Adaptive – Multilevel BDDC

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Domain decomposition (DD) - separation of a physical domain into

subdomains ≡ substructures.

Why? To allow for an efficient, parallel solution of systems of linear equations arising from discretizations of PDEs (by finite elements).

How? Using iterative substructuring (an important class of DD methods).

What is it? From the point of view of linear algebra, the solution of a large problem, $Ax = b$, is replaced by repeated solutions of a number of independent subproblems, to be solved in parallel.

The algorithms are formulated as preconditioned iterative methods such as conjugate gradients (CG), or GMRES (and we look for the preconditioner).

Outline

- Two-level BDDC
- Multilevel extension
- Adaptive selection of coarse space
- Adaptive – Multilevel BDDC
- Numerical experiments

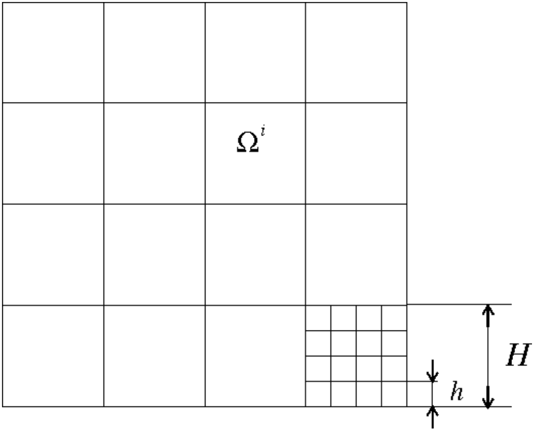
BDDC (a motivation)

BDDC – Balancing Domain Decomposition by Constraints, Dohrmann (2003)
Equiv. methods independently by Fragakis and Papadrakakis (2003), Cros (2003).

From Dohrmann (2003):

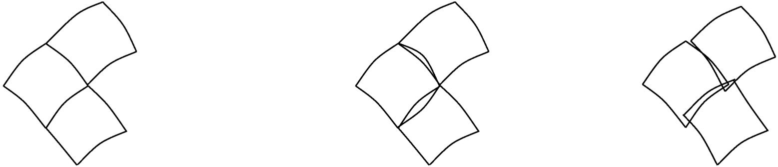
Some remaining issues need to be addressed for improvement. First, it would be useful to have an effective method for selecting additional corners and edges to improve performance for very poorly conditioned problems. Second, the performance of the multilevel extension should be investigated further. Recall that the multilevel extension is obtained by recursive application of the preconditioner to coarse problem stiffness matrices. Such an extension would be beneficial for problems with very large numbers of substructures...

Substructuring for the two-level method (with H/h=4)



BDDC description - example of spaces

$$W = \bigotimes_{i=1}^{N} W_i : \quad \text{space of block vectors, one block per substructure}$$



U	$\subset$	$\widetilde{W}$	$\subset$	W
continuous across whole substructure interfaces		continuous across corners only		no continuity required
global matrix: assembled		partially assembled		not assembled
substructures: assembled		assembled		assembled

Want to solve

$$u \in U : a(u, v) = \langle f, v \rangle \quad \forall v \in U.$$

Abstract two-level BDDC:

Variational setting of the problem and algorithm components

$$u \in U : a(u, v) = \langle f, v \rangle , \quad \forall v \in U$$

form $a(\cdot, \cdot)$ is SPD on U and **positive semidefinite** on $W \supset U$,

Example:

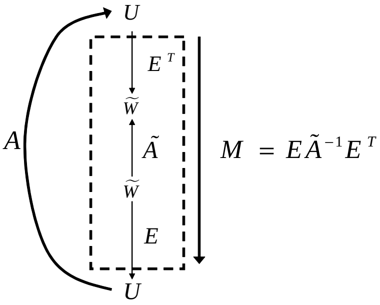
$W = W_1 \times \cdots \times W_N$ (spaces on substructures)

$U =$ functions continuous across interfaces

Choose preconditioner components:

1. space $\widetilde{W}$, $U \subset \widetilde{W} \subset W$, such that $a(\cdot, \cdot)$ is positive definite on $\widetilde{W}$.
   Example: functions with continuous coarse dofs, such as values at substructure corners
2. projection $E : \widetilde{W} \rightarrow U, \quad \mathrm{range}\, E = U$.
   Example: averaging across substructure interfaces

Algebraic view



The same bilinear form $a(\cdot,\cdot)$ defines $A : U \to U$ and $\widetilde{A} : \widetilde{W} \to \widetilde{W} \supset U$

The preconditioner M to A is obtained by:

1. solving a problem with the same bilinear form on the bigger space $\widetilde{W}$,
2. mapping back to U via the projection E and its transpose E^T.

Condition number bound

Theorem

The abstract BDDC preconditioner $M : U \longrightarrow U$ *defined by*

$$M : r \longmapsto u = Ew, \quad w \in \widetilde{W} : \quad a\left(w, z\right) = \left\langle r, Ez \right\rangle, \quad \forall z \in \widetilde{W}.$$

satisfies

$$\kappa = \frac{\lambda_{\max}(MA)}{\lambda_{\min}(MA)} \leq \omega = \sup_{w \in \widetilde{W}} \frac{\|(I - E)w\|_a^2}{\|w\|_a^2}.$$

Coarse degrees of freedom (in implementation)

The space $\widetilde{W}$ is defined using so called **coarse degrees of freedom**, as

$$\widetilde{W} = \{w \in W : C\,(I - E)\,w = 0\}\,,$$

C ... weights on the local coarse degrees of freedom, E ... averaging,

$$\Longrightarrow$$

coarse degrees of freedom on adjacent substructures coincide.

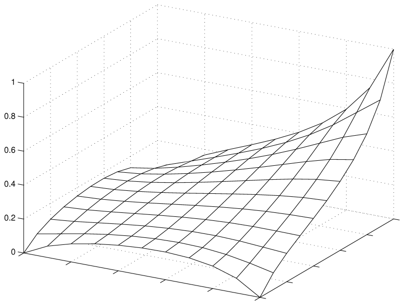
In implementation, $\widetilde{W}$ is decomposed into

$$\widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi$$

$\widetilde{W}_\Delta$ = functions with zero coarse dofs $\Rightarrow$ local problems on substructures
$\widetilde{W}_\Pi$ = functions given by coarse dofs & energy minimal $\Rightarrow$
$\Rightarrow$ global **coarse problem**.

The coarse problem

(reproduced from Mandel, Dohrmann (2003))

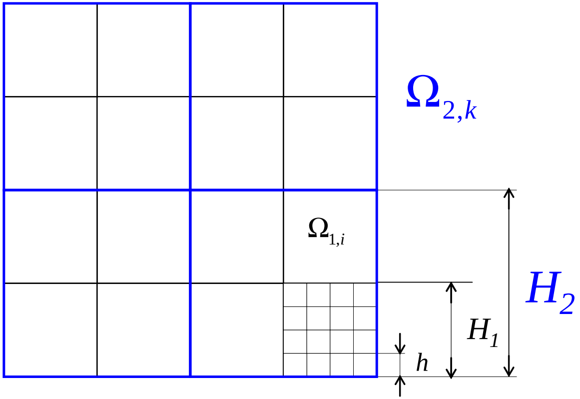


A basis function from $\widetilde{W}_\Pi$ is energy minimal subject to given values of coarse degrees of freedom on the substructure. The function is discontinuous across the interfaces between the substructures but the values of coarse degrees of freedom on the different substructures coincide.

The coarse problem has the same structure as the original FE problem $\Longrightarrow$ solve it approximately by one iteration of BDDC $\Longrightarrow$ **three-level BDDC**.

Apply recursively $\Longrightarrow$ **Multilevel BDDC**.

Substructuring for the three-level method



Condition number bound

Theorem (Mandel, Sousedík, Dohrmann (2008))

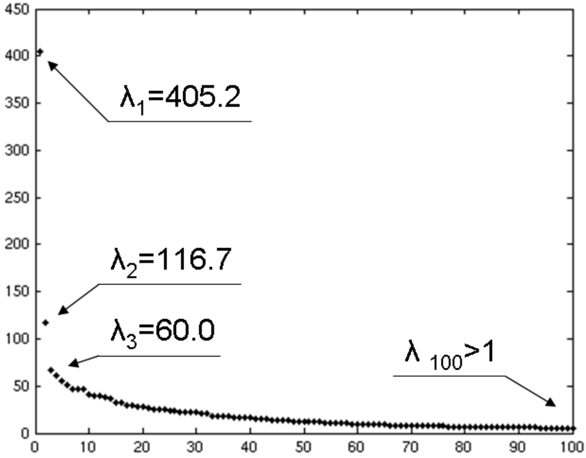
The condition number bound $\kappa \leq \omega$ of Multilevel BDDC is given by

$$\kappa \leq \omega = \Pi_{\ell=1}^{L-1} \omega_\ell \,, \qquad \omega_\ell = \sup_{w_\ell \in (I - P_\ell)\widetilde{W}_\ell} \frac{\left\| (I - E_\ell)\, w_\ell \right\|_a^2}{\left\| w_\ell \right\|_a^2}$$

Generalizes 3-level bounds by Tu (2006, 2007) to many levels.

Spectrum of a preconditioned operator

3D elasticity, 14 739 dof, 2-level method with 8 subdomains
(jagged interfaces, no jumps in coefficients)



Adaptive method idea (Mandel, Sousedik, 2006)

Condition number bound is Rayleigh quotient

$$\omega_\ell = \sup_{w \in (I - P_\ell)\widetilde{W}_\ell} \frac{\|(I - E_\ell)\, w_\ell\|_a^2}{\|w_\ell\|_a^2} = \sup_{w_\ell \in \widetilde{W}_\ell} \frac{w_\ell^T\,(I - E_\ell)^T\, S_\ell\,(I - E_\ell)\, w_\ell}{w_\ell^T S_\ell w_\ell}$$

Eigenvalues $\lambda_1 > \lambda_2 > \ldots$, stationary points are eigenvectors. Optimal decrease of the Rayleigh quotient: make the space

$$\widetilde{W}_\ell = \text{null}\, C_\ell(I - E_\ell)$$

smaller by orthogonality to the dominant eigenvector u_1:
add u_1^T to $C_\ell(I - E_\ell)$
This reduces the condition bound to the second eigenvalue λ_2.
Adding more eigenvectors "removes" more eigenvalues.

Condition number bound: Heuristic indicator

Solving the the global eigenvalue problems is expensive ... replace it by: local versions formulated for all pairs of **adjacent** substructures.

Definition: A pair of substructures is **adjacent**, if they share a **face**.

$$\omega_\ell^{st} = \max \lambda_\ell^{st} \quad : \quad \Pi_\ell^{st} \left(I - E_\ell^{st} \right)^T S_\ell^{st} \left(I - E_\ell^{st} \right) \Pi_\ell^{st} w_\ell^{st} = \lambda_\ell^{st} \Pi_\ell^{st} S_\ell^{st} \Pi_\ell^{st} w_\ell^{st}$$

E_ℓ^{st} ... locally constructed averaging operator,
S_ℓ^{st} ... block diagonal Schur complement,
Π_ℓ^{st} ... the orthogonal projection from W_ℓ^{st} into $\widetilde{W}_\ell^{st}$ (initial constraints).

The **heuristic indicator** of the condition number bound is defined as

$$\widetilde{\omega} = \Pi_{\ell=1}^{L-1} \quad \left[\max_{\{st\} \in \mathcal{A}_\ell} \omega_\ell^{st} \right].$$

Adaptive selection of face constraints

Algorithm: adding of coarse degrees of freedom to guarantee that the condition number indicator $\widetilde{\omega} \leq \tau^{L-1}$, for a given a target value τ:

for levels $\ell = 1 : L-1$

for all faces $\mathcal{F}_\ell$ **on level** ℓ

1. Compute the largest local eigenvalues and corresponding eigenvectors, until the first m^{st} is found such that $\lambda^{st}_{m^{st}} \leq \tau$, put $k = 1, \ldots, m^{st}$.
2. Compute the constraint weights c^{st}_k.
3. Add the constraint weights c^{st}_k to the matrix C_ℓ in the definition of

$$\widetilde{W}_\ell = \text{null}\, C_\ell (I - E_\ell)\,.$$

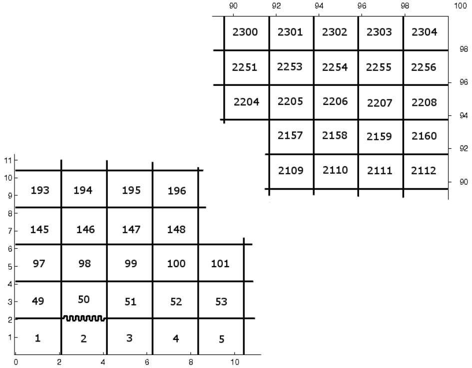
Adaptive-Multilevel BDDC: Implementation remarks

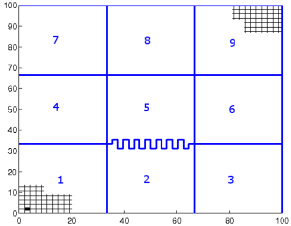
- Use the **adaptive method** for faces, with initial constraints as:
  - corners in 2D,
  - corners and arithmetic averages over edges in 3D.
- Treat substructures as (coarse) elements with:
  - energy minimal basis functions,
  - variable number of nodes per element,
  - variable number of degrees of freedom per node.
- On each decomposition level $\ell = 1, \ldots, L-1$:
  - create substructures with roughly the same number of dofs,
  - minimize the number of “cuts“ (**Metis 4**) between substructures,
  - repeat until it is suitable to factor the coarse problem directly (level L).

Numerical results in 2D: compressible elasticity, $\lambda = 1$, $\mu = 2$

Domain decomposition of the planar elasticity problem with 1 182 722 dof, 2 304 subdomains on the second level and 9 subdomains on the third-level,

The two-level decomposition (left) and the three-level decomposition (right):





The coarsening ratio on both decomposition levels is $H_i / H_{i-1} = 16$.

Non-adaptive method:

constraint	*Nc*	$\mathcal{C}$	κ	*it*
c	4 794	9.3	18.41	43
c+f	13 818	26.9	18.43	32

Adaptive constraints:

τ	*Nc*	$\mathcal{C}$	$\widetilde{\omega}$	κ	*it*
∞(=c)	4 794	9.3	-	18.41	43
10	4 805	9.4	8.67	8.34	34
3	18 110	35.3	2.67	2.44	15
2	18 305	35.7	1.97	1.97	13

c, c+f: constraints as arithmetic averages over corners, corners and faces,
Nc: number of constraints, $\mathcal{C}$: relative size of the coarse problem,
τ: condition number target, $\widetilde{\omega}$: condition number indicator,
κ: approximate condition estimate, *it*: number of iterations (tol 10^{-8}).

Numerical results in 2D: Local eigenvalues (2-level method)

Eigenvalues of the local problems for pairs of subdomains s and t

(the jagged face is between subdomains 2 and 50)

s	t	$\lambda_{st,1}$	$\lambda_{st,2}$	$\lambda_{st,3}$	$\lambda_{st,4}$	$\lambda_{st,5}$	$\lambda_{st,6}$	$\lambda_{st,7}$	$\lambda_{st,8}$
1	2	3.8	2.4	1.4	1.3	1.2	1.1	1.1	1.1
1	49	6.0	3.5	2.7	1.4	1.3	1.1	1.1	1.1
2	3	5.4	2.6	1.6	1.3	1.2	1.1	1.1	1.1
2	50	24.3	18.4	18.3	16.7	16.7	14.7	13.5	13.1
3	4	3.4	2.4	1.4	1.3	1.1	1.1	1.1	1.1
3	51	7.4	4.6	3.7	1.7	1.4	1.3	1.2	1.1
49	50	12.6	5.1	4.3	1.9	1.6	1.3	1.2	1.2
50	51	8.7	4.8	3.9	1.8	1.5	1.3	1.2	1.2
50	98	7.5	4.6	3.7	1.7	1.4	1.3	1.2	1.1

Eigenvalues of the local problems for pairs of subdomains s and t
– the second decomposition level with $\tau = 10$.
(the jagged face is between subdomains 2 and 5)

s	t	$\lambda_{st,1}$	$\lambda_{st,2}$	$\lambda_{st,3}$	$\lambda_{st,4}$	$\lambda_{st,5}$	$\lambda_{st,6}$	$\lambda_{st,7}$	$\lambda_{st,8}$
1	2	7.7	4.5	2.7	1.6	1.4	1.2	1.2	1.1
1	4	3.6	3.0	1.5	1.5	1.2	1.2	1.1	1.1
2	3	10.9	4.8	2.7	1.7	1.5	1.2	1.2	1.1
2	5	23.2	17.2	13.7	13.7	12.7	12.4	11.0	10.9
3	6	6.1	4.2	2.5	1.5	1.3	1.2	1.1	1.1
4	7	3.6	3.0	1.5	1.5	1.2	1.2	1.1	1.1
5	6	9.8	6.2	4.1	2.1	1.6	1.5	1.3	1.2
5	8	8.6	5.9	3.9	2.0	1.5	1.4	1.2	1.2
8	9	6.1	4.2	2.5	1.5	1.3	1.2	1.1	1.1

Eigenvalues of the local problems for pairs of subdomains s and t
– the second decomposition level with $\tau = 2$.
(the jagged face is between subdomains 2 and 5)

s	t	$\lambda_{st,1}$	$\lambda_{st,2}$	$\lambda_{st,3}$	$\lambda_{st,4}$	$\lambda_{st,5}$	$\lambda_{st,6}$	$\lambda_{st,7}$	$\lambda_{st,8}$
1	2	16.5	9.0	5.4	2.6	2.1	1.4	1.3	1.3
1	4	6.5	4.7	1.9	1.7	1.3	1.2	1.2	1.1
2	3	23.1	9.4	4.6	3.2	2.1	1.6	1.4	1.3
2	5	84.3	61.4	61.4	55.9	55.8	49.3	48.0	46.9
3	6	13.7	8.8	4.4	2.2	1.9	1.4	1.3	1.2
4	7	6.5	4.7	1.9	1.7	1.3	1.2	1.2	1.1
5	6	18.9	13.1	11.3	3.8	2.6	2.1	1.9	1.5
5	8	17.3	12.9	10.8	3.6	2.3	2.0	1.8	1.4
8	9	13.7	8.8	4.4	2.2	1.9	1.4	1.3	1.2

Non-adaptive method:

constraint	*Nc*	$\mathcal{C}$	κ	*it*
c	$4794 + 24$	1.0	67.5	74
c+f	$13818 + 48$	3.0	97.7	70

Adaptive constraints:

τ	*Nc*	$\mathcal{C}$	$\widetilde{\omega}$	κ	*it*
$\infty(=\text{c})$	$4794 + 24$	1.0	-	67.5	74
10	$4805 + 34$	1.0	$> (9.80)^2$	37.42	60
3	$18110 + 93$	3.9	$> (2.95)^2$	3.11	19
2	$18305 + 117$	4.0	$> (1.97)^2$	2.28	15

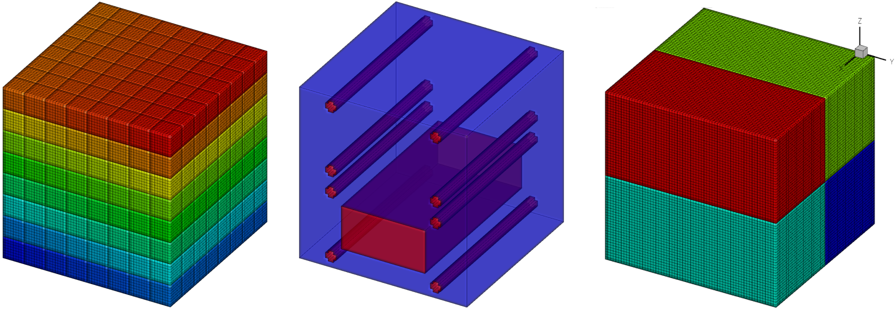
c, c+f: constraints as arithmetic averages over corners, corners and faces,
Nc: number of constraints, $\mathcal{C}$: relative size of the coarse problem,
τ: condition number target, $\widetilde{\omega}$: condition number indicator,
κ: approximate condition estimate, *it*: number of iterations (tol 10^{-8}).

Numerical results in 3D (data and visualization: J. Šístek)

dofs: 823 875

level 1: substructures: 512, corners/edges/faces: 721/1 176/1 344

level 2: substructures: 4, corners/edges/faces: 6/1/4



cube: $E = 10^6$ Pa, $\nu = 0.45$, bars: $E = 2.1 \cdot 10^{11}$ Pa, $\nu = 0.3$.

dofs: 823 875

level 1: substructures: 512, corners/edges/faces: 721/1 176/1 344

level 2: substructures: 4, corners/edges/faces: 6/1/4

constraint	*Nc*	κ	*it*
c	2 163	312 371	$>$ 3 000
c+e	5 691	45 849	1 521
e+e+f	9 723	16 384	916
c+e+f (3eigv)	9 723	3 848	367

Adaptive constraints:

τ	*Nc*	$\widetilde{\omega}$	κ	*it*	
∞(=c+e)	5 691	3.54	$o(10^4)$	45 848.60	1 521
10 000	5 883	3.66	8 776.50	5 098.60	441
1 000	6 027	3.75	5.33	9.92	32
10	6 149	3.82	6.25	6.66	28
5	9 119	5.67	<5	4.79	24
2	25 009	15.54	<2	2.92	18

Composite cube (3-level methods)

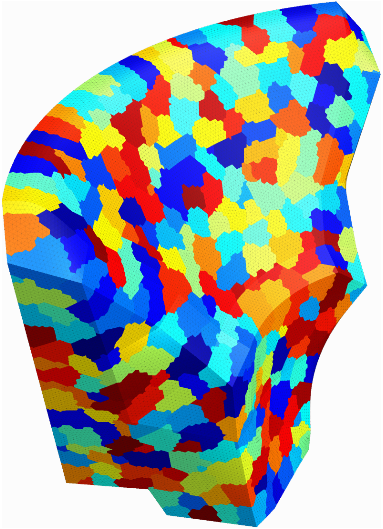
constraint	Nc	$\mathcal{C}$	κ	it
c	$2\,163 + 18$	$0.34 + 0.01$	$o(10^7)$	$>> 3\,000$
c+e	$5\,691 + 21$	$0.88 + 0.01$	$o(10^6)$	$> 3\,000$
c+e+f	$9\,723 + 33$	$1.51 + 0.02$	461 750	1 573
c+e+f (3eigv)	$9\,723 + 33$	$1.51 + 0.02$	125 305	981

Adaptive constraints:

τ	Nc	$\mathcal{C}$	$\widetilde{\omega}$	κ	it
∞(=c+e)	$5\,691 + 21$	$0.88 + 0.01$	-	$o(10^6)$	$> 3\,000$
10 000	$5\,883 + 28$	$0.91 + 0.02$	8 776.50	26 874.40	812
1 000	$6\,027 + 34$	$0.94 + 0.02$	766.82	1 449.50	145
100	$6\,027 + 53$	$0.94 + 0.03$	99.05	100.89	59
10	$6\,149 + 65$	$0.96 + 0.04$	7.93	7.91	30
5	$9\,119 + 67$	$1.42 + 0.04$	< 5	6.18	25
2	$25\,009 + 122$	$3.89 + 0.08$	< 2	3.08	18

Numerical results in 3D (data: J. Kruis, visualization: M. Brezina)

dofs: 2 006 748, substructures: 1 024, corners/edges/faces: 10 693/7 713/6 182



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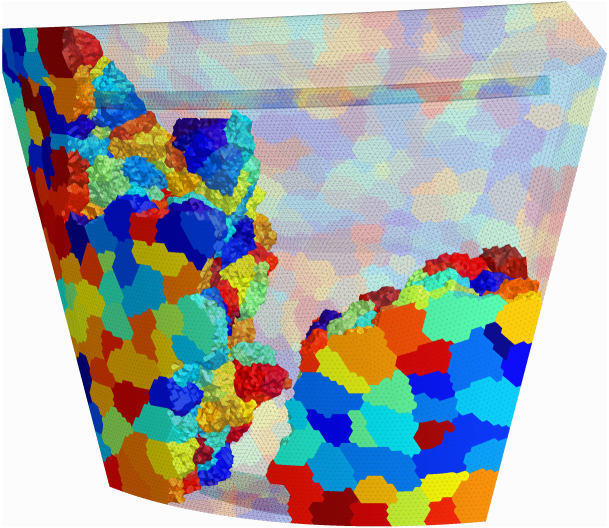
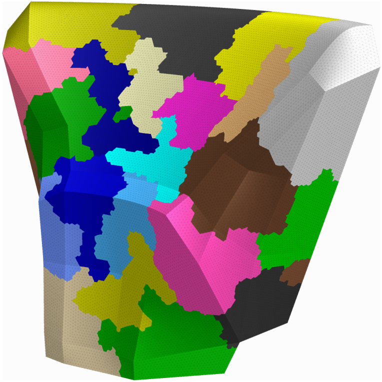
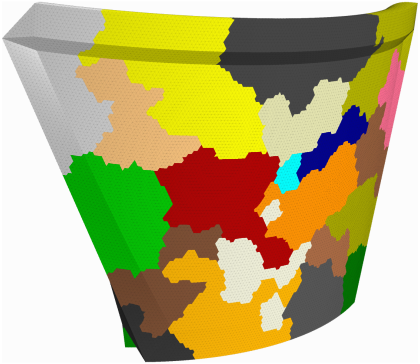


Numerical results in 3D (data: J. Kruis, visualization: M. Brezina)

substructures: $(1\,024+)\,32$





dofs: 2 006 748, substructures: 1 024, corners/edges/faces: 10 693/7 713/6 182

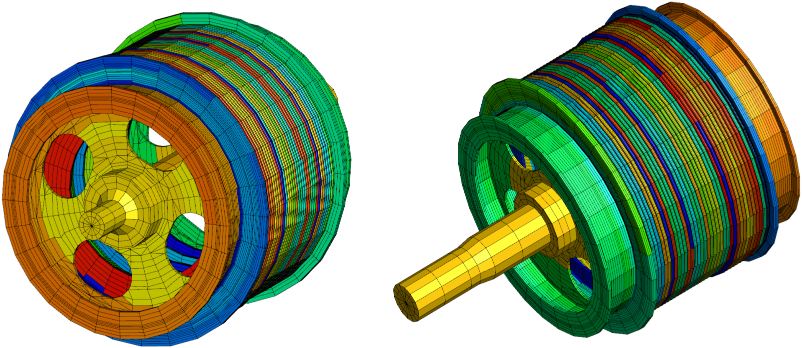
constraint	Nc	$\mathcal{C}$	κ	it
c	32079	16.37	28.53	54
c+e	55218	28.18	13.96	35
c+e+f	73764	37.64	5.00	21

3-level method with $1024 + 32$ substructures:

constraint	Nc	nc/ne/nf	$\mathcal{C}$	κ	it
c	$32079 + 768$	256/194/124	$0.51 + 0.39$	498.82	136
c+e	$55218 + 1407$	256/213/128	$0.88 + 0.71$	161.63	71
c+e+f	$73764 + 1818$	256/213/137	$1.18 + 0.93$	169.38	77

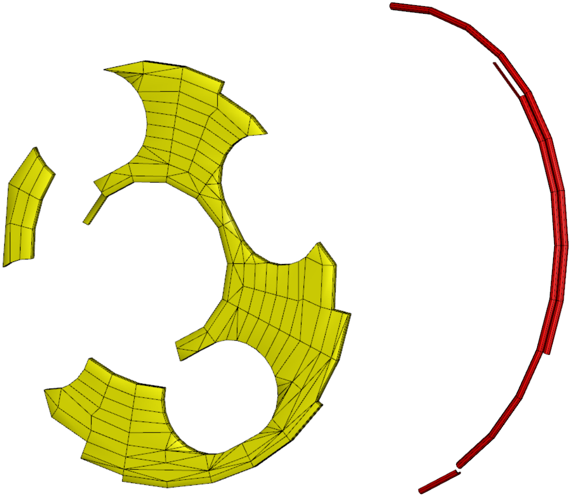
c, e, f: constraints as arithmetic averages over corners, edges and faces,
Nc: number of constraints, $\mathcal{C}$: relative size of the coarse problem,
τ: condition number target, $\widetilde{\omega}$: condition number indicator,
κ: approximate condition estimate, it: number of iterations (tol 10^{-8}).

dofs: 1 739 211, substructures: 400, corners/edges/faces: 4 010/831/1 906

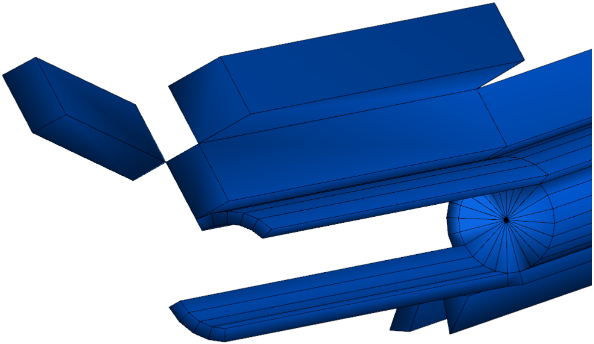


Numerical results in 3D (data and visualization: J. Šístek)

dofs: 1 739 211, substructures: 400, corners/edges/faces: 4 010/831/1 906

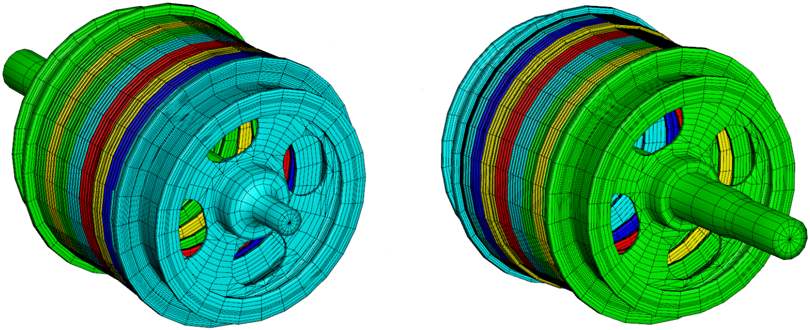


dofs: 1 739 211, substructures: 400, corners/edges/faces: 4 010/831/1 906



Numerical results in 3D (data and visualization: J. Šístek)

substructures: (400+) 8



Numerical results in 3D: Mining reel (400 substructures)

dofs: 1 739 211, substructures: 400, corners/edges/faces: 4 010/831/1 906

Adaptive method (non-adaptive fails):

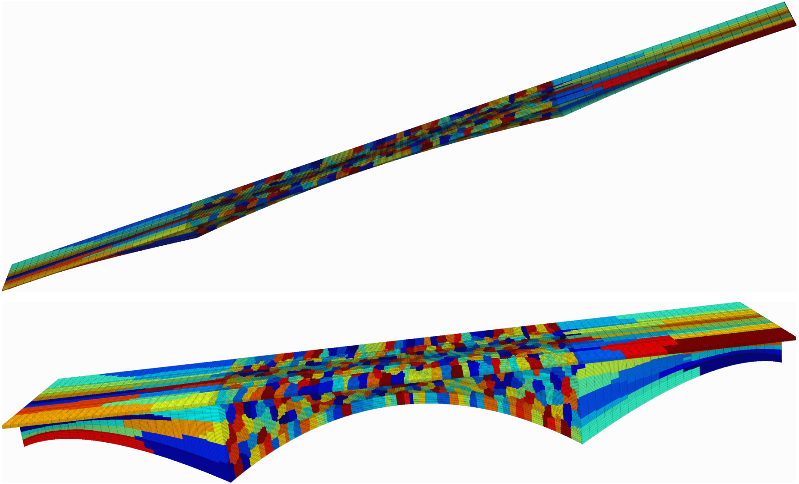
τ	Nc	$\mathcal{C}$	$\widetilde{\omega}$	κ	it
∞(=c+e)	14523	-	$o(10^7)$	-	-
10000	16080	3.70	9999.85	401441.00	1453
1000	20331	4.68	999.94	4205.79	401
100	29641	6.82	99.96	1653.31	173
10	45113	10.38	<10	1625.31	108
2	78475	18.05	<2	1608.54	80

Adaptive constraints (3-level method):

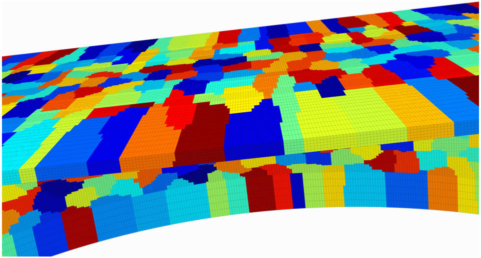
τ	Nc	nc/ne/nf	$\mathcal{C}$	$\widetilde{\omega}$	κ	it
100	$29641 + 170$	22/0/8	0.85/0.04	99.94^2	58828.8	1129
10	$45113 + 539$	22/0/8	1.30/0.12	$<10^2$	1623.18	123
2	$78\,475 + 2\,177$	22/0/8	2.26/0.50	$< 2^2$	1 607.88	79

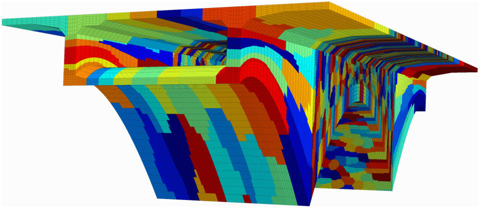
Numerical results in 3D (data: J. Kruis, visualization: M. Brezina)

dofs: 3 173 760, substructures: 1 024, corners/edges/faces: 6 051/2 099/3 034



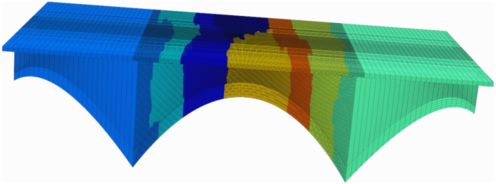
dofs: 3 173 760, substructures: 1 024, corners/edges/faces: 6 051/2 099/3 034

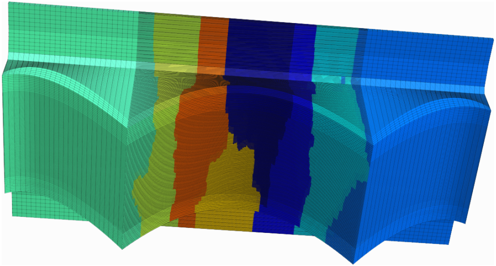




Numerical results in 3D (data: J. Kruis, visualization: M. Brezina)

substructures: $(1\,024+)\,8$





dofs: 3 173 760, substructures: 1 024, corners/edges/faces: 6 051/2 099/3 034

Adaptive method (non-adaptive fails):

τ	Nc	$\mathcal{C}$	$\widetilde{\omega}$	κ	it
∞(=c+e)	24 450	7.89	$o(10^7)$	∞	-
100	26 219	8.46	99.95	17 141.40	252
10	32 219	10.40	9.99	7 014.42	124
5	37 763	12.18	<5	6 361.90	109
2	61 497	19.84	<2	5 878.03	90

3-level method with $1024 + 8$ substructures:

τ	Nc	nc/ne/nf	$\mathcal{C}$	$\widetilde{\omega}$	κ	it
10	$32\,219 + 197$	22/2/9	$1.30 + 0.06$	$< 10^2$	7 008.82	135
5	$37\,763 + 309$	22/2/9	$1.52 + 0.10$	$< 5^2$	6 355.71	118
2	$61\,497 + 1\,007$	22/2/9	$2.48 + 0.32$	$< 2^2$	5 872.43	94

Ongoing and future research

1. Practical:
   - parallel implementation (in progress, with Dr. Šístek)
     ⇒ public availability of the developed code
   - extension to other types of problems (shells, porous media, ...)
2. Theoretical:
   - condition number bound with discrete harmonic basis functions
   - condition number bounds for multilevel linear elasticity problems
   - the adaptive algorithm (angles between subspaces + majorization?)

List of publications (related to the dissertation)

Bedřich Sousedík:
Comparison of some domain decomposition methods.
PhD thesis, Czech Technical University in Prague, Faculty of Civil Engineering, Department of Mathematics, 2008.

Jan Mandel and Bedřich Sousedík:
Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods.
Comput. Methods Appl. Mech. Engrg., 196(8):1389–1399, 2007.

Jan Mandel, Bedřich Sousedík, and Clark R. Dohrmann.
Multispace and Multilevel BDDC.
Computing, 83(2-3):55–85, 2008.

Bedřich Sousedík and Jan Mandel.
On Adaptive – Multilevel BDDC.
In *Nineteenth International Conference on Domain Decomposition*. Springer-Verlag, 2010.
Submitted.

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  - CNS-0719641
  - DMS-0713876
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  - GA ČR 106/05/2731
  - GA ČR 106/08/0403