2/13/'91

The Dynamics of Some Systems with Hysteretic Nonlinearities [1]

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KEY WORDS: *Dynamics, periodic solutions, switching systems, thermostat, relay discontinuity, hysteresis.*

This is based on a talk given at the *Binational Workshop on Optimization and Nonlinear Analysis*, held at the Technion (Haifa, Israel) in March, 1990. It will appear in the Proceedings of that workshop.

1. Introduction

We are considering some systems for which the behavior exhibits occasional discontinuous changes of 'mode' in the course of normal (autonomous) evolution. Taking the view that such a change is 'triggered' by some *event* (a 'stopping time' in probabilists' terminology — e.g., first entry time of $x(t)$ into some set $\mathcal{S}$) we will be particularly concerned here with situations which exhibit a form of *hysteresis*: the *switching surface* for transition from mode A to mode

[1]This research has been partially supported by the National Science Foundation under the grant ECS-8814788 and by the U.S. Air Force Office of Scientific Research under the grant AFOSR-91-0008.

B does not coincide with that from mode B to mode A, giving an 'overlap region' in the state space $\mathcal{X}$ for which both A and B are possible and determination of the relevant mode becomes history dependent. We refer to such systems as *switching systems*. A paradigmatic example [13] is a *thermostat*, providing two behavioral modes: $A =$'furnace on' and $B =$'furnace off', controlled by a sensor with a 'dead zone'.

Systems of the sort we consider here may arise in three fairly distinct mathematical contexts:

Situation 1. Optimal control via modal switching with 'switching cost', as in [3],

Situation 2. 'Chattering' approximation to differential equations with discontinuous right hand sides, as in [6] ([25], in a control-theoretic context),

Situation 3. Singular perturbation theory, as in [18].

[We note that the first two of these are synthetic (design issues) while the last includes the physical modeling of systems involving such devices as thermostats and relays for which the switching system corresponds precisely to an 'outer' solution, ignoring unmodeled dynamics internal to the device itself and acting on a faster time scale.] These situations suggest some of the interesting problems one might consider for switching systems. At the present state of our knowledge, a complete structural theory for such problems is out of reach — even to the extent that such a theory is now available for differential equations *without* such discontinuity — and current research takes two forms: finding useful classes of problems which are structured enough to be amenable to analysis and, contrastingly, collecting examples of interesting behavior for special systems. We will concentrate here on the *periodicity problem* (existence of periodic solutions) but will also note some open questions regarding other 'dynamical systems' aspects of switching systems.

In this paper, based largely on [20], we indicate some of the results which have been obtained on the periodicity problem. Note that [20] is essentially an up-date of the presentation in [19], and includes some earlier material from there. We take advantage of the elapsed time since the Workshop itself to include some still more recent material: Theorem 12, which reflects Prof. Krasnosel'skiĭ's talk [9] at the Workshop and some new results [24] due to a student, Mr. W. Szczechla.

A general notion of an abstract *switching system* was introduced in [13], expanding on a model of a (physical) thermostatically controlled heating system; for further detail, see [15]. For our present purposes it will be sufficient to consider a restricted class of bimodal switching systems. Intuitively, one follows a *mode* given by a differential equation $\dot{x} = f_1(x)$ until a 'switching time' at which the solution to that differential equation would cross a *switching surface* $\mathcal{S}_1$; one then switches to the other mode $\dot{x} = f_2(x)$ and proceeds by similarly following that until hitting a corresponding switching surface $\mathcal{S}_2$, etc. We will make this description more precise in the next section. Clearly, this setting is essentially related to the treatment of 'relay nonlinearities' in [11] and also to the treatment of differential equations with discontinuous right hand sides in [6].

2. The 'thermostat nonlinearity'

The key to our present considerations is the *thermostat nonlinearity* or *standard relay* $\boldsymbol{\Theta}(\cdot)$. We view this as an input/output map for which the *input* $y(\cdot)$ is to be a scalar function of t and the *output* $\jmath(\cdot)$ will be Boolean-valued (i.e., $[\boldsymbol{\Theta}(y)](t) =: \jmath(t) \in \{\text{OFF, ON}\}$) satisfying the following *switching rules* **(SR)**:

1. $\jmath(\cdot)$ switches from ON to OFF [OFF to ON] at time τ, only if $y(\tau) = 1$ $[y(\tau) = 0]$;
2. if $y(t) > 1$ $[y(t) < 0]$ then $\jmath(t) = \text{OFF}$ $[\jmath(t) = \text{ON}]$.

Thus, the output becomes OFF when the input rises across the (normalized) threshhold 1 (turning the furnace OFF when it gets too hot) and later switches to ON when/if the input again falls below the lower threshhold 0, etc. A *regular output* (corresponding to a *regular input*) is one for which the switching times are discrete. The astute reader will note that these rules are *almost* sufficient to determine the map uniquely (given a specification of the initial output state), but that there remains one possibility for ambiguity: if the output state is ON and the input y rises exactly to the value 1 at time τ, then the first rule permits switching at τ yet the second rule will not apply to make such a switch mandatory if this might be a local maximum for $y(\cdot)$. Of course, such *anomalous inputs* (including also the symmetric possibility of a local minimum occurring with the value 0) are 'unusual' — of Wiener measure 0, for example — but the analysis of periodicity inextricably involves the consideration of this possibility. This discussion has implicitly viewed the input $y(\cdot)$ as a continuous function and that context is adequate for our present purposes. We do note that in the context of control one may wish to consider more general inputs of the form $y = [y_0 - u]/\delta$ where y_0 will be continuous but the control function u is not; a more general setting for this, with the relevant topology for y, is discussed in [17]. It will be convenient to map the Boolean domain {OFF, ON} to the set $\{0, 1\}$ so as to view the output $\jmath(\cdot)$ henceforth as numerically valued.

We choose to treat anomalous inputs by accepting *both* output alternatives, taking the map $\boldsymbol{\Theta}$ to be multi-valued with two (or more) legitimate output functions for such input functions. Alternatively, the ambiguity can be resolved by enforcing an arbitrary choice — e.g., "0 when possible" (the 'minimal selection') or the complementary 'maximal selection' or "switching only when that is mandatory" (which we may refer to as the 'heavy selection', following [1], [2]).

At this point we note three important properties of the map $\boldsymbol{\Theta}$.

THEOREM 1: *For the map $\boldsymbol{\Theta}$ as defined above we have:*

1. *rate independence: If $t \mapsto \tau$ is continuous and increasing (i.e., a reparametrization of time) and $\eta(\tau) \equiv y(t)$, then $[\boldsymbol{\Theta}(\eta)](\tau) \equiv [\boldsymbol{\Theta}(y)](t)$. [Obviously $\boldsymbol{\Theta}$ is also unaffected by reparametrization of the y-values so long as $0, 1$ remain fixed.]*

2. *continuity:* *If $\bar{y}$ is not an anomalous input, then $\boldsymbol{\Theta}$ is continuous at $\bar{y}$ in the sense that the switching times for $\boldsymbol{\Theta}(y)$ converge to those for $\boldsymbol{\Theta}(\bar{y})$ when $y \to \bar{y}$ uniformly on bounded sets (with the initial state of $\boldsymbol{\Theta}(\cdot)$ fixed).*

3. *monotonicity:* *If $y_1 \geq y_2$ for all t, then $-\boldsymbol{\Theta}(y_1) \geq -\boldsymbol{\Theta}(y_2)$.*

These properties apply to the multi-valued form of $\boldsymbol{\Theta}$ and also for each of the selection rules ('minimal', 'maximal','heavy') mentioned above. Note that the glb of the 'multi-valued' set of outputs (for a single continuous input) is the 'minimal' selection, etc. ■

3. A general model of switching systems

The first of the specific 'ingredients' needed now to describe a switching system will be a *state space* $\mathcal{X}$, which we take to be a Banach space, and a continuous *sensor functional* $\Lambda : \mathcal{X} \to \mathbb{R}$. Note that, despite this terminology, $x(t) \in \mathcal{X}$ will not be a *state* since it is not an adequate descriptor for the evolution of the system: the true state evolution is given by the *pair* of functions $[x(\cdot), \jmath(\cdot)]$ where the *index function* is given by

$$\jmath(\cdot) := \boldsymbol{\Theta}(y) \qquad y(t) := \Lambda(x(t)). \tag{3.1}$$

The switching times (jumps in $\jmath(\cdot)$) obviously occur when $y(t)$ hits 0 or 1 so

$$x(t) \in \begin{cases} \mathcal{S}_0 := \{\xi \in \mathcal{X} : \Lambda(\xi) = 1\} & \text{for switching from } \jmath = 0 \text{ to } \jmath = 1, \\ \mathcal{S}_1 := \{\xi \in \mathcal{X} : \Lambda(\xi) = 0\} & \text{for switching from } \jmath = 1 \text{ to } \jmath = 0. \end{cases} \tag{3.2}$$

The final ingredient needed for the switching system is a pair of *modes* $\boldsymbol{\pi}_0$, $\boldsymbol{\pi}_1$ which are semidynamical systems on $\mathcal{X}$. For the moment we think of each $\boldsymbol{\pi}_j$ as given by an (autonomous) differential equation

$$\dot{x} = f_j(x). \tag{3.3}$$

Our only assumption about f_j need be that (3.3) does, indeed, determine a semidynamical system $\boldsymbol{\pi}_j$ on $\mathcal{X}$. In particular, one plausible form for (3.3) would be a quasilinear equation

$$\dot{x} = \mathbf{A}x + \hat{f}_j(x) \tag{3.4}$$

with $\mathcal{X}$ infinite-dimensional, $\mathbf{A}$ generating a C_0 semigroup $\mathbf{S}(\cdot)$ on $\mathcal{X}$ — i.e., (3.3) could, in fact, be a partial differential equation. We do assume here that $\mathbf{S}(\cdot)$ is exponentially stable,

$$\|\mathbf{S}(t)\| \leq Me^{-\omega t} \tag{3.5}$$

for some $M, \omega > 0$; this, e.g., excludes consideration of 'undamped wave equation' contexts which would require somewhat different technical tools. We also assume each $\hat{f}_j : \mathcal{X} \to \mathcal{X}$ is uniformly Lipschitzian (constant L).

The switching system is now given by considering (3.3) with $j = \jmath(t)$ as a coupling with (3.1). Taking an integral equation formulation, we then have

$$x(t) = x(t_0) + \int_{t_0}^{t} f_{\jmath(s)}(x(s))\, ds \tag{3.6}$$

with (3.1) and specification of initial data $[x(t_0),\, \jmath(t_0)]$; In the case of (3.4), instead of (3.6) we would have

$$x(t) = \mathbf{S}(t - t_0)x(t_0) + \int_{t_0}^{t} \mathbf{S}(t - s)\hat{f}_{\jmath(s)}(x(s))\, ds. \tag{3.7}$$

When considering a *regular solution* $x(\cdot)$, i.e., one for which $\jmath(\cdot)$ is a regular output of $\boldsymbol{\Theta}$ — piecewise constant with only finitely many switching times in any bounded interval — we have $\jmath(\cdot) = \mathit{constant} = j$ on intervals $(\tau, \hat{\tau})$ giving

$$x(t) = \boldsymbol{\pi}_j(t - \tau; x(\tau)) \qquad \text{for } \tau \le t \le \hat{\tau} \tag{3.8}$$

so one is considering a succession of independent initial value problems for the (alternating) modes, coupled by the continuity of $x(\cdot)$: solutions consist of linked pieces, each a segment of a trajectory for a mode. Note that an anomalous input $y(\cdot)$ for $\boldsymbol{\Theta}$ can arise precisely if there is a point ξ_* in a switching surface lying in the interior of some modal trajectory segment which does not cross the surface into the 'forbidden region' (i.e., in *mode* 1 does not enter $\mathcal{R}_1 := \{\xi \in \mathcal{X} : \Lambda(\xi) > 1\}$, etc.); such a ξ_* is an *anomalous point* for the system. A *normal solution* is a regular solution whose trajectory contains no anomalous points.

Certainly a sufficient condition to obtain regular solutions would be that there is a minimum interswitching interval h_*: solutions for the initial value problem for *mode* 1, starting at any point in $\mathcal{S}_0$ takes at least time h_* to reach any point in $\mathcal{S}_1$ and symmetrically.

THEOREM 2: *[15] Under rather general conditions, global solutions always exist and one has a form of well-posedness: The limit of solutions is itself a solution — with true well-posedness near normal solutions. [This well-posedness includes perturbation of the system, itself, as well as of the initial data.]* ■

We will usually assume that

$$\text{Each 'forbidden region' } \mathcal{R}_j \text{ is a global attractor for } \mathit{mode}\ j \tag{3.9}$$

so that one necessarily switches infinitely often. There is then an obvious pair of maps, the *'first encounter' maps*, $\mathbf{x}_0 : \xi \mapsto \xi'$ — defined by starting with $\xi \in [\mathcal{X} \setminus \mathcal{R}_1]$ and following *mode* 0 until the first encounter with $\mathcal{S}_0$ at ξ' — and $\mathbf{x}_1 : \xi' \mapsto \xi''$, defined correspondingly for *mode* 1; (3.9) ensures that the requisite encounters occur. Clearly, any fixpoint $\xi_* \in \mathcal{S}_1$ of $\mathbf{x} := \mathbf{x}_1 \circ \mathbf{x}_0$ gives a periodic solution since, after the evolution described above one would again switch to *mode* 0 and repeat the sequence; indeed, this is a periodic solution of a special sort, a *two-phase periodic solution* with the simplest possible structure. It is not difficult to see that $\mathbf{x}$ is continuous at ξ if neither ξ' nor ξ'' is an anomalous point.

THEOREM 3: *[19] If one can find a (topologically suitable) compact invariant set $\mathcal{K}$ for which the evolution does not encounter anomalous points, then there exists a two-phase periodic solution.*

PROOF: This is an application of the Schauder Fixpoint Theorem to $\mathbf{x}$ on $\mathcal{K}$, comparable to the standard argument using the Poincaré map. ■

In contrast, we note the following:

EXAMPLE 4: *[19] There is a switching system (with, e.g., $\mathcal{X} = \mathrm{I\!R}^2$) for which one can show that there is no periodic solution at all — although the 'nice' topological hypotheses are verified (e.g., there is a compact, convex invariant set and each of the flows is smooth).*

We emphasize the importance to the theory of being able to avoid anomalous points since the construction here depends on the existence of a (single) anomalous point to permit a reformulation of the dynamics as equivalent to an irrational rotation of a circle. Cf., also, [22] for some variations and extensions of this.

Useful for the compactness requirement of Theorem 3 is the following, noting that periodic solutions will necessarily be defined on all of $\mathrm{I\!R}$.

THEOREM 5: *[20] Consider (3.4) on $\mathrm{I\!R}$ under the assumptions above; suppose $L < M/\omega$ (M, ω as in (3.5)). With mild boundary conditions as $t \to -\infty$, this has a unique solution for each $\jmath \in \mathcal{J} := L^\infty(\mathrm{I\!R} \to \{0,1\})$, satisfying the integral equation*

$$x(t) = \int_{-\infty}^{t} \mathbf{S}(t-s)\hat{f}_{\jmath(s)}(x(s))\,ds. \tag{3.10}$$

Further,

$$\mathcal{K}_0 := \{x(t) : t \in \mathrm{I\!R} \text{ using (3.10) with } \jmath \in \mathcal{J}\} \tag{3.11}$$

is precompact in $\mathcal{X}$ and the set $\boldsymbol{\mathcal{X}} := \{x(\cdot) : \text{(3.10) with } \jmath \in \mathcal{J}\}$ is (uniformly in t) equicontinuous, hence is precompact in $C(\mathrm{I\!R} \to \mathcal{X})$, using the topology of uniform convergence on bounded sets. Finally, for every solution $x(\cdot)$ of (3.7) (i.e., starting with $x(t_0) = \xi$) one has $\|x(t) - \mathcal{K}_0\| \to 0$ as $t \to \infty$.

PROOF: The solvability of (3.10) for fixed $\jmath$ is a standard 'Contraction Mapping' argument. For the precompactness of $\mathcal{K}_0$ we apply the Second Lemma of [14], showing that it is *totally* *bounded*: it can be covered by finitely many ε-balls for arbitrary $\varepsilon > 0$. This now gives uniformity (in $t_0 \in \mathbb{R}$, $\jmath \in \mathcal{J}$) of the convergence $\mathbf{S}(t-t_0)x(t_0) \to x(t_0)$ as $t \to t_0$ in (3.7) and so, easily, the uniform equicontinuity; precompactness of $\boldsymbol{\mathcal{X}}$ then follows by Arzelà–Ascoli. For the final assertion, note the exponential convergence $\|x(t) - \hat{x}(t)\|$ for solutions of (3.7) under our present assumptions and take $\hat{x}$ to be the solution of (3.10) for some extension of $\jmath$ to $s < t_0$ so it is also a solution of (3.7). ■

Under the assumptions of Theorem 5 all solutions of (3.7) are regular.

4. Linear switching systems

In this section we consider a restricted class of switching systems with additional special structure: *linear* switching systems. The modes are now given by

$$\dot{x} = \mathbf{A}x + \jmath z \tag{4.1}$$

where, as in (3.4), $\mathbf{A}$ generates a C_0 semigroup $\mathbf{S}(\cdot)$ satisfying (3.5); we are here taking $\hat{f}_j := jz$ with $z \in \mathcal{X}$ fixed so Theorem 5 applies. Also, the sensor functional Λ in (3.1) is now to be linear ($\Lambda \in \mathcal{X}^*$). [It may be noted that the assumptions made here can be weakened.] We will normalize Λ so that $\langle \Lambda, -\mathbf{A}^{-1}z \rangle = 1$ and will then modify (3.1) to

$$\jmath(\cdot) := \boldsymbol{\Theta}([y-u]/\delta) \qquad y(t) := \langle \Lambda, x(t) \rangle, \tag{4.2}$$

admitting a constant *offset* u and *zone width* $\delta > 0$. The switching surfaces are thus a pair of parallel hyperplanes in $\mathcal{X}$ and — noting that $-\mathbf{A}^{-1}z$ is the stable rest point for *mode* 1 as 0 is for *mode* 0 — we observe that in the present context (3.9) takes the form

$$0 < u < 1 - \delta. \tag{4.3}$$

Remark 6: It was shown in [13] that anomalous points cannot occur for linear switching systems when $\mathcal{X} = \mathbb{R}^2$ so Theorems 3, 5 apply to ensure existence of a two-phase periodic solution in this case. B. Stoth [23] has analyzed the stability of these two-phase periodic solutions in some detail. More generally, we note that one may be able to analyze the stability of a two-phase periodic solution by linearization [20]. Such a solution $x(\cdot)$ will have the form: start at $\xi_0 \in \mathcal{S}_0$ and proceed in *mode* 1 for a time τ_1, arriving at $\xi_1 \in \mathcal{S}_1$, and then proceed in *mode* 0 for a time τ_0, arriving back at ξ_0. We will suppose that ξ_0, ξ_1 are in the domain of $\mathbf{A}$ and that neither is an anomalous point so (for $j = 0, 1$) $d_j := [\mathbf{A}\xi_j + jz]$ is not in $\mathcal{N}$, the nullspace of Λ. In this case we can define $\mathbf{P}_j : \mathcal{X} \to \mathcal{X}$ as the projection onto the span of $d = d_j$ along $\mathcal{N}$, i.e., $\mathbf{P}_j : \xi \mapsto \frac{\langle \Lambda, \xi \rangle}{\langle \Lambda, d \rangle} d$. Then ([20]; compare [8]) $x(\cdot)$ is (locally) exponentially stable if the operator

$$\mathbf{M} := [1 - \mathbf{P}_0]\mathbf{S}(\tau_0)[1 - \mathbf{P}_1]\mathbf{S}(\tau_1) : \mathcal{N} \to \mathcal{N} \tag{4.4}$$

has spectral norm < 1 (i.e., has a contractive iterate). Note that this condition is robust under small perturbations of the system. Obviously, a similar analysis is possible for the stability of periodic solutions having more complicated structure. ■

It will be convenient at this point to introduce the *impulse response function* $\varphi(\cdot)$ for this switching system, given by

$$\varphi(s) := \langle \Lambda, \mathbf{S}(s)z \rangle \qquad \text{for } s \geq 0. \tag{4.5}$$

LEMMA 7: *Under the assumptions above, one has*

1. *normalization:* $\int_0^\infty \varphi(s)\, ds = 1$;
2. *exponential decay:* $\varphi(s) \leq \tilde{M} e^{-\omega s}$;
3. *continuity:* $|\varphi(t+h) - \varphi(t)| \leq \tilde{M} e^{-\omega t} \|\mathbf{S}(h)z - z\|$.

■

For solutions on $\mathbb{R}$ of the switching system (e.g., periodic solutions), one has the *representation formula*

$$y = \varphi * \jmath \qquad \text{with } \jmath = \boldsymbol{\Theta}([y-u]/\delta) \tag{4.6}$$

('$*$' denoting *convolution*) and $x(\cdot)$ can then be recovered from $\jmath(\cdot)$ by

$$x(t) = \int_{-\infty}^{t} [\mathbf{S}(t-s)z] \jmath(s)\, ds = \int_0^{-\infty} [\mathbf{S}(r)z] \jmath(t-r)\, dr \tag{4.7}$$

with convergence ensured by (3.5); in particular, periodic solutions of (4.6) correspond precisely to periodic solutions of the switching system. Quite generally, the dynamical behavior of the switching system is determined by the impulse response function φ and the parameters u, δ.

In terms of φ, u, δ, we have the following sufficient condition for the existence of a two-phase periodic solution:

THEOREM 8: *[19] Under our hypotheses, including (4.3), suppose there is some* $t_* \geq 0$ *and* $\alpha > 0$ *such that:*

1. $\varphi(\cdot)$ *is (strictly) decreasing to* 0 *on* (t_*, ∞) *and*
2. $\int_t^\infty \varphi(s)\, ds \geq \alpha$ *for* $0 < t < t_*$.

Then there exists a two-phase periodic solution of (4.6) — and so of the switching system — whenever δ is close enough to 1.

PROOF: From a left-infinite sequence $\mathbf{s} = \{\dots, s_{-1}, s_0\}$ of 'possible interswitching intervals' (each $s_k > 0$), define $\jmath(\cdot)$ on $\mathbb{R}_-$ (fixing $\jmath(s_0-) = 0$), and consider the map which adjoins to $\mathbf{s}$ on the right the next two interswitching intervals. The compact set $[a, b]^\infty$ will be invariant under this map for suitably estimated $0 < a < b$ and for δ close enough to 1 one will have $a > t_*$; the conditions then ensure that no anomalous points arise so the map is continuous. One now can apply the Schauder Theorem. ■

The condition on δ appears in Theorem 8 in order to concentrate on the 'tail' of $\varphi(\cdot)$. This does simplify the analysis but is somewhat unfortunate in excluding a particularly interesting case, the limit $\delta \to 0$ in which the switching system becomes a 'differential equation with discontinuous right hand side' [6].

On considering Remark 6, Theorem 8, and other positive results (as, e.g., [5], [8], [12], along with the computational investigation in [7] for a closely related model) it seems plausible to conjecture that linear switching systems *always* have two-phase periodic solutions. Indeed, we recall that Prof. Krasnosel'skiĭ, in his talk [9], described a 'general' existence result for linear switching systems with periodic forcing in the context of ordinary differential equations

$$\mathbf{L}y + \boldsymbol{\Theta}(y) = g(t) \tag{4.8}$$

where $\mathbf{L}$ is a 'suitable' n^{th}-order differential operator; cf. [10], [11] which, however, omit specific information of the statement and proof; compare Theorem 12, below. Writing (4.8) as a 1^{st}-order system, it takes the form of (4.1) with an external input included and, for a dense set of matrices $\mathbf{A}$, the converse is true. It was noted at the Workshop that, as a consequence of the Müntz-Szász Theorem, the set of finite linear combinations of decaying exponentials (which arise as the impulse response functions for such finite dimensional systems) is dense for a relevant topology and this, together with the well-posedness of Theorem 2, permits a reduction of the general case to this if one could suitably bound the period — and if 'enough' operators $\mathbf{L}$ would be 'suitable' for application of Krasnosel'skiĭ's result. Recently, however, a student of mine, Witold Szczechla, has shown that this conjecture is, in fact, *false.*

EXAMPLE 9: *[24] One can construct a family of linear switching systems, subject to (4.3) with $\delta \to 1$, for each of which there cannot exist any two-phase periodic solution.*

The analysis still leaves open the possible existence of periodic solutions of more complicated structure.

REMARK 10: We also mention a complementary positive result. Under (3.9), each trajectory for *mode* j necessarily encounters the switching surface $\mathcal{S}_j$ but may do so more than once. Suppose we drop the second switching rule and now require only that (e.g., as a 'control action') we intervene to select some one of these possibilities for switching; one might refer to this setting as a *switching system with choice*. Szczechla has shown [24] that, with this modification, two-phase periodic solutions always exist for linear switching systems; the argument extends to a class of nonlinear switching systems as well. ■

5. Multilinear switching systems

The general framework of switching systems can be generalized in a variety of ways but in this section we consider specifically a generalization of the linear setting of the previous section to consider the effect of several relays, acting independently, and with the possibility of an external input $g(t)$. We now have K sensor functionals $\Lambda_1, \ldots, \Lambda_K$ given in $\mathcal{X}^*$ and $z_1, \ldots, z_K$ given in $\mathcal{X}$; instead of (4.1) we have

$$\dot{x} = \mathbf{A}x + \jmath_1 z_1 + \cdots + \jmath_K z_k + g(t) \tag{5.1}$$

and (4.2) is replaced by

$$\jmath_k(\cdot) := \boldsymbol{\Theta}(y_k - u_k) \qquad y_k(t) := \langle \Lambda_k, (x(t)) \rangle \tag{5.2}$$

for $k = 1, \ldots, K$. [No normalization is imposed on the Λ_k so there is no consideration here of zone widths.] The 'independent action' of the relays means that the sensor functional and the jump in the right hand side of (5.1) for switching of the k^{th} relay $\jmath_k$ is entirely independent of the rest of the *index vector* $\jmath = [\jmath_1, \ldots, \jmath_K]$ (compare [21]). We do note, however, that a new kind of anomalous point can arise when one has several sensor functionals. At the intersection of switching surfaces one will not, in general, have a well-defined evolution since nearby trajectories will encounter one or the other of the switching surfaces first (although one still obtains a well-defined evolution if each continuation assures a proximal encounter with the other surface).

We will continue to assume (3.5) and will need the following.

LEMMA 11: *Suppose the input $g : \mathbb{R} \to \mathcal{X}$ is uniformly locally integrable, i.e., $\int_t^{t+1} \|g\| \leq M$. Then*

$$x^g(t) := \int_{-\infty}^t \mathbf{S}(t-s) g(s)\, ds \tag{5.3}$$

is continuous and uniformly bounded.

We will be particularly concerned with periodic g in which case the bound is automatic and $x(\cdot)$ is uniformly continuous. We also note a modified version of Theorem 5:

THEOREM 5': *[20] For $\jmath = [\jmath_1, \ldots, \jmath_K] \in \mathcal{J}^K$, define*

$$x_\jmath(t) = \int_{-\infty}^t \mathbf{S}(t-s)[j_1(s) z_1 + \cdots + j_K(s) z_K]\, ds. \tag{5.4}$$

Then the set

$$\mathcal{K} := \{x_\jmath(t) : t \in \mathbb{R},\ \jmath \in \mathcal{J}^K\} \tag{5.5}$$

is precompact in $\mathcal{X}$ *and* $\boldsymbol{\mathcal{X}} := \{x_\jmath(\cdot) : \jmath \in \mathcal{J}^K\}$ *is (uniformly in* t*) equicontinuous, hence is precompact in* $C(\mathbb{R} \to \mathcal{X})$*, using the topology of uniform convergence on bounded sets. Finally, under the hypotheses of Lemma 11, for every solution* $x(\cdot)$ *of (5.1) one has* $[x(t) - x^g(t)] \to \mathcal{K}$ *as* $t \to \infty$*.*

In the present multilinear context the impulse response function φ will be matrix valued with entries

$$\varphi_{\ell,m}(s) := \langle \Lambda_m, \mathbf{S}(s) z_\ell \rangle \qquad \text{for } \ell, m = 1, \ldots, K \tag{5.6}$$

to which Lemma 7 applies. The *representation formula* is now

$$y = \jmath * \varphi + y^g \qquad \text{with } \jmath_m(\cdot) := \boldsymbol{\Theta}(y_m - u_m) \text{ for } m = 1, \ldots, K \tag{5.7}$$

where, of course, $y = [y_1, \ldots, y_K]$ and $y^g = [y_1^g, \ldots, y_K^g]$ with x^g given by (5.3) and setting $y_\ell^g(t) := \langle \Lambda_\ell, (x^g(t)) \rangle$ for $\ell = 1, \ldots, K$. We now recover $x(\cdot)$ from $\jmath(\cdot)$ by

$$x(t) = x^g(t) + \int_{-\infty}^{t} \sum_{k=1}^{K} [\mathbf{S}(t-s) z_k] \jmath_k(s)\, ds. \tag{5.8}$$

The next Theorem is a generalization of the result [9], [10] mentioned above for (4.8). The proof sketched below clearly differs in detail from that intended in [10] but is necessarily essentially similar, being indebted to [9] for the idea of using the property 3 of Theorem 1 and the fixpoint theorem for order-preserving maps to avoid the necessity of considering continuity of the map.

THEOREM 12: *Assume (3.5) and let* g *be any locally integrable* $\mathcal{X}$*-valued periodic function with period* P*. Suppose the impulse response function* φ *is nonpositive:* $\varphi_{\ell,m}(s) \le 0$ *for* $s > 0$ *and* $\ell, m = 1, \ldots, K$*. Then the multilinear switching system (5.1), (5.2) has a periodic solution with period* P *which does not involve anomalous points.*

PROOF: Let $\boldsymbol{\mathcal{Y}}$ be the set of P-periodic functions with the bound and uniform modulus of continuity obtained from $\boldsymbol{\mathcal{X}}$ in Theorem 5' on applying the switching surfaces, with the obvious partial order. This is a *complete* poset in that every nonempty subset of $\boldsymbol{\mathcal{Y}}$ has a *glb* and *lub* in $\boldsymbol{\mathcal{Y}}$. Consider the map

$$\mathbf{Y} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Y}} : \tilde{y} \mapsto \jmath := -\boldsymbol{\Theta}_*(\tilde{y} - \tilde{u}^g) \mapsto \jmath * [-\varphi] \tag{5.9}$$

so $y_* = \tilde{y}_* + y^g$ is a periodic solution of (5.7) for any fixpoint $\tilde{y}_*$ of $\mathbf{Y}$; here, $\boldsymbol{\Theta}_*$ is the vector form of the 'maximal selection' version of $\boldsymbol{\Theta}$ and $\tilde{u}_k^g := u_k - y_k^g$. Theorem 5' ensures the invariance of $\boldsymbol{\mathcal{Y}}$; the first factor of $\mathbf{Y}$ is order-preserving by Theorem 1 (property 3) and the second by the assumed nonpositivity of φ. Thus, there is a least fixpoint $\tilde{y}_*$ by Theorem 4.15 of [4], giving a periodic solution of (5.7) and so of (5.1), (5.2). This involves no anomalous points since, otherwise, there would be another, smaller, output $\hat{\jmath}_-$ giving $\hat{\jmath}_- * \varphi =: \tilde{y}_- < \tilde{y}_*$ and applying the fixpoint theorem on $\{\tilde{y} \in \boldsymbol{\mathcal{Y}} : \tilde{y} \leq \tilde{y}_-\}$ would contradict the minimality of $\tilde{y}_*$. ■

The requirement of nonpositivity for the impulse response function in Theorem 12 is clearly inconsistent with (3.9). For perspective, we note that our orientation, colored by the 'homeostatic' thermostat interpretation and related applications, has been to view the imposition of (3.9) as typical. A contrasting orientation, however, might correspond to the consideration of bi-stable devices, as 'flip-flop' circuits, for which the action of the relay would, indeed, be reversed as here. Note that for the autonomous case, $g \equiv constant$, the result is trivial: one then has a stable rest point without switching. More generally, the periodic solution need not involve switching when $g(\cdot)$ has small range.

Acknowledgments

I would like to express my appreciation to the organizers, and especially to Prof. A. Ioffe, for the opportunity to participate in such an exceptionally stimulating Workshop and to thank Z. Artstein, A. Ben-Tal, J. Greenberg, and M. Teboulle for their help in making this possible.

Acknowledgment is also due to the United States National Science Foundation and to the U.S. Air Force Office of Scientific Research for their support of this research under the grants ECS-8814788 and AFOSR-101-0008, respectively. Finally, acknowledgment is due to UMBC for its support.

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7. Some open problems

It is clear from the above that the long term behavior of switching systems may involve various possibilities (e.g., periodic solutions of several forms, chaotic behavior, etc.) with a complicated structure for the dependence on parameters for any detailed description of the asymptotics. In particular, despite the Examples 4 and 9, it remains entirely open as to whether *linear* switching systems would always have periodic solutions. One would be particularly interested in the existence of periodic solutions for small zone widths ($\delta \to 0$) and, if these should be stable, in the extent of their basins of attraction.

A detailed understanding of this complicated structure seems far off. On the other hand, one might hope that for many purposes it would be sufficient to have a less detailed description involving only certain asymptotic averages. For example, one would expect to be interested in the asymptotic mean and variance of the sensor functional

$$\begin{aligned} \bar{y} &:= \lim_{T\to\infty} \frac{1}{T} \int_0^T y\, dt, \\ \sigma_y^2 &:= \lim_{T\to\infty} \frac{1}{T} \int_0^T [y - \bar{y}]^2\, dt \end{aligned}$$

provided, of course, these limits exist. Similarly, one might be interested in the asymptotic statistics for the lengths of interswitch intervals (separately for each mode). E.g., each 'mean interval length' $\bar{\tau}_j$ would be given by

$$\bar{\tau}_j := \lim_{T\to\infty} \left\{ \frac{\text{time spent in mode } j}{\text{number of intervals in mode } j} : \text{ during } [0,T] \right\}. \tag{7.1}$$

[For a two-phase problem, $1/[\bar{\tau}_+ + \bar{\tau}_-]$ would then be the *average switch rate*.] One could also ask for the *asymptotic measure* μ_y

$$\mu_y(\mathcal{A}) := \lim_{T\to\infty} \{ \text{time during } [0,T] \text{ for which } y \in \mathcal{A} \}/T \tag{7.2}$$

for suitable sets $\mathcal{A} \subset \mathbb{R}$ or for the asymptotic distribution for interval lengths, etc.

Such averages, assuming they are well-defined at all, apparently depend on the particular solution of (4.1), i.e., on the initial data. At this point, it is only an attractive conjecture that (except possibly for a negligible set of initial values) these are indeed well-defined and, further, that they are essentially independent of the particular solution. If verified, the conjectured ergodicity would mean that (at the level of this description) the long term behavior would depend only on the system parameters: almost all solutions of any particular system (4.1) would be asymptotically alike. Assuming this, one would then like to know that the averages vary smoothly with perturbation of the system parameters so one might, for example, effectively use such averages in performance measures and seek to optimize the system parameters.

Differentiability would then permit the determination of first order necessary conditions for optimality.