**Thomas I. Seidman**[1]

# Equations with hysteretically discontinuous right hand sides[2]

## 1. Introduction

We consider, here, *switching systems* having the form

$$\dot{x} - \mathbf{A}x = f_k(x), \qquad x(0) = \xi_0,\ k(0) = j_0 \tag{1.1}$$

where $x(\cdot)$ takes values in a Banach space $\mathcal{X}$ on which $\mathbf{A}$ is infinitesimal generator of a $C_0$ semigroup $\mathbf{S}(\cdot)$ and the index $k = k(\cdot)$ evolves subject to the rules (1.3), below. The point of the present paper is to avoid the imposition of any continuity conditions on the functions $f_k$ and still to be able to show the existence of suitable 'solutions'. Following Filippov [1] (see, also, [5]), this will involve a re-interpretation, replacing (1.1) by

$$\dot{x} - \mathbf{A}x \in \mathcal{F}_k(x), \qquad x(0) = \xi_0,\ k(0) = j_0\ . \tag{1.2}$$

For the logic of the particular $\mathcal{F}_j$, see [1]; note that $\mathcal{F}_j(\xi) = \{f_j(\xi)\}$ where $f_j$ is continuous.

Following [3], [4], the function $k(\cdot)$ in (1.1), (1.2) takes its values in a finite index set $\mathcal{J}$ and evolves subject to *switching rules* given in terms of sets $\{\mathcal{R}_j\}$ and $\{\mathcal{S}_{i,j}\}$:

$$\begin{array}{ll} (i) & x(t) \in \mathcal{R}_j \ \Rightarrow\ k(t) \neq j \\ (ii) & k(t-) = i,\ k(t+) = j \neq i \ \Rightarrow\ x(t) \in \mathcal{S}_{i,j} \end{array} \tag{1.3}$$

so $\mathcal{R}_j$ is a 'forbidden set' for mode $j \in \mathcal{J}$ and $\mathcal{S}_{i,j}$ is a 'switching set' for transitions from the $i$-th to the $j$-th mode. We assume throughout that each $\mathcal{R}_j$ is open and each $\mathcal{S}_{i,j}$ is closed with the consistency condition

$$\begin{array}{ll} (i) & \mathcal{S}_{i,j} \cap \mathcal{S}_{j,k} = \emptyset = \mathcal{S}_{i,j} \cap \overline{\mathcal{R}_j} \\ (ii) & \partial\mathcal{R}_i \subset \cup_j \mathcal{S}_{i,j}. \end{array} \tag{1.4}$$

It is the set of switching rules (1.3) which produces the essential character of *hysteresis*: the evolution of the 'visible state' $x(\cdot)$ depends on a 'hidden' state component $k(\cdot)$ whose evolution is both rate-independent and causal.

Since there are known examples of continuous $f : \mathcal{X} \to \mathcal{X} =$ Banach space for which the simpler equation $\dot{x} = f(x)$ has no solutions, it is clear that *some* condition must be imposed on the functions $f_j$. To this end we assume throughout the *structural hypothesis*:

**(H1)** *There is a finite set of $N$ Lipschitzian functions $\psi_\nu : \mathcal{X} \to \mathcal{X}$ (uniform Lipschitz constant $L$) such that, $\forall j \in \mathcal{J}$, $\forall \xi$, one has $f_j(\xi) \in \overline{\mathit{hull}}\{\{\psi_\nu(\xi)\}_{\nu=1}^N\}$.*

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[1] Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21228, USA. e-mail: ⟨seidman@math.umbc.edu⟩. This research has been partially supported by the National Science Foundation under the grant ECS-8814788.

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One easily sees that in a finite dimensional setting (H1) restricts $f_j$ only by imposing a linear growth, ensuring that blow-up cannot occur in finite time. In the general setting, however, it is a genuine restriction which is critical for our approach.

Our goals in this paper will be to show, in Section 3, existence of mild solutions and at the same time to obtain a 'stability' result: that *the limit of (approximate) solutions is a solution.* The principal tools for this will be provided in Section 3: a representation obtained from semigroup theory and (H1) together with a compactness result based on Lemma 2 of [2].

## 2. Formulation and Preliminaries

Following [1], [5]), we may begin with any (densely defined) function $f : \mathcal{X} \supset \mathcal{D} \to \mathcal{X}$ and, for $\hat{\xi} \in \mathcal{X}$ and $\varepsilon > 0$, define $\mathcal{F}(\xi, \varepsilon) = \mathcal{F}_*(\xi, \varepsilon; f)$ and $\mathcal{F}(\xi) = \mathcal{F}^*(\xi; f)$ by

$$
\begin{aligned}
\mathcal{F}_*(\xi,\varepsilon;f) &:= \overline{\text{hull}}\,\{\eta \in \mathcal{X} : \exists \hat{\xi} \in \mathcal{D} \text{ with } |\xi - \hat{\xi}| \le \varepsilon,\ |\eta - f(\hat{\xi})| \le \varepsilon\} \\
\mathcal{F}^*(\xi;f) &:= \textstyle\bigcap_{\varepsilon>0} \mathcal{F}_*(\xi,\varepsilon;f) \\
&= \overline{\text{hull}}\,\{\eta \in \mathcal{X} : \exists \{\xi_n\} \ni \xi_n \to \hat{\xi},\ f(\xi_n) \to \eta\}.
\end{aligned} \tag{2.1}
$$

For our present purposes, we then set $\mathcal{F}_j(\xi,\varepsilon) := \mathcal{F}_*(\xi,\varepsilon; f_j)$ and $\mathcal{F}_j(\xi) := \mathcal{F}^*(\xi; f_j)$ for each $j \in \mathcal{J}$. It is this $\mathcal{F}_j$ which is to be used in (1.2). Note that (2.1) and the Lipschitz condition on the $\psi_\nu$ in (H1) give

$$
\mathcal{F}_j(\xi,\varepsilon) \subset \left[\overline{\text{hull}}\,\{\{\psi_\nu(\hat{\xi})\}\,|_1^N : |\hat{\xi} - \xi| \le \varepsilon\} + \mathcal{B}_\varepsilon\right] \subset \left[\overline{\text{hull}}\,\{\{\psi_\nu(\xi)\}\,|_1^N\} + \mathcal{B}_{(L+1)\varepsilon}\right] \tag{2.2}
$$

where $\mathcal{B}_r := \{\rho \in \mathcal{X} : |\rho| \le r\}$. It is convenient to set

$$
\begin{aligned}
\boldsymbol{\mathcal{F}}_j(x,\varepsilon;\mathcal{I}) &:= \{y \in L^\infty(\mathcal{I}) : y(t) \in \left[\mathcal{F}_j(x(t),\varepsilon) + \overline{B_{a\varepsilon}}\right] \text{ ae on } \mathcal{I}\} \\
\boldsymbol{\mathcal{F}}_j(x;\mathcal{I}) &:= \{y \in L^\infty(\mathcal{I}) : y(t) \in \mathcal{F}_j(x(t)) \text{ ae}\} = \textstyle\bigcap_{\varepsilon>0} \boldsymbol{\mathcal{F}}_j(x,\varepsilon;\mathcal{I})
\end{aligned} \tag{2.3}
$$

for $x \in \boldsymbol{\mathcal{X}}$ and intervals $\mathcal{I} \subset [0,T]$; we say $y \in \boldsymbol{\mathcal{F}}(x,k,\varepsilon)$ if $y \in \boldsymbol{\mathcal{F}}_j(x,\varepsilon;\mathcal{I})$ for each interval of constancy $k(\cdot) = j$ and similarly for $y \in \boldsymbol{\mathcal{F}}(x,k)$.

We introduce $\underline{\mathcal{C}} = \{\mathbf{c} \in [0,1]^N : \Sigma_\nu\ c_\nu = 1\}$ and the set $\boldsymbol{\mathcal{C}}$ of all measurable $\underline{\mathcal{C}}$-valued functions on $[0,T]$. Essential for us is the following implication of (2.2), etc.:

**LEMMA 1:** *Suppose $k : [0,T] \to \mathcal{J}$, $x \in \boldsymbol{\mathcal{X}}$, and $y \in \boldsymbol{\mathcal{F}}(x,k,\varepsilon)$ for some $\varepsilon > 0$. Then there is some $\mathbf{c}(\cdot) \in \boldsymbol{\mathcal{C}}$ and measurable $\rho : [0,T] \to \mathcal{X}$ such that $|\rho(t)| \le (a+L)\varepsilon$ and*

$$
y(t) = \rho(t) + \sum_{\nu=1}^{N} c_\nu(t)\psi_\nu(x(t)) \quad \text{ae on } [0,T]. \tag{2.4}
$$

From semigroup theory, we note that (1.2) is equivalent to the integral equation

$$
x(t) = \mathbf{S}(t)\xi_0 + \int_0^t \mathbf{S}(t-s)y(s)\,ds \qquad\qquad y \in \boldsymbol{\mathcal{F}}(x,k) \tag{2.5}
$$

Definition: If (2.5) and (1.3) are satisfied, we say that we have a *mild solution* of the original system (1.1), (1.3).

The significance of Lemma 1 is that we can write the integral equation (2.5) in the form

$$x(t) = \mathbf{S}(t)\xi_0 + \int_0^t \mathbf{S}(t-s)\Sigma_\nu c_\nu(s)\psi_\nu(x(s))\,ds \tag{2.6}$$

for some (unknown) coefficient function $\mathbf{c}(\cdot) \in \boldsymbol{\mathcal{C}}$ and use the Lipschitzian nature of the $\psi_\nu(\cdot)$. Introduce the families of operators $\mathbf{Y}(\cdot, x) : L^2([0,T] \to \mathbb{R}^N) \to L^2([0,T] \to \mathcal{X})$ (parametrized by $x \in \boldsymbol{\mathcal{X}}$) defined by

$$[\mathbf{Y}(\mathbf{c}; x)](t) = \Sigma_\nu c_\nu(t)\psi_\nu(x(t)). \tag{2.7}$$

and $\mathbf{G}(\cdot; \mathbf{c}) : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ (parametrized by $\mathbf{c} \in \boldsymbol{\mathcal{C}}$) defined by the right hand side of (2.6):

$$\begin{aligned} [\mathbf{G}(x(\cdot); \mathbf{c})](t) &:= \mathbf{S}(t)\xi_0 + \textstyle\int_0^t \mathbf{S}(t-s)\Sigma_\nu c_\nu(s)\psi_\nu(x(s))\,ds \\ &= \mathbf{S}(t)\xi_0 + \textstyle\int_0^t \mathbf{S}(t-s)y(s)\,ds \qquad \text{with } y = \mathbf{Y}(\mathbf{c}, x) \end{aligned} \tag{2.8}$$

with $\xi_0$ as in (1.1) and $x(\cdot) \in \boldsymbol{\mathcal{X}} := C([0,T] \to \mathcal{X})$ so that (1.2) is equivalent to the system (for $[x, \mathbf{c}] \in \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{C}}$ and $k$)

$$x = \mathbf{G}(x; \mathbf{c}) \qquad\qquad \mathbf{Y}(\mathbf{c}; x) \in \boldsymbol{\mathcal{F}}(x, k) \tag{2.9}$$

together with (1.3). Note that if we norm $\boldsymbol{\mathcal{X}}$ by

$$\|x(\cdot)\|_* := \max\{e^{-2Lt}\|x(t)\|_{\mathcal{X}} : t \in [0,T]\}, \tag{2.10}$$

then a standard estimate gives contractivity (uniform in $\mathbf{c} \in \boldsymbol{\mathcal{C}}$):

$$\|\mathbf{G}(x; \mathbf{c}) - \mathbf{G}(\hat{x}; \mathbf{c})\|_* \leq \tfrac{1}{2}\|x - \hat{x}\|_* \tag{2.11}$$

so, for each $\mathbf{c}$, the Contractive Mapping Principle (CMP) gives existence of a unique solution $x^* = x^*(\mathbf{c})$ of (2.6), i.e., $x^* = \mathbf{G}(x^*; \mathbf{c})$. Then (2.9) becomes

$$\mathbf{Y}(\mathbf{c}; x^*(\mathbf{c})) \in \boldsymbol{\mathcal{F}}(x^*(\mathbf{c}), k). \tag{2.12}$$

At this point we assert a technical lemma:

$\{\mathbf{G}(x; \mathbf{c}) : x \in \mathcal{A}, \mathbf{c} \in \underline{\mathcal{C}}\}$ *is precompact in* $\boldsymbol{\mathcal{X}}$ *for every compact* $\mathcal{A} \subset \boldsymbol{\mathcal{X}}$.

and note that this and (2.11) provide precisely the two hypotheses needed for application of Lemma 2 of [2] to give the desired compactness result:

**LEMMA 2:**

$\mathcal{G}_* := \{x^*(\mathbf{c}) : \mathbf{c} \in \boldsymbol{\mathcal{C}}\}$ *is precompact in* $\boldsymbol{\mathcal{X}}$, *whence*
$\mathcal{K}_* := \overline{hull}\{x^*(t; \mathbf{c}) : t \in [0,1], \mathbf{c} \in \boldsymbol{\mathcal{C}}\}$ *is compact in* $\mathcal{X}$.

It is then not difficult to show that

**LEMMA 3:** *Suppose* $\mathbf{c} \rightharpoonup \bar{\mathbf{c}}$ *(weak convergence in* $L^2([0,T] \to \mathbb{R}^N)$*) and* $x \to \bar{x}$ *in* $\boldsymbol{\mathcal{X}}$. *Then* $\mathbf{Y}(\mathbf{c}, x) \rightharpoonup \mathbf{Y}(\bar{\mathbf{c}}, \bar{x})$ *(weak convergence in* $L^2([0,T] \to \mathcal{X})$*) and* $\mathbf{G}(x, \mathbf{c}) \to \mathbf{G}(\bar{x}, \bar{\mathbf{c}})$ *in* $\boldsymbol{\mathcal{X}}$ *— with* $x^*(\mathbf{c}) \to x^*(\bar{\mathbf{c}})$ *in* $\mathcal{G} \subset \boldsymbol{\mathcal{X}}$ *if* $\mathbf{c} \in \boldsymbol{\mathcal{C}}$.

At this point we introduce a definition of approximate solution for the system. Given $\varepsilon > 0$, we consider $x(\cdot) \in \boldsymbol{\mathcal{X}}$ together with functions $y(\cdot)$, $k(\cdot)$ on $[0,T]$ taking values in $\mathcal{X}$, $\mathcal{J}$. With $a, b > 0$ conveniently fixed, we couple

$$\left| x(t) - \left[ \mathbf{S}(t)\xi_0 + \int_0^t \mathbf{S}(t-s) y(s)\, ds \right] \right| \leq b\varepsilon \tag{2.13}$$

with the requirement that $y \in \boldsymbol{\mathcal{F}}(x, k, \varepsilon)$ and an approximate version of (1.3):

$$\begin{array}{lrcl} (i) & [x(t) + \mathcal{B}_{a\varepsilon}] \subset \mathcal{R}_j & \Rightarrow & k(t) \neq j \\ (ii) & k(t-) = i,\ k(t+) = j \neq i & \Rightarrow & |x(t) - \mathcal{S}_{i,j}| \leq \varepsilon. \end{array} \tag{2.14}$$

Definition: If these are satisfied, we say that we have an *$\varepsilon$–approximate solution of the system* (1.1), (1.3).

## 3. Principal Results

Our goals in this section are to prove two results, one on the stability of the solution notion here and one on existence. Our assumptions throughout are (H1) and (1.4).

**THEOREM 1:** *Let $x_n$ be any sequence of $\varepsilon$–approximate solutions of (1.1), (1.3) with $\varepsilon = \varepsilon_n \to 0$. Then there is a subsequence $\left(x_{n(m)}\right)$ converging to a mild solution $\bar{x}$ of (1.1), (1.3).*

**THEOREM 2:** *There is always a mild solution of (1.1), (1.3).*

Proof (of Theorem 1): From (2.13) and the $\mathbf{c}_n$ provided by Lemma 1, we have the representation: $x_n = \mathbf{G}(x_n, \mathbf{c}_n) + \hat{\rho}_n$ with $\hat{\rho}_n \to 0$ in $\boldsymbol{\mathcal{X}}$. A standard CMP estimate then gives $\|x_n - x^*(\mathbf{c}_n)\| \leq 2\|\hat{\rho}_n\| \to 0$. Using the weak compactness of $\boldsymbol{\mathcal{C}}$ in $L^2$ and Lemmas 2,3, there is a subsequence such that $\mathbf{c}_n \rightharpoonup \bar{\mathbf{c}}$ and $x_n \to \bar{x} = x^*(\bar{\mathbf{c}})$.

Since $x_n \approx x^*(\mathbf{c}_n) \in \mathcal{G}_*$, one can effectively use the compactness of $\mathcal{G}_*$ and $\mathcal{K}_*$ (once $\varepsilon$ is small enough). From (1.4-$i$) and the fact that each $\mathcal{S}_{i,j}$ is closed, one then sees that there is a uniform minimum separation of consecutive switching points for an $\varepsilon$–approximate solution whence, by equicontinuity of $\mathcal{G}_*$, a minimum time separation. Thus, the switching times are uniformly isolated. We can then define the convergence of $k_n(\cdot)$ by convergence of the first switching time, the second, etc. Clearly one can find a suitably convergent subsequence $k_n \to \bar{k}$ — with fixed choices of the sequence of modes since $\mathcal{J}$ is finite. Thus, we may suppose (for the $m$-th modal transition) that $k_n(t_{m,n}-) = i$, $k_n(t_{m,n}+) = j$, independent of $n$, and that $t_{m,n} \to \bar{t}_m$ as $n \to \infty$. Since $x_n(t_{m,n}) \to \bar{x}(\bar{t}_m)$, (2.14) gives (1.3) in the limit.

Finally, to show that $\bar{x}$ is a mild solution we must show that this $\bar{\mathbf{c}}$ satisfies $\mathbf{Y}(\bar{\mathbf{c}}; \bar{x}) \in \boldsymbol{\mathcal{F}}(\bar{x}, \bar{k}; \mathcal{I})$ when $\mathcal{I}$ lies in an interval of constancy for the mode so $k_n(\cdot) \equiv j$ on $\mathcal{I}$ for each (large) $n$. Note that

$$\mathcal{F}_j(x_n(s), \varepsilon_n) \subset \mathcal{F}_j(\bar{x}(s), \hat{\varepsilon}_n)$$

with $\hat{\varepsilon}_n = \varepsilon_n + \|x_n - \bar{x}\|_\infty \to 0$ so, once $\hat{\varepsilon}_n < \hat{\varepsilon}$,

$$\mathbf{Y}(\mathbf{c}_n; x_n) \in \boldsymbol{\mathcal{F}}_j(x_n, \varepsilon_n; \mathcal{I}) \subset \boldsymbol{\mathcal{F}}_j(x, \hat{\varepsilon}_n; \mathcal{I}) \subset \boldsymbol{\mathcal{F}}_j(x, \hat{\varepsilon}; \mathcal{I}). \tag{3.1}$$

Using Lemma 3 on $\mathcal{I}$ and the convexity of $\mathscr{F}_j(x,\hat{\varepsilon};\mathcal{I})$, taking the weak limit $\mathbf{c}_n \rightharpoonup \bar{\mathbf{c}}$ gives $\mathbf{Y}(\bar{\mathbf{c}},\bar{x}) \in \mathscr{F}_j(x,\hat{\varepsilon};\mathcal{I})$ for each such $\mathcal{I}$ and each $\hat{\varepsilon} > 0$ whence (2.12) holds. ∎

Proof (of Theorem 2): By Theorem 1, it is sufficient to construct $\varepsilon$–approximate solutions for arbitrarily small $\varepsilon > 0$. We do this in the form $x = x^*(\mathbf{c})$ to have the benefit of Lemma 2 — recursively constructing $k(\cdot) \equiv j_m$ and then $\mathbf{c}(\cdot) \equiv \mathbf{c}_m$ constant on each interval $\mathcal{I}_m := [t_{m-1}, t_m]$ for $m = 1,\dots,M$. Here, $t_m = m\delta$ with $\delta = T/M$, having chosen $M$ large enough that $\bar{\varepsilon}(\delta) < \varepsilon$ where $\bar{\varepsilon}(\delta)$ is the uniform modulus of equicontinuity for functions in $\mathcal{G}_*$. The point of the construction is to have both (2.14) and

$$y(t) := \sum_\nu c_{m,\nu}\psi_\nu(x(t)) \in \mathcal{F}_j(x(t),\varepsilon) + \mathcal{B}_{a\varepsilon} \qquad (j = j_m) \tag{3.2}$$

ae on each $\mathcal{I}_m$. We are given $\xi_0 = x(t_0)$ and $j_0$. In general, given $j_{m-1} =: j$ and $\xi_{m-1} =: \xi$, choose $\eta_m \in \mathcal{F}_j(\xi)$ and take $\mathbf{c}_m \in \underline{\mathcal{C}}$ such that $\eta_m = \Sigma_\nu c_{m,\nu}\psi_\nu(\xi)$. We now can obtain the partial solution $x(\cdot)$ extended to $[0, t_m]$ and so get $\xi' := \xi_m := x(t_m)$. Since $\bar{\varepsilon}(\delta) < \varepsilon$, we have $|x(t) - \xi| < \varepsilon$ on $\mathcal{I}_m$ so $\eta_m \in \mathcal{F}_j(x(t),\varepsilon)$ and also $|y(t) - \eta_m| < L\varepsilon$ on $\mathcal{I}_m$, giving (3.2) — provided $a \geq L$, else use this construction with a suitably smaller $\varepsilon$. If ever $x(t_*) \in \partial\mathcal{R}_j$ on $\mathcal{I}_m$, then (1.4-*ii*) ensures $x(t_*) \in \mathcal{S}_{j,j'}$ for some $j' \in \mathcal{J}$ and (2.14-*i*) permits us to switch at $t_m$, setting $j_{m+1} = j'$. Otherwise we may set $j_{m+1} = j_m$ and, since $|x(t) - \xi'| < \varepsilon$, (2.14-*ii*) will not be violated. On reaching $t = t_M = T$ we will have then constructed an $\varepsilon$–approximate solution of (1.1), (1.3). ∎

It is not difficult to see that when each $f_j$ is continuous the 'mild solutions' discussed here are precisely the solutions discussed in [3], [4]. Conversely, if one considers switching systems with switching surfaces 'moving to coincidence' (so in the limit the modes correspond to a single rhs with a discontinuity at the common switching surface), then limits of the switching system solutions are just the Filippov solutions. On the other hand, while switching system solutions are obtainable as limits for singular perturbations, the interchange of limits involved in using that in conjunction with our present formulation is not always valid.

# References


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