OPTIMAL CONTROL FOR SWITCHING SYSTEMS [1]

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ABSTRACT:

The notion of *switching system* generalizes the idea of a 'differential equation': a switching system consists of a number of *modes* (e.g., differential equations) together with a set of *switching rules.* Such systems arise in modeling thermostats (and other devices involving 'hysteresis'), in considering the evolution of a multimodal (variable structure) system for which 'feedback control' has been implemented by 'switching surfaces', and as reduced order models for systems involving certain singular perturbation effects. Here, we formulate some optimal control problems involving switching systems and, under appropriate asumptions, prove the existence of optimal controls.

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1.INTRODUCTION

Very roughly, a *switching system* is a system which can operate in any of a number of *modes*, e.g., given by differential equations

(1) $$\dot{x} = f_j(x)$$

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indexed by $j \in \mathcal{J}$, together with certain *switching rules*. Some background on switching systems, adapted to the consideration of *controlled switching systems*, will be presented in the next section; for more detail see [7], [8].

While there are various other interesting control-theoretic questions which might be considered for switching systems, our concern in this talk is with the existence of optimal controls for a controlled switching system i.e., a switching system with modes of the form:

$$\dot{x} = f_j(x, u) \tag{2}$$

where $u(\cdot)$ is a *control function* subject to the pointwise constraint[2]:

$$u(t) \in \mathcal{U}_j(x(t)) \qquad a.e.\ on\ [0, T]. \tag{3}$$

Since the switching rules admit some ambiguity in the 'dynamics' (i.e., some switches may be *optional*), we include the choices made on such occasions in the category of *control*. If we have some *cost functional* **j** (depending on $u(\cdot)$, such available choices, and the corresponding solution $[x(\cdot), j(\cdot)]$, all subject to the switching system dynamics and the control constraints), then we can seek to minimize **j** over admissible controls for which, e.g., some *terminal condition* $x(T) = \xi$ is satisfied.

Our object, here, is to show, under suitable hypotheses, the existence of admissible controls (i.e., controls for which a 'regular' solution exists and the constraints are satisfied) and the existence of an optimal control. For simplicity of exposition we will restrict attention to a control-theoretic setting for which the technical details are relatively standard[3]. We will also impose simplifying restrictions on the setting for the switching rules.

The standard pattern for proving existence of an *optimal control* is:

- Consider a *minimizing sequence* for the cost functional **j**;
- Use some compactness property to extract a convergent subsequence;
- Show (i) that the limit is itself a solution and (ii) that it minimizes **j**.

[2]Note that, since one switches modes, one really has $j = j(t)$ in considering this constraint.

[3]It would, of course, have been still simpler to have considered only $\mathcal{U}_j(\xi)$ independent of ξ but we wish to draw attention to the results of [2].

To follow this pattern, we topologize control functions $u(\cdot)$ in some L^p space, trajectories $x(\cdot)$ by uniform convergence, and index functions $j(\cdot)$ by convergence of the switching times. The technical details are then standard (using results of [2]) — except for the treatment of $j(\cdot)$.

2.CONTROLLED SWITCHING SYSTEMS

While a fuller discussion of switching systems appears in [8], that discussion was not specifically addressed to the consideration of *controlled switching systems* and we will present here the relevant definitions adapted to the present context (somewhat restricted for simplicity). We also demonstrate the existence of a suitable (nonempty) set of controlled *regular solutions* with a uniform estimate for the *interswitching intervals*.

DEFINITION: For the **controlled switching system**

$$\dot{x}(t) = f_{j(t)}(x(t), u(t)) \tag{4}$$

$$u(t) \in \mathcal{U}_{j(t)}(x(t)) \subset \mathcal{V} \tag{5}$$

a **regular solution** will be a function:

$$t \mapsto [u(t), x(t), j(t)] : [0, T] \to \mathcal{V} \times \mathcal{X} \times \mathcal{J} \tag{6}$$

with $u(\cdot)$ measurable, $x(\cdot)$ absolutely continuous, and $j(\cdot)$ piecewise constant with discrete (i.e., finitely many) *switching times*. We require that (5) holds a.e. on $[0, T]$, that the differential equation (4) holds — say, in integrated form — on the intervals of constancy of the *index function* $j(\cdot)$), and that $j(\cdot)$ satisfies the *switching rules* below.

DEFINITION: We assume that we are given, as part of the specification of the switching system, *forbidden regions* $\{\mathcal{R}_j \subset \mathcal{X} : j \in \mathcal{J}\}$ and *switching sets* $\{\mathcal{S}_{j,k} \subset \mathcal{X} : j, k \in \mathcal{J};\ j \neq k\}$. In terms of these, the **switching rules** are:

(SR_1) If $x(t) \in \mathcal{R}_k$ for some $t \in [0, T]$ and some $k \in \mathcal{J}$, then $j(t) \neq k$.

(SR_2) If τ is a *switching time*, then $x(\tau) \in \mathcal{S}_{j(\tau-), j(\tau+)}$.

HYPOTHESES:

(H_1) For our present purposes we consider switching systems with only two modes: $\mathcal{J} = \{1,2\}$. We assume that $\mathcal{R}_1$, and $\mathcal{R}_2$ are open sets in $\mathcal{X} = \mathbb{R}^n$ with disjoint closures and that $\mathcal{S}_{1,2} = \partial\mathcal{R}_1, \quad \mathcal{S}_{2,1} = \partial\mathcal{R}_2$. We assume that each set $\mathcal{U}_k(\xi)$ (for $\xi \in \mathcal{X}$, $k = 1,2$) is compact and convex in $\mathcal{V} = \mathbb{R}^m$ and that the two (set-valued) mappings $\xi \mapsto \mathcal{U}_k(\xi)$ are upper semicontinuous and uniformly bounded. Finally, we take f_j to be affine in the control:

$$f_k(\xi,\omega) = g_k(\xi) + B_k(\xi)\omega \tag{7}$$

with $g_k : \mathcal{X} \to \mathcal{X}$ and $B_k : \mathcal{X} \to [n \times m \ matrices]$ continuous and of linear growth so there are constants α, β giving

$$\begin{aligned} \max\{\|f_k(\xi,\omega)\| \quad &: \quad \omega \in \mathcal{U}_k(\xi);\ k = 1,2\} \\ &\leq \quad \alpha + \beta\|\xi\|. \end{aligned} \tag{8}$$

Theorem 1: *Assume* (H_1). *Then, for any initial conditions* $[x(0), j(0)]$ *consistent with* (SR_1)*, the switching system has global solutions. Further, (fixing the initial conditions) these solutions are uniformly regular, i.e., there is a uniform lower bound* $\delta > 0$ *on the lengths of interswitching intervals (hence a bound on the number of switches) for all such solutions.*

Proof In view of (7), we may rewrite (4) (for fixed index k) as a differential inclusion: $\dot{x} \in F_k(x)$, satisfying conditions given in [2] for local existence of solutions with $x(\cdot)$ absolutely continuous and $u(\cdot)$ measurable, satisfying (4), (5) with $j(\cdot) \equiv k$. With the estimate (8) this existence is global.

Now suppose, without loss of generality, that $j(0) = 1$ (for simplicity assuming that $\xi_0 = x(0) \notin \bar{\mathcal{R}}_1$, although this is not essential) and proceed as above with $k = 1$; set $\tau_0 = 0$. Now set

$$\begin{aligned} \mathcal{T}_1^0 \quad &= \quad \{t \in (\tau_0, T] : x(s) \notin \mathcal{R}_1 \ for\ s \in (\tau_0, t]\}, \\ \mathcal{T}_1 \quad &= \quad \{t \in \mathcal{T}_1^0 :\ t = T \ or\ x(t) \in \partial\mathcal{R}_1\} \end{aligned} \tag{9}$$

and arbitrarily choose $\tau_1 \in \mathcal{T}_1$. We take this (with $j(\cdot) \equiv k = 1$) to define the solution $[u, x, j]$ on the interval $[0, \tau_1]$.

If $\tau_1 = T$ we are done; if $\tau_1 < T$ then $\xi_1 = x(\tau_1)$ is in $\partial\mathcal{R}_1 = \mathcal{S}_{1,2}$ so we can switch to the second mode and 'restart the problem' at the switching time $t = \tau_1$, proceeding as above (now with $k = 2$) until some τ_2 in a correspondingly defined $\mathcal{T}_2$, etc. (accepting the indeterminacy of $j(\cdot)$ at the switching times).

We next obtain a lower bound $\delta > 0$ on the lengths of the *interswitching intervals* (τ_k, τ_{k+1}); observe that this ensures that the 'successive restart' procedure ends (with the solution globally defined) in no more than $\bar{\nu} = 2 + [T/\delta]$ steps.

From (8) and the Gronwall inequality we obtain a fixed bound on x and, using (8) again, a uniform Lipschitz bound for all controlled trajectories $x(\cdot)$. With this bound on x, the relevant portions of the switching sets $\mathcal{S}_{1,2}$, $\mathcal{S}_{2,1}$ are compact and disjoint so one has a lower bound on $\|\xi_{k+1} - \xi_k\|$ as these are necessarily in different switching sets. With the uniform Lipschitz bound, this bounds $(\tau_{k+1} - \tau_k)$ from below.

One sees that $[x(\cdot), j(\cdot)]$ so constructed must satisfy the switching rules and that $[u(\cdot), x(\cdot)]$ satisfies (4), (5) on each interswitching interval; hence $[u, x, j]$ is a solution of the controlled switching system on $[0, T]$. ■

REMARKS: (1) We note that essentially the same conclusions could have been obtained in more general settings and under weaker hypotheses than (H_1).
(2) For future reference, we also note from the proof above that the bound and Lipschitz bound on x obtained put the set of trajectories in a *compact* subset of $C([0, T]; \mathcal{X})$. Further, the assumed upper semicontinuity of the set-valued maps:

$$\xi \mapsto \mathcal{U}_k(\xi) : \mathcal{X} \to [compact \subset \mathcal{V}]$$

then ensures that the controls $u(\cdot)$ take values in a fixed compact set.
(3) The choice of $\tau \in \mathcal{T}_1$ is here considered an element of 'control' but we note that it is a significant aspect of the theory of switching systems; see [8], [9], [10].

3.COMPACTNESS

We assume,now, that we are considering a controlled switching system with initial conditions $[x(0), j(0)]$, subject to the hypotheses of Theorem 1. Thus there is a nonempty set of solutions $\{[u, x, j]\}$ which we denote by $\mathcal{A}_0$. We will topologize $\mathcal{A}_0$ by using weak convergence in $L^p(0, T; \mathcal{V})$ (with $1 < p < \infty$ — say, $p = 2$) for the controls $u(\cdot)$, uniform convergence in

$C([0,T] \to \mathcal{X})$ for the trajectories $x(\cdot)$, and 'convergence of switching times' for the index functions $j(\cdot)$ — i.e., $j^i(\cdot) \to j(\cdot)$ means that the switching times τ_ν^i converge in $[0,T]$ to τ_ν for each ν (with the interpretation that $\tau_\nu^i = T$ for 'large' ν) and the sequence of index values is fixed.

Suppose now that a *target set* is specified and admit only such solutions as 'terminate in the target' $\mathcal{S}_T \subset \mathcal{X}$, i.e., the set of *admissible solutions* is just

$$\mathcal{A} = \{[u,x,j] \in \mathcal{A}_0 : x(T) \in \mathcal{S}_T\}. \tag{10}$$

HYPOTHESES:

(H_2) Assume that the initial conditions $[x(0), j(0)]$ are consistent with (SR_1), that the specified target set $\mathcal{S}_T$ is closed (compact) in $\mathcal{X}$, and that it is *attainable*: the set $\mathcal{A}$ is nonempty.

Theorem 2: *Assume* (H_1) *and* (H_2). *Then the set $\mathcal{A}$ of admissible solutions is compact in the indicated topology.*

PROOF Given any sequence $\{\mathbf{s}^i = [u^i, x^i, j^i]\}$ in $\mathcal{A}$, the bound on controls (recall Remark 2, following Theorem 1) permits extraction of a subsequence for which $u^i \rightharpoonup u$ (weak convergence in the reflexive space $L^p(0,T;\mathcal{V})$. One can also extract so $\{\tau_\nu^i\}$ converges in $[0,T]$ (to some τ_ν) for each ν so $j^i \to j$. Finally, Remark 2 permits extraction so $x^i \to x$ uniformly on $[0,T]$. We must show that $\mathbf{s} = [u,x,j]$ is again in $\mathcal{A}$. Since the target set is closed, the convergence $x^i(T) \to x(T)$ ensures $x(T) \in \mathcal{S}_T$; we must just show that $\mathbf{s} \in \mathcal{A}_0$.

The uniform regularity of the solutions $\{\mathbf{s}^i\}$ gives the lower bound δ for the limit interswitching intervals; consider an open interval $\mathcal{I} \subset [\tau_\nu, \tau_{\nu+1}]$ so $\mathcal{I} \subset [\tau_\nu^i, \tau_{\nu+1}^i]$ and $j^i(\cdot) \equiv k$ (k fixed) on $\mathcal{I}$. Restricting to $\mathcal{I}$, application of the Convergence Theorem (p. 60 of [2]) gives (5) for the limit. Next, (H_1), in view of the affine form (7) ensures weak convergence: $f_k(x^i, u^i) \rightharpoonup f_k(x,u)$ on $\mathcal{I}$ so standard arguments give (4)in the limit on $\mathcal{I}$. This, for each such $\mathcal{I}$, gives (4), (5) as desired. Finally, we need only verify the switching rules. Consider some t in an interval $\mathcal{I}$ as above for which $j(\cdot) \equiv k$. Since $\mathcal{R}_k$ is open, $\lim_i x^i(t) = x(t)$ cannot be in $\mathcal{R}_k$. This gives (SR_1). For a switching time $t = \tau_\nu$ we have $\tau_\nu^i \to \tau_\nu$ so, in view of the equicontinuity of the functions

$\{x^i(\cdot)\}$, we have $x^i(\tau^i_\nu) \to x(\tau_\nu)$. Since the switching sets are closed, this gives $x(\tau_\nu)$ in the appropriate switching set, verifying (SR_2). Thus, $\mathbf{s} \in \mathcal{A}_0$ and so $\mathbf{s} \in \mathcal{A}$. ■

4.OPTIMAL CONTROL

Suppose we have a controlled switching system with specified initial conditions and target set, as above. In this section we introduce a cost functional $\mathbf{j}$ and consider the **optimal control problem** of minimizing $\mathbf{j}$ over the set of admissible controlled solutions. Under suitable hypotheses we show the existence of an **optimal control**, attaining this minimum.

DEFINITION:
We consider a **cost functional** of the form:

$$\begin{aligned} \mathbf{j} \;=\; & \int_0^T \phi_{j(t)}(x(t), u(t))\, dt + \Phi(x(T)) \\ & + \sum_\tau \hat{\phi}_{k-,k+}(\tau, x(\tau)) \end{aligned} \tag{11}$$

where the final term on the right is a sum over switching times τ with $k- = j(\tau-)$ and $k+ = j(\tau+)$.

HYPOTHESES:
(H_3) For $k = 1, 2$, assume that $\phi_k(\xi, \cdot)$ is convex on $\mathcal{V}$ for each $\xi \in \mathcal{X}$; for the various indices, assume that the functions Φ, ϕ_k, and $\hat{\phi}_{k,k'}$ are continuous in their assorted variables.

Lemma 1: *Let $\mathcal{A}_0$, including topology, be as above; let $\mathbf{j}$ be as in (11). Assume (H_1) and (H_3). Then the cost functional: $\mathbf{s} \mapsto \mathbf{j} : \mathcal{A}_0 \to \mathbb{R}$ is lower semicontinuous.*

PROOF : With the compactness noted in Remark 2 for trajectories and controls, each $\phi_k(\cdot)$ is uniformly continuous on its (compact) relevant domain. With the assumed convexity one easily sees lower semicontinuity of the integral term in (11). Continuity of the endpoint term is immediate and we need only check the switching costs. Since $\mathbf{s}^i \to \mathbf{s}$ gives uniform convergence: $x^i \to x$ and $\tau^i \to \tau$, one has $\hat{\phi}_{k-,k+}(\tau^i, x^i(\tau^i)) \to \hat{\phi}_{k-,k+}(\tau, x(\tau))$ so the (finite) sum is continuous. ■

Theorem 3: *Consider the setting above and assume* (H_1), (H_2), (H_3). *Then* $\mathbf{j}$ *attains its minimum: there exists at least one optimal controlled solution.*

PROOF : By (H_2) the set $\mathcal{A}$ of admissible controlled solutions is nonempty so we can find a minimizing sequence $\{\mathbf{s}^i\} \subset \mathcal{A}$ for $\mathbf{j}$, i.e., $\mathbf{j}^i \to \inf \mathbf{j}$. By Theorem 2 we may extract a subsequence convergent in $\mathcal{A}$ so $\mathbf{s}^i \to \mathbf{s}$. By Lemma 1, we have $\mathbf{j}(\mathbf{s}) \le \lim \mathbf{j}(\mathbf{s}^i) = \inf \mathbf{j}$ so $\mathbf{j}$ attains its minimum at this admissible controlled solution $\mathbf{s}$. ■

REMARK: (4) In this paper we do not consider the derivation of 'first order necessary conditions' which might be used to characterize optimally controlled solutions. We can, however, note an approach to such characterization by reduction to a more familiar setting. The form of any solution consists of a finite number of 'segments' governed by ordinary differential equations (4) with j constant and 'endpoint conditions' $x(\tau_\nu) = \xi_\nu$. For each of these *separately* one can treat the problem of optimal control for these as independent problems with $\{[\tau_\nu, \xi_\nu \,:\, \nu = 1, \ldots, \bar{\nu}]\}$ taken, temporarily as fixed parameters and the usual approaches to neccessary conditions apply. The 'optimal' solution obtained is, of course, a function of these parameters and one completes the computation of the final result by optimizing over the (finite dimensional) manifold of admissible sets of parameters — with, e.g., the constraint that $\xi_1 \in \mathcal{S}_{1,2}$, etc.

5.DISCUSSION

In this section we conclude with some remarks on motivation and some possible connections of this theory to other problems. Some, though not all, of this is treated briefly in [8] and will be treated more fully in [10].

The theory of switching systems, as presented in [8], was originally motivated by an attempt to model a thermostat (compare, e.g., [4], [1]) and to study the existence of time-periodic solutions (cf., [9]). Switching systems can also arise in analysis of the feedback-controlled behavior of a multimodal system for which 'optimal' feedback is implemented through consideration of switching surfaces, here corresponding to the sets $\mathcal{S}_{k,k'}$, (compare, e.g., [3]).

Another possible source of switching system models, to be treated in more detail in [10], is *singular perturbation theory* (compare, e.g., [6], [5]). For a control problem

$$\dot{x} = F(x, y, u) \tag{12}$$

$$\varepsilon\dot{y} = G(x, y) \tag{13}$$

with 'very small' ε, one formally approximates by setting $\varepsilon = 0$ in (13): solve $G(\xi, \eta) = 0$ for η to get $\eta = Y(\xi)$ and substitute this into (12). If there may be *multiple solution branches* $\eta = Y_k(\xi)$, then this leads to the controlled switching system:

$$\dot{x} = f_k(x, u) := F(x, Y_k(x), u) \tag{14}$$

with switching sets corresponding to the bifurcations involved in solving $G(\xi, \eta) = 0$ for η. It is not unreasonable to hope (subject, of course, to 'suitable hypotheses' — e.g., assuming certain stability properties of the 'fast dynamics') that the analysis of (14) would provide a good approximation to the corresponding behavior for (12), (13).

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