BRIEF INTRODUCTION TO LINEAR SYSTEM THEORY

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ABSTRACT. The purpose of this paper is to provide the reader with basics of linear system theory. We consider a system given by ordinary differential equation of n-th order. We show its transformation into the state equation (system of equations). Next, we study existence of solution of state equation and its stability. We also explain notions of controllability, observability and realizability. Finally we study linear feedback and its effect on stability. In the paper mostly systems with constant coefficients are considered, but we try to give a comparison with time-varying case when necessary or interesting. The explanation is brief, but the purpose is to offer short introduction to the reader that has some background in mathematics, but basically no knowledge of system theory.

1. INTRODUCTION

In engineering and mathematics, control theory deals with the behavior of dynamical systems over time. The desired output of a system is called the reference variable. When one or more output variables of a system need to show a certain behavior over time, a controller tries to manipulate the inputs of the system to realize this behavior at the output of the system. There are two basic ways of connection of a controller to a system: (1) **open-loop**, when there is no direct connection between the output of the system and its input. The main disadvantage of this type of control is the lack of sensitivity to the dynamics of the system under control. So every time it it possible, a (2) **closed-loop** control is preferred. In this implementation so called feedback is used, which means that the output (reference) variables of a system are plugged back into the input of a controller that changes accordingly values of primary variables in order to keep system in the desired state. Another issue might be to perform in some sense the **optimal control** of a system. Such control minimizes a certain cost functional and satisfying some constraints at the same time. Such control can be deduced using specific optimization algorithms.

Example 1: As the first example, let's consider cruise control. In this case, the system is a car. The goal of cruise control is to keep the car at a constant speed, which is the output variable. The primary variable is the amount of fuel being fed into the engine. The simplest way to implement cruise control is simply to lock the throttle in a certain position. This is fine if the car is driving on perfectly flat terrain, but not in hilly terrain. This would be an example of the open-loop controller because there is no direct connection between the output of the system and its input. But, the actual way that cruise control is implemented involves feedback control, whereby the speed is monitored and the amount of gas is increased if the car is driving slower than the intended speed and decreased if the car is driving faster. This feedback makes the car less sensitive to disturbances to the system, such as changes in slope of the ground or wind speed.

Example 2: Again, consider a car traveling through a hilly road. Now the question is, how should the driver press the accelerator pedal in order to minimize the total traveling time? Clearly in this example, the term control law refers specifically to the way in which the driver presses the accelerator and shifts the gears. The system is intended to be both the car and the hilly road, and the optimality criterion is the minimization of the total traveling time. The problem formulation usually also contains constraints. For example the amount of available fuel might be limited, the accelerator pedal cannot be pushed through the floor of the car, etc. This is an example of optimal control.

In the next sections we will formulate our intuitive understanding of such problems in the somehow more rigorous manner.

2. Basic definitions

2.1. **State equation representation.** Consider a system described by a linear differential equation of n-th order in the independent variable y as

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ... + a_0(t)y(t) = b_0(t)u(t) \tag{2.1.1}$$

defined for $t_0 \leq t$, with $b_0(t)u(t)$ as forcing variables and initial conditions

$$y(t_0), \quad y'(t_0), \quad ..., y^{(n-1)}(t_0). \tag{2.1.2}$$

A simple transformation can be used to restate the diff. equation into the form of a system by setting

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad ..., \quad x_n(t) = y^{(n-1)}(t) \tag{2.1.3}$$

and

$$x_1'(t) = x_2(t), \quad ..., \quad x_{n-1}'(t) = x_n(t), \tag{2.1.4}$$

where $x(t)$ is the vector of state variables.

Definition 2.1.1. The state of a system is a collection of variables that at any time completely describes the system.

Writing this in vector-matrix form

$$x'(t) = \begin{bmatrix} 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & 1 \\ -a_0(t) & -a_1(t) & ... & -a_{n-1}(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0(t) \end{bmatrix} u(t) \tag{2.1.5}$$

and the output equation can be written as

$$y(t) = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix} x(t). \tag{2.1.6}$$

We can write the linear system in the form of *linear state equation* in the standard (general) form as

$$\begin{aligned} x'(t) &= \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= \mathbf{C}(t)x(t) + \mathbf{D}(t)u(t), \end{aligned} \tag{2.1.7}$$

where matrix $D(t)$ can be involved in the certain realizations of the control loop. For the sake of simplicity in the next, we will consider the matrices $\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t)$

to be independent of time and matrix $\mathbf{D}(t) = 0$, i.e. our equations will be *time-invariant*

$$\begin{aligned} x'(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \quad x(t_0) = x_0 \\ y(t) &= \mathbf{C}x(t). \end{aligned} \tag{2.1.8}$$

2.2. **Continuous vs. discrete time modeling.** Let us mention here that there are two basic approaches to modeling with respect to time, namely continuous time and discrete time modelling. The first one is somehow classical from the point of view of analysis, whereas the second one is more common mainly because of use of digital computers in the practical realization of control loops today, i.e. the state of a system is described in certain discrete steps k in time

$$\begin{aligned} x(k+1) &= \mathbf{A}x(k) + \mathbf{B}u(k), \quad x(0) = x_0 \\ y(k) &= \mathbf{C}x(k). \end{aligned} \tag{2.2.1}$$

In the following sections we will consider continuous time models.

3. State equation solution and Stability

3.1. **State equation solution.** The basic question of the existence and uniqueness of solution of our system are first addressed to linear state equation of the system

$$x'(t) = \mathbf{A}\, x(t), \qquad x(t_0) = x_0, \tag{3.1.1}$$

where the initial time t_0 and initial state x_0 are given. If A is an $n \times n$ constant matrix, then the transition matrix is

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k (t - t_0)^k. \tag{3.1.2}$$

If $t_0 = 0$ then we work with

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}. \tag{3.1.3}$$

3.2. **Internal stability.** Internal stability deals with boundedness properties and asymptotic behavior (as $t \to \infty$) of solutions of the zero-input linear state equation 3.1.1. It would be natural to begin by characterizing stability properties of the linear state equation in terms of bounds on the transition matrix $\mathbf{\Phi}(t, t_0)$ for $\mathbf{A}$. However, this does not provide a generally useful stability test for specific examples because of the difficulty of computing $\mathbf{\Phi}(t, t_0)$ for time-variant $\mathbf{A}(t)$. Therefore we consider useful to state here some more general issues on stability.

Definition 3.2.1. The linear state equation 3.1.1 is called *uniformly stable* if there exists a finite positive constant $\gamma \geq 1$ such that for any t_0 and x_0 the corresponding solution satisfies

$$\|x(t)\| \leq \gamma \|x_0\|, \quad t \geq t_0. \tag{3.2.1}$$

Definition 3.2.2. The linear state equation 3.1.1 is called *uniformly exponentially stable* if there exist a finite positive constants $\gamma \geq 1$, λ such that for any t_0 and x_0 the corresponding solution satisfies

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)} \|x_0\|, \quad t \geq t_0. \tag{3.2.2}$$

and the adjective *uniform* refers to the fact that γ and λ are independent of t_0.

This definition allows us to immediately formulate following theorem.

Theorem 3.2.3. *The linear state equation 3.1.1 is uniformly exponentially stable if and only if there exist a finite positive constants γ and λ such that*

$$\|\mathbf{\Phi}(t, t_0)\| \leq \gamma e^{-\lambda(t-t_0)} \tag{3.2.3}$$

for all t, t_0 such that $t \geq t_0$.

Next, we state theorem that has essential importance for stability in case when the matrix $\mathbf{A}$ is time-invariant. Proof can be found in [3].

Theorem 3.2.4. *The linear state equation 3.1.1 with constant $\mathbf{A}(t) = \mathbf{A}$ is exponentially stable if and only if all eigenvalues of $\mathbf{A}$ have negative real parts.*

Corollary 3.2.5. *Necessary and sufficient condition for uniform exponential stability in the time-invariant case is that $\lim_{t\to\infty} e^{\mathbf{A}t} = 0$.*

However the corresponding corollary is not sufficient for time-varying linear state equations. So, what extra condition to $\lim_{t\to\infty} \mathbf{\Phi}(t, t_0) = 0$ do we need in time-varying case in addition to uniform exponential stability? The answer follows from the next definition and theorem.

Definition 3.2.6. The linear state equation 3.1.1 with $\mathbf{A}(t)$ is called *uniformly asymptotically stable* if it is uniformly stable, and if given any positive constant δ there exists a positive T such that for any t_0 and x_0 the corresponding solution satisfies

$$\|x(t)\| \leq \delta\|x_0\|, \quad t \geq t_0 + T. \tag{3.2.4}$$

Note that the elapsed time T until the solution satisfies the bound (3.2.4) must be independent of the initial time.

Theorem 3.2.7. *The linear state equation 3.1.1 with $\mathbf{A}(t)$ is uniformly asymptotically stable if and only if it is uniformly exponentially stable,*

3.3. **Lyapunov stability (an introduction).** The origin of Lyapunov's so-called *direct method* for stability assesment is the notion that total energy of an unforced, dissipative mechanical system decreases as the state of the system evolves in time. To illustrate thi basic idea consider again our linear state equation 3.1.1. For any solution of 3.1.1, the derivative of the scalar function $\|x(t)\|^2 = x^T(t)x(t)$ can be written as

$$\frac{d}{dt}\|x(t)\|^2 = x'^T(x)x(t) + x^T(x)x'(t) = x^T(t)\left[\mathbf{A}^T(t) + \mathbf{A}(t)\right]x(t). \tag{3.3.1}$$

Suppose that the matrix $\mathbf{A}^T(t)+\mathbf{A}(t)$ is negative definite at each t. Then $\|x(t)\|^2$ decreases at t increases. Further it can be shown, that if this negative definiteness does not asymptotically vanish, that is, if there is a constant $\mu > 0$ such that $\mathbf{A}^T(t) + \mathbf{A}(t) \leq -\mu\mathbf{I}$ for all t, then $\|x(t)\|^2$ goes to zero as $t \to \infty$. This forms a basis for deeper study of stability in the sense of Lyapunov.

4. Observability and Controllability

4.1. **State Equation Review.** In this section we will work again with formulas

$$\begin{aligned} x'(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \quad x(t_0) = x_0 \\ y(t) &= \mathbf{C}x(t), \end{aligned} \tag{4.1.1}$$

for the the $n-$dim., time-invariant, linear system (with m inputs and p outputs).

4.2. **Controllability.** The concept of controllability involves the influence of the input signals on the state vector, and does not involve the output signal. In other words, controllability means the ability to move a system around in its entire configuration space using only certain admissible manipulations. We say that the system is controllable, if it is possible by admissible inputs to steer the states from any initial value to any final value within some time window. In particular, no information on the past of a system will help in predicting the future, if the states at the present time are known. However, controllability does not mean that once we reach a desired state that we will be able to keep the system there.

Theorem 4.2.1. *The time-invariant linear state equation 4.1.1 is controllable on the time interval* $[t_0, t_f]$ *if and only if the* $n \times nm$ *controllability matrix satisfies*

$$\text{rank} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} = n. \tag{4.2.1}$$

4.3. **Observability.** The second concept of interest in this section deals with the influence of the state vector on the output of the linear state equation. It is simplest to consider without loss of generality zero input signal. Therefore we consider the unforced state equation of the form

$$\begin{aligned} x'(t) &= \mathbf{A}(t)x(t), \quad x(t_0) = x_0 \\ y(t) &= \mathbf{C}(t)x(t). \end{aligned} \tag{4.3.1}$$

We formalize the concept of observability by following definition and theorem.

Definition 4.3.1. The linear state equation (4.3.1) is called *observable* if any initial state x_0 is uniquely determined by the corresponding response $y(t)$, for $t \in [t_0, t_f]$.

Theorem 4.3.2. *If* $\mathbf{A}(t) = \mathbf{A}$ *and* $\mathbf{C}(t) = \mathbf{C}$ *in 4.3.1 then 4.1.1 is observable on the time interval* $[t_0, t_f]$ *if and only if the* $np \times n$ *observability matrix satisfies*

$$\text{rank} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = n. \tag{4.3.2}$$

As for controllability, the concept of observability for time-invariant linear state equations is independent of the particular (nonzero) time interval. Let us close this section by statement (without proof) that controllability and observability are dual aspect of the same problem.

5. Realizability and Input-Output Stability

5.1. **Realizability.** Once we are familiar with basic concept of stability of a system, one can ask the question, what is the relation between input and output, i.e. what is the input-output behavior of the time-varying linear system 2.1.7 ? In other words, with zero initial state assumed, the output signal $y(t)$ corresponding to a given input signal $u(t)$, defined for $t \geq t_0$, is described by

$$y(t) = \int_{t_0}^{t} \mathbf{G}(t,\sigma)u(\sigma)d\sigma + \mathbf{D}(t)u(t), \quad t \geq t_0 \tag{5.1.1}$$

where

$$\mathbf{G}(t,\sigma) = \mathbf{C}(t)\mathbf{\Phi}(t,\sigma)\mathbf{B}(\sigma) \tag{5.1.2}$$

and the function $\mathbf{G}(t,\sigma)$ is called *weighting pattern*. Of course, computing $\mathbf{G}(t,\sigma)$ based on knowledge of 2.1.7 is generally possible. But here the question is rather inverse. Given weighting pattern $\mathbf{G}(t,\sigma)$, find a system that realizes output $y(t)$. This task is of great practical importance, because in practical life, we often have to find an equation describing a 'black-box' system. One of ways is, to reconstruct equations governing a system from an experimentally measured weighting pattern. This is called as an *identifications of systems* and can be studied elsewhere.

For the time invariant case, realizability conditions on a weighting pattern $\mathbf{G}(t)$ give us for input-output behavior

$$y(t) = \int_{t_0}^{t} \mathbf{G}(t-\tau)u(\tau)d\tau. \tag{5.1.3}$$

This is well-known as a *convolution* and using *Laplace transform*, the input-output relation can be written as

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s) \tag{5.1.4}$$

and

$$\mathbf{G}(s) = \int_{0}^{\infty} \mathbf{G}(t)e^{-st}dt. \tag{5.1.5}$$

However, Laplace transform and it's properties can be studied elsewhere. As the last expression here, we give weighting pattern for time-invariant system as

$$\mathbf{G}(t,t_0) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{B} = \mathbf{C}e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{B}. \tag{5.1.6}$$

5.2. **Input-Output Stability.** As the last (but not least important) issue on stability, we would like to study stability properties appropriate to the input-output behavior (so called zero-state response) of the linear state equation 4.1.1. The main task is to prove, that the stability of weighting pattern, i.e. taking $t_0 = 0$

$$\int_{0}^{\infty} \|\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\|dt \tag{5.2.1}$$

implies the stability of the system, i.e.

$$\int_{0}^{\infty} \|e^{\mathbf{A}t}\|dt. \tag{5.2.2}$$

This relation is fully established by the next theorem. Proof can be found in [3].

Theorem 5.2.1. *Suppose the time-invariant linear state equation 4.1.1 is controllable and observable. Then the state equation is uniformly bounded-input, bounded-output stable if and only if it is exponentially stable.*

6. Linear Feedback

6.1. **Effect of Linear Feedback.** The theory of linear systems provides the tools for linear control theory. As mentioned in the introduction, main tool is use of feedback for stabilization or (in some sense) optimal performance of a system. There are several ways of construction of feedback, but we will restrict ourselves to *linear state feedback*. Given state equation of the system 4.1.1 we can see from the figure 1, that *linear state feedback* replaces the input $u(t)$ by an expression of the form

$$u(t) = \mathbf{K}x(t) + \mathbf{N}r(t), \tag{6.1.1}$$

where $r(t)$ is the new name for $m \times 1$ input signal, $\mathbf{K}$ is $m \times n$ matrix and $\mathbf{N}$ is $m \times m$ matrix of coefficients. After substituting for $u(t) = \mathbf{K}x(t) + \mathbf{N}r(t)$ into 4.1.1, we get a new linear static equation, called the *closed-loop state equation*, described by

$$\begin{aligned} x'(t) &= [\mathbf{A} + \mathbf{BK}]x(t) + \mathbf{BN}r(t) \\ y(t) &= \mathbf{C}x(t). \end{aligned} \tag{6.1.2}$$

If the coefficient matrices are constant, then the feedback is called *time invariant*. In any case the feedback is called *static*, because at any t the value of $u(t)$ depends only on the values of $r(t)$ and $x(t)$ at the same time (and not on the history). So called *dynamic feedback* can be studied elsewhere.

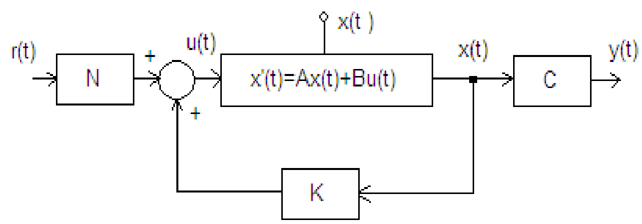

FIGURE 1. Structure of linear feedback.

6.2. **Linear Feedback Stabilization.** The most important objective that arises in considering the capabilities of feedback involves stabilization. The basic problem is to choose a *state feedback gain* $\mathbf{K}$ so that the resulting closed loop state equation is uniformly exponentially stable. Actually, somewhat more than uniform exponential stability can be achieved and for this purposes we slightly modify the definition of uniform exponential stability.

Definition 6.2.1. The linear state equation 4.1.1 is called *uniformly exponentially stable with rate* λ, where λ is positive constant, if there exists a constant γ such that for any t_0 and x_0, the corresponding solution of 4.1.1 satisfies

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)}\|x_0\|, \quad t \geq t_0. \tag{6.2.1}$$

Lemma 6.2.2. *The linear state equation 4.1.1 is uniformly exponentially stable with rate* $\lambda + \alpha$*, where* λ *and* α *are positive constants, if the linear state equation*

$$z'(t) = [\mathbf{A} + \alpha\mathbf{I}]z(t) \tag{6.2.2}$$

is uniformly exponentially stable with rate λ*.*

The answer to setting constant state feedback gain that stabilizes can be given with use of the following theorem.

Theorem 6.2.3. *Suppose the time-invariant linear state equation 4.1.1 is controllable, and let*

$$\alpha_m = \|\mathbf{A}\|. \tag{6.2.3}$$

Then for any $\alpha > \alpha_m$ the constant state feedback gain

$$\mathbf{K} = -\mathbf{B}^T \mathbf{Q}^{-1}, \tag{6.2.4}$$

where $\mathbf{Q}$ *is the positive definite solution of*

$$(\mathbf{A} + \alpha \mathbf{I})\mathbf{Q} + \mathbf{Q}(\mathbf{A} + \alpha \mathbf{I})^T = \mathbf{B}\mathbf{B}^T, \tag{6.2.5}$$

is such that the resulting closed-loop state equation is exponentially stable with the rate α.

6.3. **Eigenvalue Assignment.** Stabilization in the time-invariant case can be developed in several directions. Next, we show one interesting example. Given a set of eigenvalues, the objective is to compute a constant state feedback gain $\mathbf{K}$ such that the closed-loop state equation

$$x'(t) = (\mathbf{A} + \mathbf{B}\mathbf{K})x(t) \tag{6.3.1}$$

has exactly these eigenvalues. Of course in almost all situations eigenvalues are specified to have negative real parts for exponential stability. The capability of assigning specific values for the real parts directly influences the rate of decay of the zero-input response component, and assigning imaginary parts influences the frequencies of oscillations that occur (to eliminate vibrations, etc.).

Because the eigenvalues of a real-coefficient state equation must occur in complex-conjugate pair, it is convenient to specify, instead of eigenvalues, a real-coefficient characteristic polynomial of degree n for 6.3.1. This possibility is ilustrated by the next theorem.

Theorem 6.3.1. *Suppose that the time invariant linear state equation 4.1.1 is controllable and rank* $\mathbf{B} = m$. *Given any monic polynomial $p(\lambda)$ of degree n, there is a constant feedback gain* $\mathbf{K}$ *such that* $det(\lambda\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}) = p(\lambda)$.

7. Conclusion

In the paper we tried to give a brief introduction to the theory of linear system theory, to show state equation formulation and give definitions of stability, controllability, observability, realizability and feedback control-loop stabilization. In the Introduction we gave two motivation examples: one on feedback control and other on optimal control. However, optimal control should be studied elsewhere. The purpose of this paper is mainly to satisfy Applied Linear Algebra Project, but since the original background of the author is Automatic Control, personally takes it as a revision and partly extension of his knowledge and will be happy to extend this paper in the future (esp. into optimal control direction) and possibly combine his knowledge and some experience in this field with his current studies of applied mathematics.

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