilde{W}$ is obtained as the solution to problem

$$w \in \widetilde{W} : \widetilde{a}(w,z) = (r, Ez) \quad \forall z \in \widetilde{W}, \tag{9}$$

or in terms of matrices as

$$v = E\widetilde{A}^{-1}E^T r. \tag{10}$$

3. Numerical results

Our parallel implementation of the BDDC preconditioner has been extensively tested on problems with symmetric positive definite matrices arising from linear elasticity (e.g. [11]). The current version is based on the multifrontal massively parallel sparse direct solver MUMPS [1], which is used for factorization of matrix $\widetilde{A}$ in (10). The parallelism is obtained through the parallel direct solver used for factorization of the matrix of preconditioner, in combination with parallel PCG method.

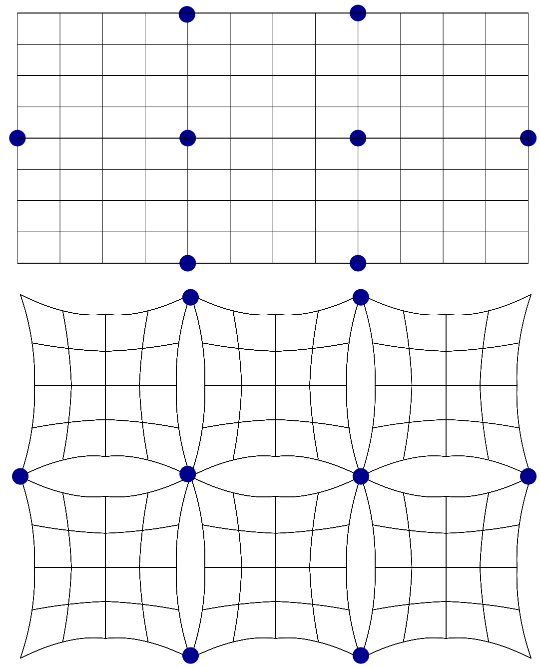

FIGURE 1. Example of an actual mesh (top) and the corresponding fictious mesh for construction of BDDC preconditioner (bottom), blue dots mark corners

The applicability of the preconditioner to the steady problem of Stokes flow was tested, and results are presented in this contribution. The system matrix of the Stokes problem is symmetric, but indefinite. For this reason, the standard theory of BDDC does not cover this case. A way to assure positive definiteness of the preconditioned operator based on BDDC was presented by Li and Widlund [6]. Their method relies on coarse degrees of freedom consisting of special averages on edges of subdomains. However, the approach is limited to piecewise constant pressure approximation. For P2/P1 and Q2/Q1 Taylor-Hood finite elements (e.g. [2]) used in our computations, we were not able to obtain contributive results by that method. Instead, presented problems were successfully solved with basic constraints as continuity at corners in the BDDC preconditioner setup. For the Stokes problem, matrix $\widetilde{A}$ is symmetric indefinite and as such is factorized by the MUMPS solver. Thus, the method leads to an indefinite preconditioner.

The method was first tested on the problem of lid driven cavity, a popular benchmark problem for methods for viscous flow. The domain is a unit square with homogeneous boundary conditions except horizontal velocity prescribed on the upper side. Thus, the entire motion in the cavity is driven by viscosity of the fluid. The case of uniform mesh of 128 ×128 Q2/Q1 elements was chosen. It was divided into 8 subdomains by METIS package (Figure 2).

Resulting streamlines and plot of pressure for Reynolds number $10,000$ are presented in Figure 3. Streamlines are symmetric along the vertical centreline for the Stokes problem.

Solution of the problem by our earlier solver based on a serial frontal algorithm took 231 seconds on one 1.5 GHz Intel Itanium 2 processor of SGI Altix 4700 computer in CTU Supercomputing Centre, Prague, compared to 17.2 seconds on 8 processors of the same computer necessary for the solution by the new implementation of BDDC. The stopping criterion of PCG was chosen as $\|r\|_2/\|g\|_2 < 10^{-3}$, resulting in 59 PCG iterations.

In the second example, a geometry with a sudden reduction of diameter is considered. Flow in this geometry described by the Navier-Stokes model was studied in [3], with respect to precise solution of corner singularities. Due to the symmetry of the channel, only the upper part is considered in the computation. Division into 4 subdomains obtained by METIS is presented in Figure 4.

Solution obtained by BDDC method at Reynolds number 250 for the Stokes flow is presented in Figure 5. The stopping criterion of PCG was chosen as $\|r\|_2/\|g\|_2 < 10^{-3}$, resulting again in 59 PCG iterations. Note, that fluid flows from right to left in the plot of pressure in order to show the situation at corners of domain.

To investigate the performance of the BDDC preconditioner in combination with standard iterative methods for general matrices, namely BICGSTAB and GMRES, we have also performed several preliminary experiments with our serial code written in MATLAB. In Table 1, we compare the resulting number of iterations of these methods

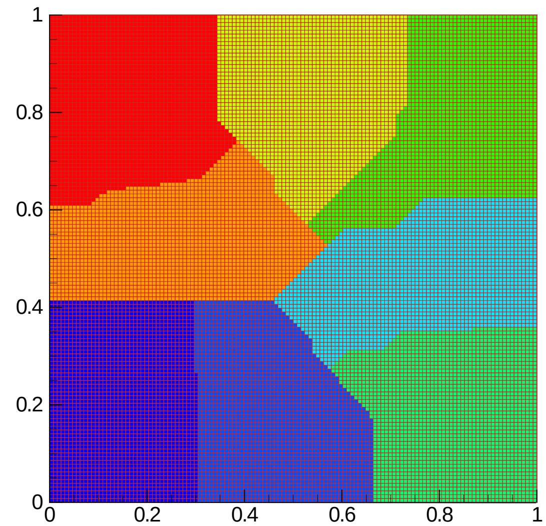

FIGURE 2. Mesh and its division into 8 subdomains for lid driven cavity, 128×128 elements

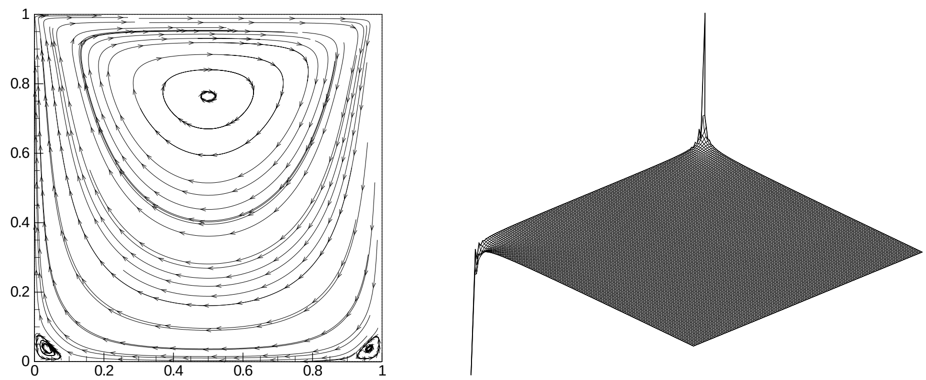

FIGURE 3. Streamlines (left) and pressure (right) for lid driven cavity, 128×128 elements

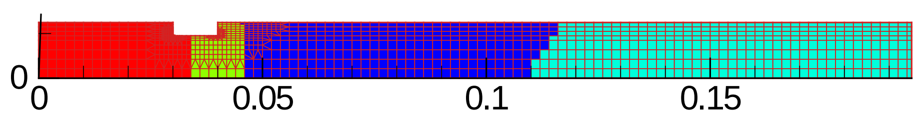

FIGURE 4. Mesh and its division into 4 subdomains for channel with sudden reduction of diameter, only the upper part of the channel is considered for symmetry.

preconditioned by BDDC and by the ILU preconditioner for several values of treshold τ for dropping entries in incomplete factorization for the cavity problem. The desired tolerance of relative residual for these methods was chosen as $\|r\|_2/\|g\|_2 < 10^{-8}$. Where 'n/a' is present in the table, BICGSTAB failed to converge.

iterative method	without preconditioner	BDDC corners only	BDDC corners+faces	ILU $\tau = 10^{-3}$	ILU $\tau = 10^{-4}$	ILU $\tau = 10^{-5}$
BICGSTAB	n/a	45	22	n/a	331	10
GMRES	759	49	38	472	87	18

TABLE 1. Number of iterations for BICGSTAB and GMRES without preconditioning, and preconditioned by BDDC and ILU, lid driven cavity.

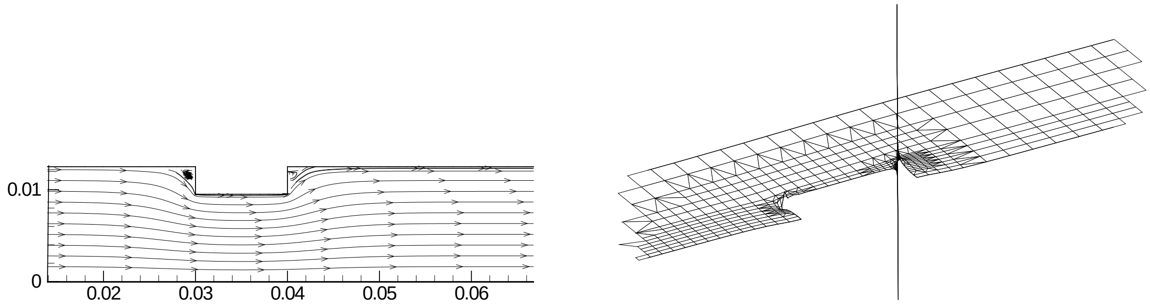

FIGURE 5. Detail of streamlines (left) and pressure (right) for channel with sudden reduction of diameter, Re = 250

4. CONCLUSION

In our contribution, we present a parallel implementation of the BDDC preconditioner. After a verification of the solver on a number of problems from linear elasticity analysis, we explore the application of BDDC to problems with indefinite matrices, namely the Stokes problem. Although the available theory either does not cover this case, or treats it differently [6], the presented experiments suggest promising ways for this effort. Without claiming that this is the general case, we have performed several experiments, for which PCG was successfully used even if the system was indefinite. The reason why a breakdown was not observed lies probably in the indefiniteness of the BDDC preconditioner for this case and deserves further investigation. Our serial experiments also led to promising results for combination of BDDC method with standard iterative methods for solving systems with general matrices, such as BICGSTAB and GMRES.

Acknowledgement: This research has been supported by National Science Foundation under grant DMS-0713876, by Czech Science Foundation under grant GA CR 106/08/0403, and by the Grant Agency of the Academy of Sciences of the Czech Republic under grant IAA200600801. A part of this work was done during Jakub Šístek's visit at the University of Colorado Denver.

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