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SWITCHING SYSTEMS AND PERIODICITY[1]

Thomas I. Seidman
Department of Mathematics and Statistics
University of Maryland Baltimore County
Baltimore, MD 21228

ABSTRACT:
We consider a class of bimodal systems which alternate modes on hitting 'switching surfaces': the canonical model is a physical thermostat. A sufficient condition is obtained for the existence of *periodic* solutions. The significance for this theory of certain *anomalous points* on the switching surface is shown by an example with *no* periodic solutions. Further discussion is presented for the important case of *linear switching systems*, including a new existence theorem.

Key words: *switching system, thermostat, periodicity, existence.*

1 Introduction

The intention of this paper is to present the periodicity problem for switching systems and to indicate some of the results which have been obtained.

The abstract notion of *switching system* was introduced in [7] to generalize a model of a (physical) thermostat; for further detail, see [7], [8], [9], [10]. Here it will be sufficient to consider a much more intuitive restricted class of switching systems, including those which have been used for thermostat models. Thus, a switching system, here, will be an autonomous bimodal system in which one follows a *mode* given by a differential equation: $\dot{x} = f_1(x)$ until a *switching time* at which the trajectory hits the boundary $\mathcal{S}_1 = \partial\mathcal{R}_1$ of a *forbidden region* $\mathcal{R}_1$ and one then proceeds by similarly following the other mode $\dot{x} = f_2(x)$ until hitting the other *switching surface* $\mathcal{S}_2 := \partial\mathcal{R}_2$, etc. We will be somewhat more precise in the next section.

The initial stimulus to these investigations came from a conversation with K. Glashoff and J. Sprekels regarding the model [3]. Their computational experience suggested that, for any initial state, one rapidly settled down to a periodic regime, cycling between the two modes (ON/OFF). Direct experience with (physical) thermostats also suggests such behavior. An attempt to demonstrate analytically the

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existence of *exactly* periodic solutions for the rather different model considered in [3] both succeeded and failed: the model supports certain (physically spurious) constant solutions due to a convexification and it was not clear how to show existence of *nontrivial* periodic solutions. After developing in [7] the switching system model, the concern for periodicity remains. Also, compare [1].

In the next section we discuss topologically general results for periodicity of switching systems. The principal thrust of this material is the importance to the theory of considering certain *anomalous points* on the switching surfaces. This is demonstrated, in particular, by the construction of an example of a switching system on $\mathbb{R}^2$ for which there exists no periodic solution at all although all the 'nice' topological hypotheses are verified (e.g., there is a compact, convex invariant set and the flows are smooth) except that there is a single anomalous point.

For a thermostat, the modes corresponding to [furnace ON] and [furnace OFF] are linear (given by the same pde with different inhomogeneities)and the switching surfaces $\mathcal{S}_\jmath$ are defined by the temperature set-points (evaluated at the sensor). This will be our model example of the important subclass of *linear switching systems*, discussed in Section 3.

2 General Results

Our aim in this section is to consider the extent to which such standard techniques of analysis of periodicity for odes as the Schauder Fixpoint Theorem apply to 'general' (nonlinear) switching systems. Since we are considering autonomous systems, the 'natural' map to consider is that from an initial point to the position at a subsequent switching time.

For our present purposes we consider a somewhat less general definition of a switching system than is treated in [8], thus avoiding some technical details while still including the principal problems of interest for our discussion of the periodicity problem. Thus, we assume we have two *modes*, given by continuous semi-flows $\boldsymbol{\pi}_k$ indexed by $k = 1, 2$. (The formulation in terms of the solution maps $\boldsymbol{\pi}_k$ rather than differential equations simplifies consideration of hypotheses on the equations; the state space $\mathcal{X}$ will be a Banach space which we may think of as $\mathbb{R}^d$ but which may be infinite-dimensional for, e.g., an example such as the thermostat problem where the equation is actually a pde.) We introduce, also, a pair of *forbidden regions* $\mathcal{R}_k$ — open sets in $\mathcal{X}$ with disjoint closures whose boundaries will be the admitted *switching surfaces*. Thus, the quadruple $\sum = [\boldsymbol{\pi}_k, \mathcal{R}_k]_{k=1,2}$ 'is' a switching system.
DEFINITION: A **solution** of the switching system $\sum$ is then any function pair

$$[x, \jmath] : \mathbb{R}^+ \to \mathcal{X} \times \{1, 2\} \tag{2.1}$$

satisfying the following set of rules:

- On an interswitching interval $[t_1, t_2)$ for which $\jmath(\cdot)$ has the constant value k we have $x(\cdot)$ following the mode $\boldsymbol{\pi}_k$, i.e., $x(t) = \boldsymbol{\pi}_k(t - t_1)x(t_1)$ for $t_1 \le t < t_2$.

- One is forbidden to have $\jmath(t) = 1$ if $x(t) \in \mathcal{R}_1$ and, similarly, $\jmath(t) = 2$ is forbidden if $x(t) \in \mathcal{R}_2$.

- $\jmath(\cdot)$ is piecewise constant (say, left-continuous) with a *switch* permitted *only* at $\partial\mathcal{R}_\jmath$ — i.e., one may have $\jmath(\tau-) = 1$ with $\jmath(\tau) = \jmath(\tau+) = 2$ only if $x(\tau) \in \partial\mathcal{R}_1$ and similarly for switching from $\jmath = 2$ to $\jmath = 1$.

We assume here that one is always assured of global solutions with separated switching times; see [8] or [10] for more detail as to sufficient hypotheses for this although this will be clear for the settings we now consider.

Implicit in this set of rules is a possible anomaly: if a trajectory for $\boldsymbol{\pi}_k$ (starting at a point $\xi_0 \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$) could first intersect $\partial\mathcal{R}_k$ at a time τ and then continue without actually entering the open set $\mathcal{R}_k$, then these rules would, somewhat ambiguously, accept as solutions *both* those which switch at this τ and also those which do not! This indeterminacy is necessary to preserve as an underlying principle that: THE LIMIT OF SOLUTIONS IS AGAIN A SOLUTION.

DEFINITION: We call a point $\xi \in \partial\mathcal{R}_k$ **anomalous** (for $\boldsymbol{\pi}_k$) if there is some $\xi_0 \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$ and $\tau > 0$ such that: $\boldsymbol{\pi}_k(\tau)\xi_0 = \xi$; $\boldsymbol{\pi}_k(t)\xi_0 \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$ for $0 < t < \tau$; there are sequences $\xi_\nu \to \xi_0$ and $t_\nu \to \tau$ with $\boldsymbol{\pi}_k(t_\nu) \in \mathcal{R}_k$; there is also a sequence $\xi'_\nu \to \xi_0$ and $\varepsilon > 0$ for which $\boldsymbol{\pi}_k(t)\xi'_\nu \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$ for $0 < t < \tau + \varepsilon$.

Thus, ξ is called anomalous if it is a (potential) switching point on a solution which is the limit both of a sequence of solutions for which (nearby) switching is mandatory *and also* of a sequence of solutions for which there cannot be any such switching— these are the points at which switching is optional. Even if the indeterminacy at such points were to be resolved by some (arbitrary) 'selection principle', the existence of anomalous points is of fundamental significance to our theory: see Example 2.1 below.

DEFINITION: Given $k = 1$ or 2 and some $\xi \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$, suppose there is a $\tau > 0$ such that: $\boldsymbol{\pi}_k(\tau)\xi \in \partial\mathcal{R}_k$ with $\boldsymbol{\pi}_k(t)\xi \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$ for $0 < t < \tau$. We set $\mathbf{t}(\xi) = \mathbf{t}_k(\xi) := \tau$ and $\mathbf{x}(\xi) = \mathbf{x}_k(\xi) := \boldsymbol{\pi}_k(\tau)\xi$. Thus, $\mathbf{x}$ is the first point at which the trajectory from ξ (in mode k) hits $\bar{\mathcal{R}}_k$ and $\mathbf{t}$ is the first hitting time.

Lemma 2.1 *Given $\xi \in [\mathcal{X}\backslash\bar{\mathcal{R}}_k]$, suppose $\mathbf{x}_k(\xi)$ is defined and is not an anomalous point. Then the functions $\mathbf{x}_k(\cdot)$ and $\mathbf{t}_k(\cdot)$ are defined and continuous in a neighborhood of ξ.*

The proof is a rather straightforward application of the definitions, recalling the continuity in t, ξ of $\boldsymbol{\pi}_k$. The somewhat messy detail will thus be omitted here. □

The principal *positive* result available for 'general' switching systems is essentially a corollary of this lemma.

DEFINITION: A **minimal periodic solution (mps)** has the following form: Suppose we were to have a point $\xi_0 \in \partial\mathcal{R}_2$ and imagine solving the switching system with initial data $[\xi_0, 1]$. If there is a solution segment which proceeds in mode $\jmath = 1$

until it switches to mode $\jmath = 2$ at a point $\xi_1 \in \partial\mathcal{R}_1$ and then continues in that mode until returning to ξ_0, then one can switch back to mode $\jmath = 1$ and continue periodically — repeating this 'two-phase' segment, alternating modes, 'forever'. This is the simplest possible nontrivial periodic solution of the switching system.

Theorem 2.2 *Let $\mathcal{K}$ be a compact, topologically convex subset of $\mathcal{X}\backslash\bar{\mathcal{R}}_1$. Suppose $\mathbf{x}_1(\xi)$ is defined for each $\xi \in \mathcal{K}$ and set $\mathcal{K}' := \{\mathbf{x}_1(\xi) : \xi \in \mathcal{K}\} \subset \partial\mathcal{R}_2$. Next, suppose also that $\mathbf{x}_2(\xi')$ is defined for each $\xi' \in \mathcal{K}'$ and set $\mathcal{K}'' := \{\mathbf{x}_2(\xi') : \xi' \in \mathcal{K}'\}$. If there are no anomalous points in either $\mathcal{K}'$ or $\mathcal{K}''$ and if $\mathcal{K}'' \subset \mathcal{K}$, then there exists an mps of the switching system with 'initial value' $[\xi, 1]$ for some $\xi \in \mathcal{K}'' \subset \mathcal{K}$.*

PROOF: Define the map $F = (\mathbf{x}_1 \circ \mathbf{x}_2)$. Applying Lemma 2.1 twice, using the assumption that there are no anomalous points, we see that F is well-defined and continuous from $\mathcal{K}$ to $\mathcal{K}''$. By assumption F is then a continuous selfmap of $\mathcal{K}$ and applying the Schauder Theorem shows existence of a fixpoint $\xi \in \mathcal{K}$ (Necessarily, of course, $\xi \in \mathcal{K}''$.) so that, by our definition, $[\xi, 1]$ will be initial data for such an mps of the switching system. □

Example 2.1 *We conclude this section by constructing a class of examples to show that the hypothesis in Theorem 2.2 excluding anomalous points cannot be omitted.*

We take $\mathcal{X}$ to be the plane $\mathbb{R}^2$ and let $\mathcal{R}_k$ be a pair of separated (open) disks — say, with $\mathcal{R}_1$ on the 'left' as shown in Figure 1 — and take each semiflow $\boldsymbol{\pi}_k$ to have a global attractor in the corresponding $\mathcal{R}_k$ so all solutions of the switching system will necessarily alternate modes infinitely often.

It will be sufficient to describe the trajectories associated with certain critical semiflows; the rest will smoothly fill out the intervening space and 'speed' along the trajectories is irrelevant for our present considerations.

For the semiflow $\boldsymbol{\pi}_1$ there will be no anomalous points on $\partial\mathcal{R}_1$. Clearly, in discussing periodicity we are here only concerned with those orbits which intersect $\partial\mathcal{R}_2$. The critical orbits are the 'upper orbit' $[\alpha_1]$, which (moving from 'right infinity') touches $\partial\mathcal{R}_2$ tangentially at a point $\mathbf{a}$ before intersecting $\partial\mathcal{R}_1$ at the point $\mathbf{A}$, and the 'lower orbit' $[\delta_1]$, which (again moving from 'right infinity') touches $\partial\mathcal{R}_2$ tangentially at a point $\mathbf{d}$ before intersecting $\partial\mathcal{R}_1$ at the point $\mathbf{D}$. We take $[\delta_1]$ to touch $\partial\mathcal{R}_2$ tangentially at a point $\mathbf{d}'$ and then loop away somewhat from $\partial\mathcal{R}_2$ before again touching at $\mathbf{d}$ and continuing to $\mathbf{D}$. All orbits which intersect $\partial\mathcal{R}_2$ must then lie between $[\alpha_1]$ and $[\delta_1]$ and we identify, in particular, one such orbit $[\gamma_1]$ (slightly 'above' $[\delta_1]$) which enters $\mathcal{R}_2$, loops out and then re-enters at a point $\mathbf{c}'$ between $\mathbf{d}'$ and $\mathbf{d}$, crosses $\mathcal{R}_2$ exiting at a point $\mathbf{c}$, and finally enters $\mathcal{R}_1$ at a point $\mathbf{C}$ (slightly above $\mathbf{D}$) in the arc $\overline{\mathbf{AD}}$.

Now consider the semiflow $\boldsymbol{\pi}_2$. There are now three critical orbits: the 'upper orbit' $[\alpha_2]$ passing through $\mathbf{A}$, the 'lower orbit' $[\delta_2]$ passing through $\mathbf{D}$, and an orbit $[\beta_2]$ containing the only anomalous point of the switching system. This orbit $[\beta_2]$

(moving from 'left infinity') passes through a point **B** of $\overline{\mathbf{AD}}$, then tangentially touches $\partial\mathcal{R}_2$ at the point **a** — which is thus anomalous — and loops somewhat away from $\partial\mathcal{R}_2$ before entering $\mathcal{R}_2$ at $\mathbf{d}'$. The orbit $[\alpha_2]$ enters $\mathcal{R}_2$ at $\mathbf{c}'$ (swinging slightly above $[\beta_2]$ 'around' $\mathcal{R}_2$) while $[\delta_2]$ enters $\mathcal{R}_2$ at **c**.

$\boldsymbol{\pi}_1$

$[\alpha_1]$

$\mathcal{R}_1$ $\mathcal{R}_2$

$[\gamma_1]$

$[\delta_1]$

$\boldsymbol{\pi}_2$

$[\alpha_2]$

$[\beta_2]$

$\mathcal{R}_1$ $\mathcal{R}_2$

$[\delta_2]$

Figure 1

If one now *identifies* $\mathbf{A} \approx \mathbf{D}$, then $\overline{\mathbf{AD}}$ becomes, topologically, a circle. The 'round trip map' $(\mathbf{x}_2 \circ \mathbf{x}_1)$ gives $[\mathbf{A} \approx \mathbf{D}] \mapsto \mathbf{C}$ by construction and, with our identification, gives $\mathbf{B} \mapsto [\mathbf{A} \approx \mathbf{D}]$ (independently of the option selected at $\mathbf{a}$. Thus, this 'round trip' determines a well-defined map $\boldsymbol{\wp}$ of the 'circle' $\overline{\mathbf{AD}}$ to itself; one easily verifies, using the continuity of π_1, π_2, that this map $\boldsymbol{\wp}$ is continuous as well as orientation-preserving and injective, hence a homeomorphism as the circle is compact.

Within the framework above one can (homotopically) adjust the construction so that $(\mathbf{x}_2 \circ \mathbf{x}_1)$ is *any* desired (orientation-preserving) homeomorphism $\boldsymbol{\wp} : \overline{\mathbf{AD}} \to \overline{\mathbf{AD}}$. To see this, parametrize $\overline{\mathbf{AD}}$ as $[0, 2\pi]$ with $0 \approx 2\pi$ and modify $\boldsymbol{\pi}_2$ so the 'initial point' $\mathbf{B} \in \partial\mathcal{R}_2$ of the orbit $[\beta_2]$ is just $\boldsymbol{\wp}^{-1}(\mathbf{A} \approx \mathbf{D})$. One can then modify $\boldsymbol{\pi}_1$ (only in the 'rectangle' between $[\alpha_1]$ and $[\delta_1]$ and from $\overline{\mathbf{ad}}$ going into $\mathcal{R}_1$) so each 'initial point' in $\overline{\mathbf{AD}}$ is mapped by $(\mathbf{x}_2 \circ \mathbf{x}_1)$ as desired to produce $\boldsymbol{\wp}$. In particular, one can construct $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ so $\boldsymbol{\wp}$ is a 'rotation' of the parametric circle through an arbitrarily specified angle ω. (This can even be accomplished with C^∞ flows on $\mathbb{R}^2$.)

If the construction has, indeed, been 'tuned' so that $\boldsymbol{\wp}$ corresponds to a rotation with $\omega/2\pi$ irrational, this is the classic example of a map such that no iterate has a fixed point. The resulting switching system then has *no periodic solutions at all* since, clearly, *any* periodic solution must entail existence of a fixed point of some iterate of $\boldsymbol{\wp}$. Thus, this construction shows the essential significance of the prohibition (in the hypotheses of Theorem 2.2) of anomalous points in $\mathcal{K}', \mathcal{K}''$; note that the inclusion of a single anomalous point has here 'prevented' the existence not only of *minimal* periodic solutions but of *all* periodic solutions.

We note that this construction is not really limited to $\mathbb{R}^2$. Let $\mathcal{X}$ be $\mathbb{R}^m$ $(m > 2)$ or even an infinite-dimensional Banach space which can be written as $\mathcal{X} = \mathbb{R}^2 \oplus \mathcal{Y}$. Define

$$\hat{\boldsymbol{\pi}}_k(t)[\xi, y] := [\boldsymbol{\pi}_k(t)\xi, \mathbf{S}(t)y]$$

where the $\boldsymbol{\pi}_k$ are as in the two-dimensional construction above and $\mathbf{S}(\cdot)$ is a compact semigroup on $\mathcal{Y}$ going exponentially to 0. The regions $\hat{\mathcal{R}}_k$ are, say, balls with centers $[\xi_k, 0]$ where the ξ_k are the centers used in $\mathbb{R}^2$ above and the radii are the same; in the infinite-dimensional case this could be modified to get compact closure. Since $\mathbf{S}(t) \to 0$ as $t \to \infty$, the only points of interest for possible periodicity lie in the subspace $(y = 0)$ where the system reduces to the previous construction for $\mathbb{R}^2$.

3 Linear Systems

The consideration of *linear switching systems* permits the application of a number of more specialized modes of analysis which provide the deeper results available in that case.

DEFINITION: For a **linear switching system** the dynamics are given by a pair of (abstract) linear odes:

$$\dot{x} = \mathbf{A}x + u_k \qquad (k = \pm 1) \tag{3.2}$$

where $\mathbf{A}$ is the infinitesimal generator of a C_0 linear semigroup $\mathbf{S}(\cdot)$ on the state space $\mathcal{X}$ (cf, e.g., [5]) and the *forbidden regions* are given by

$$\mathcal{R}_k = \begin{cases} \{\xi \in \mathcal{X} : \langle \Lambda, \xi \rangle > \vartheta_+\} & \text{if } k = 1 \\ \{\xi \in \mathcal{X} : \langle \Lambda, \xi \rangle < \vartheta_-\} & \text{if } k = -1 \end{cases} \tag{3.3}$$

for some $\Lambda \in \mathcal{X}^*$ and $\vartheta_- < \vartheta_+$. (Note that, for reasons which will shortly be apparent, we have chosen to index the modes by ± 1 when considering *linear* switching systems.)

We will restrict our attention to (exponentially) *stable* semigroups $\mathbf{S}(\cdot)$ so there are global attractors $z_k = -\mathbf{A}^{-1} u_k$ for the two modes and we assume $z_k \in \mathcal{R}_k$; then all solutions of the switching system will alternate modes infinitely often. Observe, also that without loss of generality we may make a (linear) change of variables to have $z_{\pm 1} = \pm z$ and $\langle \Lambda, z \rangle = 1$. From the dynamics (and noting that the assumed stability gives independence of the 'infinite past') we obtain the representation:

$$x(t) = \int_{-\infty}^{t} \mathbf{S}(t-s)[\jmath(s) z]\, ds. \tag{3.4}$$

If we set

$$\vartheta(t) := \langle \Lambda, x(t) \rangle, \qquad \varphi(t) := \langle \Lambda, \mathbf{S}(t) z \rangle, \tag{3.5}$$

then we see from (3.4) that the *sensor function* $\vartheta(\cdot)$ is the convolution of the *switching function* $\jmath(\cdot)$ and the *impulse response function* $\varphi(\cdot)$:

$$\vartheta(t) = \int_{-\infty}^{t} \varphi(t-s) \jmath(s)\, ds = \int_{0}^{\infty} \varphi(r) \jmath(r-t)\, dr. \tag{3.6}$$

One easily sees that the rules imply switching for such a linear system when $\vartheta(\cdot)$ crosses (touches?) the critical values $\vartheta_\pm$. Indeed, it is possible to construct an essentially equivalent new switching system in which the 'state' is the semi-infinite past history of ϑ, the dynamics is given by (3.6), and the functional Λ is now evaluation of ϑ at the current time. Thus the function $\varphi(\cdot)$ and the critical values $\vartheta_\pm$ completely characterize the behavior of such a stable linear switching system. See [8], [10].

We henceforth assume that φ is in $L^1(\mathbb{R})$ (defined as 0 for $t < 0$) and in $C_0(\mathbb{R}^+)$. This is automatically true for φ coming from an exponentially stable semigroup as above but also includes some cases in which the semigroup is merely asymptotically stable. We easily see from the above that our construction has imposed the normalizations:

$$\int_0^\infty \varphi = 1, \qquad -1 < \vartheta_- < \vartheta_+ < 1. \tag{3.7}$$

It is interesting to consider the finite-dimensional situation $\mathcal{X} = \mathbb{R}^m$. The stability assumption makes it easy to obtain a bounded (and so a compact) invariant set but we must consider the possibility of anomalous points. An anomalous point ξ on the switching surface $\vartheta = \vartheta_+$ would typically be such that:

$$\langle \Lambda, \xi \rangle = \vartheta_+, \quad \langle \Lambda, \mathbf{A}\xi \rangle = 0, \quad \langle \Lambda, \mathbf{A}^2\xi \rangle = \zeta < 0.$$

If e.g. $\{\Lambda, \mathbf{A}^*\Lambda, [\mathbf{A}^*]^2\Lambda\}$ is linearly independent, then such points will always exist — indeed, they then form an affine half-space of dimension $(m-2)$. This need not make our general existence theorem above inapplicable but does make the determination of a suitable $\mathcal{K}$ much more difficult. For $m = 2$, however, one has linear dependence and an argument [7] using the quadratic characteristic polynomial of $\mathbf{A}$ shows that there cannot be any anomalous points and so one has an mps for 'arbitrary' $\vartheta_\pm$; recent work by Stoth [11] shows that this solution (for $\mathcal{X} = \mathbb{R}^2$) is always unique and stable.

It is worth noting a somewhat different analysis, used in [11] but treated more deeply in [2]. If one were to have an mps, say with period p, then (shifting so the switch from $\jmath = 1$ to $\jmath = -1$ occurs at $t = 0$) the switching function $\jmath(\cdot)$ will be -1 for $0 < t < \alpha$ and 1 for $\alpha < t < p$ (some switching time $0 < \alpha < p$) and, by (3.6), the sensor function is given by

$$\vartheta(t) = \int_0^p \psi(\tau;p)\jmath(t-\tau)\,d\tau \text{ where } \psi(\tau;p) := \sum_{n=0}^{\infty} \varphi(\tau + np). \tag{3.8}$$

Now consider the function $F : [p, \alpha] \mapsto [u, v]$ given by

$$\begin{aligned} u := \ \vartheta(0) &= 2\int_0^{p-\alpha} \psi(\cdot;p) - 1 = 1 - 2\int_{p-\alpha}^{p} \psi(\cdot;p) \\ v := \ \vartheta(\alpha) &= 1 - 2\int_0^{\alpha} \psi(\cdot;p) \end{aligned} \tag{3.9}$$

and note that this $\jmath(\cdot)$ gives a solution of the switching system through (3.4) if $u = \vartheta_+$, $v = \vartheta_-$ — provided that (3.8) gives

$$\vartheta(\cdot) > v \text{ on } (0, \alpha), \qquad \vartheta(\cdot) < u \text{ on } (\alpha, p). \tag{3.10}$$

To seek an mps for such switching systems (for varying $\vartheta_\pm$) is thus an attempt to invert this function F, subject to (3.10).

In treating the (physical) thermostat problem as a switching system we would have a pde governing the dynamics: the heat equation with either of two source terms depending on whether the furnace is ON or OFF. Depending on the modeling, this source may appear either in the equation or in the boundary data. Especially in the latter case, the inhomogeneity may not be in the state space $\mathcal{X}$ but the appropriate semigroup is so strongly smoothing that we can expect $\mathbf{S}(t)z$ to be well-behaved for positive t. Similarly, the sensor measures temperature at a point; this is not a functional in $\mathcal{X}^*$ but nevertheless is well-behaved on the range of $\mathbf{S}(t)$ for $t > 0$. Assuming the sensor is not actually placed in the furnace, the impulse response function φ can be expected to be analytic for $t > 0$ (C^∞ vanishing at 0) and vanishing at ∞ with exponential decay. A physical thermostat has a pair of 'set points' whose effect in switching the furnace is, indeed, more-or-less[2] as presented here. In practical operation the separation of the switching values is very small compared to the potential range of variation. The desirable stability analysis of this situation is still lacking. The (one-dimensional) models for which results have been obtained regarding periodicity [6], [2] have been of this form (i.e., with dynamics given by (3.6) and switching according to the crossings of $\vartheta_\pm$ by $\vartheta(t)$). In addition, they have involved boundary conditions consistent with the use of the maximum principle: in particular, in each case one had

$$\varphi(t) > 0 \ (t > 0), \qquad \varphi'(t) < 0 \ (t > t_*). \tag{3.11}$$

Prüss [6] obtained existence when $\vartheta_\pm$ are far apart while, more recently, Friedman and Jiang [2] provided the best results currently available:

- Existence of an mps for *every* choice of switching values $\vartheta_\pm$.
- Uniqueness of this solution when $\vartheta_\pm$ are far enough apart.

We note that the paper [2] is not in the present framework and uses special properties of the pde setting, some of which do not correspond to hypotheses easily formulated in terms of φ.

The existence result in [6] is for $\vartheta_\pm$ far enough apart and another of the results in [11] is general existence when φ is strictly decreasing to 0 on $\mathbb{R}^+$. Note also a quite recent paper by Gripenberg [4] which proves existence, under quite general conditions, of periodic functions 'weakly controlled' by the thermostat; these need not be solutions in the sense considered here. We now conclude this paper by presenting a new 'general' existence result for periodic solutions.

Theorem 3.1 *Suppose $\varphi(\cdot) \in L^1$ is continuous on $[0, \infty)$; assume the normalization (3.7). Suppose, also, that there is some $t_* \geq 0$ such that: (i) $\varphi(\cdot)$ is (strictly) decreasing to 0 on (t_*, ∞) and (ii) Φ is bounded away from 0 on $[0, t_*]$ where*

$$\Phi(t) := \int_t^\infty \varphi(s)\, ds = 1 - \int_0^t \varphi(s)\, ds; \tag{3.12}$$

without loss of generality we take t_* so $\Phi(t) > \Phi(t_*) =: \beta_*$ for $t < t_*$. Then there always exists an mps of the switching system whenever $\vartheta_\pm$ are far enough apart.

PROOF: Consider a sequence $\mathbf{s} := \{\sigma_1, \sigma_2, \ldots\}$ of positive numbers. With $s_0 = 0$, recursively set $s_k = s_{k-1} - \sigma_k$ and define $\jmath = \jmath_{\mathbf{s}}(\cdot)$ on $(-\infty, 0)$ as $(-1)^k$ on each

[2]Strictly speaking, this ignores unmodeled 'fast dynamics' within the thermostat and furnace. The interesting connection between switching systems and this sort of bifurcating singular perturbation problem is not under consideration here.

interval (s_k, s_{k-1}). Assuming $\vartheta(0) < \vartheta_+$, extend $\jmath$ on $(0, \tau)$ as $+1$ until $\vartheta(\sigma) = \vartheta_+$ with $\vartheta = \vartheta_{\mathbf{s}}(\cdot)$ obtained from $\jmath_{\mathbf{s}}$ by (3.6); this defines $\sigma = \sigma_{\mathbf{s}}$. Similarly, we then continue to extend $\jmath_{\mathbf{s}}$ on $(\sigma, \sigma + \tau)$ as -1 until $\vartheta(\sigma + \tau) = \vartheta_-$, defining $\tau = \tau_{\mathbf{s}}$. Denote by $\mathbf{M}$ the map: $\mathbf{s} \to \mathbf{s}'$ where $\sigma_1' := \tau_{\mathbf{s}}$, $\sigma_2' := \sigma_{\mathbf{s}}$ and then $\sigma_k' := \sigma_{k-2}$ for $k = 2, 3, \ldots$; note that a fixpoint of $\mathbf{M}$ provides an mps of the switching system.

Our first task is to obtain a suitable lower bound for $\sigma = \sigma_{\mathbf{s}}$. Note that the construction of $\jmath_{\mathbf{s}}$ gives

$$\begin{aligned} \vartheta_+ = \vartheta(\sigma) &= \int_0^\sigma \varphi - \int_\sigma^{\sigma+\sigma_1} \varphi + \int_{\sigma+\sigma_1}^\infty \varphi(s)\jmath(\sigma - s) \\ &\leq \int_0^\sigma \varphi - \int_\sigma^{\sigma+\sigma_1} \varphi + \int_{\sigma+\sigma_1}^\infty \varphi \\ &= 1 - 2\int_{\sigma+\sigma_1}^\infty \varphi \qquad\qquad \text{so} \end{aligned}$$

$$\Phi(\sigma) - \Phi(\sigma + \sigma_1) \leq \frac{1 - \vartheta_+}{2} =: \alpha_+ \tag{3.13}$$

provided we can assume $\sigma_1 \geq t_*$ so φ will be positive on $(\sigma + \sigma_1, \infty)$. Now (temporarily) fix $t' > t_*$ so $\Phi(t') =: \beta < \beta_*$ and, for $\alpha \leq \beta_* - \beta$, define

$$s(\alpha) = s(\alpha; t') := \min\{s \geq t_* : \Phi(s) - \Phi(s + t') \leq \alpha\}. \tag{3.14}$$

Clearly, $s(\alpha)$ is well-defined and tends to 0 as $\alpha \to 0+$. Hence, for any $t' > t_*$ as here there will exist $\alpha(t')$ such that:

$$0 < \alpha(t') \leq \beta_* - \Phi(t'), \qquad s(\alpha(t')) \geq t'; \tag{3.15}$$

without loss of generality we might fix $t' > t_*$ so $\alpha' = \alpha(t')$ is as large as possible. From this definition we easily see that if

$$\vartheta_+ \geq 1 - 2\alpha', \qquad \vartheta_- \leq -1 + 2\alpha', \tag{3.16}$$

then $\sigma_1 \geq t' > t_*$ implies $\sigma_{\mathbf{s}} \geq t'$ and, repeating the analysis, further implies $\tau_{\mathbf{s}} \geq t'$. This shows, subject to (3.16), that $\{\mathbf{s} : \sigma_k \geq t'\}$ is invariant under the map $\mathbf{M}$. Now fixing $\vartheta_\pm$ subject to (3.16), a simple estimate like (3.13) gives

$$2\Phi(\sigma) \geq |1 - \vartheta_+|, \qquad 2\Phi(\tau) \geq |-1 - \vartheta_-|,$$

providing an upper bound: $\sigma, \tau \leq t''$; hence, $\mathbf{M}$ is a well-defined selfmap of $\mathcal{K}_* := [t', t'']^\infty$.

Note that $\mathcal{K}_*$ is convex in $\mathbb{R}^\infty$ and is compact by the Tychonov Theorem. The function: $\mathbf{s} \mapsto \sigma_{\mathbf{s}}$ will clearly be continuous — essentially as in Lemma 2.1 — if (3.6) gives $\dot\vartheta(\sigma-) > 0$ (with $\vartheta = \vartheta_{\mathbf{s}}$, $\sigma = \sigma_{\mathbf{s}}$). Assuming, for the moment, that φ is differentiable on $\mathbb{R}^+$ with $\varphi' \in L^1$, we have

$$\dot\vartheta(t) = \varphi(0) + \int_{-\infty}^t \varphi'(t - s)\jmath(s)\, ds \quad \text{ for } 0 < t < \sigma,$$

$$
\begin{aligned}
(3.17) \qquad \dot{\vartheta}(\sigma-) &= \varphi(0) + \int_0^\infty \varphi'(s) \jmath_{\mathbf{s}}(\sigma - s)\, ds \\
&= \varphi(0) + \int_0^\sigma \varphi' - \int_\sigma^{\sigma+\sigma_1} \varphi' + - \ldots \\
&= 2\left([\varphi(t_1) - \varphi(t_2)] + [\varphi(t_3) - \varphi(t_4)] + \ldots \right)
\end{aligned}
$$

where $t_1 = \sigma$ and then $t_{k+1} = t_k + \sigma_k$. (A density argument then shows that (3.17) holds without the differentiability assumption on φ; one can similarly avoid the assumption of *strict* decrease on (t_*, ∞).) Since our assumption (3.16) ensures that $t_1 = \sigma_{\mathbf{s}} \geq t' > t_*$, each term on the right of (3.17) is strictly positive by the hypotheses so $\dot{\vartheta}(\sigma-) > 0$ as desired. Essentially the same argument shows $\dot{\vartheta}(\sigma + \tau-) < 0$ so the function: $\mathbf{s} \mapsto \tau_{\mathbf{s}}$ is also continuous.

From the above, $\mathbf{M}$ is a *continuous* selfmap of $\mathcal{K}_*$ (subject to (3.16) — i.e., if the switching values $\vartheta_\pm$ are far enough apart as assumed) so, by the Schauder Theorem, there is necessarily a fixpoint giving the desired mps for the switching system. □

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