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SOME PROBLEMS WITH THERMOSTAT NONLINEARITIES [1]

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Abstract

The *thermostat nonlinearity* is the simplest form of history-dependent feedback control with a discontinuity. We consider here a class of distributed parameter control problems with feedback involving a thermostat whose time-dependent setting $u(\cdot)$ is now taken as a control to be optimized. Existence of an optimal control is shown first for (known) input, and then considered for a formulation of the problem with a stochastically modeled input.

1. Introduction

The type of problem we wish to consider may be described, somewhat vaguely, as follows:

> Consider the heat distribution in a region Ω (e.g., a house), coupled through the boundary conditions with the external heat flux. This is to be regulated for comfort by a furnace or air conditioner actuated by a thermostat. Here we assume the thermostat has a clock so the (time-dependent) setting can be specified. This advance specification of the thermostat setting is our control —

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to be chosen so as to optimize a suitable comfort criterion. If we might model the uncertainty in our prediction of the external heat flux as a stochastic input, then we would seek to optimize the *expectation* of this criterion.

One may view this as an open loop optimal control problem or, alternatively, view the thermostat itself as a feedback control, with the choice of setting as an optimal design problem.

For mathematical analysis, of course, this description must be made more precise. In particular, we must pay considerable attention to the stochastic nature assumed for uncertainty in the predicted external input and to the prescription of a reasonable set $\mathcal{U}_{ad}$ of admissible controls. We will work here with the 'standard thermostat' — see, e.g., [1], [10], [8], [9]; note that the control now enters in quite a different way than was treated in [8].

For the present, we are considering the basic questions of existence (of solutions to the controlled problem and of an optimal control). The characterization of optimal controls through 'necessary conditions' — which originally motivated the particular choice of problem — remains to be considered later. This paper is somewhat of a 'report on work in progress': we have some partial results but much yet remains to be done.

The problem we will consider was stimulated by the analysis of [3], [2]. There, a thermostat model of a somewhat different sort (the *Preisach model* for hysteresis, cf. [7], [11]) was introduced as a regularization with the idea of obtaining necessary optimality conditions for optimal controls. For essentially similar reasons, as well as the direct applicability, it seemed of interest to introduce a stochastic element in the problem in the hope that one effect of the 'noise' would be to smooth out the discontinuity inherent in the consideration of a 'standard' thermostat nonlinearity.

The general mathematical framework for this analysis then takes the form:

$$\dot{x} = \mathbf{A}x + v\alpha + k\beta \tag{1.1}$$

$$v = \boldsymbol{\Theta}_u[y] \tag{1.2}$$

$$y = \langle \Lambda, x \rangle \tag{1.3}$$

where $\mathbf{A}$ is the operator giving evolution of the heat distribution, $k\beta$ is the effect of the external input, and $v\alpha$ is the effect of the output v of the thermostat $\boldsymbol{\Theta}_u$ with setting $u = u(t)$ given input y coming from a sensor described by the functional Λ. It is $k(\cdot)$ which will here be taken to be stochastic and $u(\cdot)$ which is our (deterministic) control.

As a more concrete example, one may think of (1.1) as an abstract form of the parabolic partial differential equation

$$x_t = \Delta x \qquad\qquad \text{on } \mathcal{Q} := (0,T)\times\Omega \tag{1.4}$$

with, say, first order boundary conditions:

$$\begin{aligned} \mathbf{B}_1 x &= k(t)\varphi_1(\cdot) \qquad \text{on } \Sigma_1 := (0,T)\times\Gamma_1 \\ \mathbf{B}_2 x &= v(t)\varphi_2(\cdot) \qquad \text{on } \Sigma_2 := (0,T)\times\Gamma_2 \end{aligned}$$

where the boundary Γ of the spatial domain Ω consists of two disjoint parts Γ_j $(j = 1,2)$ with corresponding first order boundary operators $\mathbf{B}_j$. Then $\mathbf{A}$ would be the Laplace operator Δ, say, acting as a closed (selfadjoint) unbounded operator on a suitable space $\mathcal{H}$ of functions on Ω —e.g., $\mathcal{H} = L^2(\Omega)$ with the domain $\mathcal{D} = \mathcal{D}(\mathbf{A})$ given by the homogeneous boundary conditions. Here, α, β are symbolically given by: $\alpha = -\mathbf{A}e_1$, $\beta = -\mathbf{A}e_2$ with

$$\begin{array}{ll} \Delta e_1 = 0 & \qquad\qquad \Delta e_2 = 0 \\ \mathbf{B}_1 e_1 = \varphi_1(\cdot) & \qquad\qquad \mathbf{B}_1 e_2 = 0 \\ \mathbf{B}_2 e_1 = 0 & \qquad\qquad \mathbf{B}_2 e_1 = \varphi_2(\cdot). \end{array}$$

Note that if the boundary conditions are subsumed in the specification of $\mathcal{D}$ then $e_j \notin \mathcal{D}$ so α, β are, at best, 'generalized elements' of $\mathcal{H}$. For this example we take the sensor functional $\langle \Lambda, \cdot \rangle$ of (1.3) to be given by point evaluation at some interior point of Ω, the location of the thermostat. For simplicity we have included no convection in (1.4). Γ_2 may be interpreted as the interface between the heating system (furnace and radiators) and the house (Ω) while Γ_1 is interpreted as [walls, windows, etc.].

We will impose the following set of hypotheses regarding the abstract model (1.1):

(H)

1. $\mathbf{A}$ generates an analytic semigroup $\mathbf{S}(\cdot)$ of compact linear operators on $\mathcal{H}$;

2. $\mathbf{S}(t)$ extends by continuity, for $t > 0$, as an operator on a space $\mathcal{W}$ such that $\alpha, \beta \in \mathcal{W} \subset \mathcal{D}(\Lambda)$ and $\mathbf{AS}(\cdot)\beta \in L^1((0,T) \to \mathcal{W})$;

3. $\varphi(\cdot) := \langle \Lambda, \mathbf{S}(\cdot)\alpha \rangle \in L^1(0,T)$.

(We note that $\psi(\cdot) := \langle \Lambda, \mathbf{AS}(\cdot)\beta \rangle \in L^1(0,T)$ by **(H-2)**.)

These hypotheses are weak enough to include, e.g., the example above: take $\mathcal{W}$ to be a space of functions with suitable regularity near the boundary but smooth near the sensor location.

It is reasonable to include three kinds of costs in the *cost functional* $\mathcal{J}$: costs of 'discomfort' $\mathcal{J}^0$, 'operating costs' $\mathcal{J}^1$, and 'setup costs' $\mathcal{J}^2$ so $\mathcal{J} = \mathcal{J}^0 + \mathcal{J}^1 + \mathcal{J}^2$. We will consider $\mathcal{J}^0$ of the form:

$$\mathcal{J}^0 = \mathcal{J}^0(x) = \int_0^T g(t, x(t))\, dt \tag{1.5}$$

where $g : [0,T] \times \mathcal{W} \to \mathbb{R}_+$ is measurable in t, lower semicontinuous (in the $\mathcal{W}$-topology) with respect to its second variable, and of suitable growth in $|x|_{\mathcal{W}}$. For example, one might have

$$g(t, \xi) = p(t) |\langle \Lambda, \xi \rangle - \bar{y}(t)|$$

where $\bar{y}(\cdot)$ is a desired value and $p(\cdot)$ a weight. We take $\mathcal{J}^1$ to include a cost of having the furnace (or the air conditioner) 'on' plus a 'switching cost':

$$\begin{aligned} \mathcal{J}^1 = \mathcal{J}^1(v) \quad &= \quad c_1 \,\mathrm{meas}\{v = 1\} \\ &\qquad + c_2 \#\{t \in (0,T) : v \text{ changes at } t\}. \end{aligned} \tag{1.6}$$

Finally, we take $\mathcal{J}^2 = \mathcal{J}^2(u)$ to be some (lower semicontinuous?) measure of the 'complexity' of the (admissible) control [= setting function] $u(\cdot)$ — e.g., the number of 'jumps' permitted for u plus the number of parameters (with a bound on their range) for a spline representation on each 'segment'. (One might remark that a measure of this sort would enforce a compactness condition with respect to the $\mathcal{U}$-topology we will use.)

Clearly, if we consider a stochastic model for the input $k(\cdot)$ in (1.1), then $\mathcal{J}$ will be a random variable, indicated for emphasis by writing $\mathcal{J} = \mathcal{J}_\omega$, and we will wish to choose the control u in order to minimize the *expectation*

$$\bar{\mathcal{J}} = \boldsymbol{\mathcal{E}}[\mathcal{J}_\omega] := \int \mathcal{J}_\omega \, \boldsymbol{\mathcal{P}}(d\omega) \tag{1.7}$$

where $\boldsymbol{\mathcal{P}}(\cdot)$ denotes the probability measure associated with the stochastic nature of $k = k_\omega$.

2. The Thermostat Nonlinearity

What is our thermostat model? We take the thermostat as having an *internal state* which we can identify with the output; with no loss of generality we may normalize to take this as being 0 or 1 at each t. Now, for an input $y = y(\cdot)$, the evolution of this internal state is governed by the *switching rules*:

(SR)

1. If $v = 1$ and y crosses the value ϑ_+ from below at $t = \tau$, then (and only then) v switches to 0 at $t = \tau$;
2. If $v = 0$ and y crosses the value ϑ_- from above at $t = \tau$, then (and only then) v switches to 1 at $t = \tau$.

Note that **(SR)** makes sense provided we would know that $v(\tau-)$ makes sense, e.g., that there would be only finitely many switches in any interval $[0, \tau)$ — which would follow from a lower bound on the length of the 'interswitching interval'.

There is some delicacy, a problem with continuous dependence, if one would have $v = 1$ and y would rise from below to take the value ϑ_+ at $t = \tau$ but (assuming no change in v) would not *cross* ϑ_+; we refer to such a τ as an *anomalous point*. In the deterministic theory, the possible existence of such points — at which one has a nondeterministic evolution, permitting *both* solutions: with a switch at $t = \tau$ and without — is of particular significance: see, e.g., [1], [9]. More precisely, if one has $\tau \in (0, T)$ with $y(\tau) = \vartheta_+$, then we call τ an anomalous point for this input unless either $v \equiv 0$ in a neighborhood of τ or y crosses ϑ_+ 'strictly': one has $y < \vartheta_+$ on some interval $(\tau - \varepsilon, \tau)$ and $y > \vartheta_+$ on $(\tau, \tau + \varepsilon)$ — and similarly, of course, for the 'reflected' situation in which y is decreasing to take the value ϑ_- without crossing. Modulo these technical questions, we take **(SR)** as defining the 'thermostat nonlinearity'.

For the stochastic theory one imposes suitable hypotheses to ensure (e.g., applying the Borel-Cantelli Lemma) that one would almost surely have finitely many switching times and also that the possible difficulty with anomalous points can only occur with probability 0 (3.4), hence can be ignored.

The thermostat model is then characterized by the pair ϑ_-, ϑ_+. Typically, the width $[\vartheta_+ - \vartheta_-] > 0$ of the 'dead band' is fixed; with no loss of generality this can then be normalized to 1 and we then take ϑ_- as the 'setting'. It is clear from **(SR)** that it is entirely equivalent to take $y = y(\cdot)$ as input to the thermostat with setting $\vartheta_- = u(\cdot)$ or to take $[y - u]$ as input to the 'standard thermostat' $\boldsymbol{\Theta}$ with fixed $\vartheta_- = 0$, $\vartheta_+ = 1$ — i.e., $\boldsymbol{\Theta}_u[y] = \boldsymbol{\Theta}[y - u]$. We will adopt the latter formulation for (1.1).

At this point we may ask: *What are the 'reasonable' setting functions* $\vartheta_- = u(\cdot)$*?* From our intuition as to practical application we would like to include piecewise continuous functions in this class $\mathcal{U}$ but our choice of $\mathcal{U}_{ad}$ is essentially dictated by the technical demands of our existence theorems. As to the output, since v appears on the right hand side of the differential equation (1.1) it will be reasonable to consider it in the set $\mathcal{V}$ of piecewise constant functions on $[0, T]$, topologized by considering the (finite) set of switching times in some $\mathbb{R}^K$ (with the provision that a pair of adjacent switching times moving to coincidence may be simply deleted — so the $\mathcal{V}$ topology is just that of weak-* convergence in $L^\infty(0, T)$ with a bound on the number of switching times.)

Lemma 1: Assume **(H)**. Suppose $k(\cdot)$ is the (distributional) derivative of a function $K \in \mathcal{C}[0, T]$ and $v(\cdot) \in \mathcal{V}$. Then there is a unique solution $x = x(\cdot; k, v)$ of the (abstract) differential equation

$$\dot{x} = \mathbf{A}x + v\alpha + k\beta, \qquad x(0) = x_0 \in \mathcal{W} \tag{2.1}$$

with $y := \langle \Lambda, x \rangle \in \mathcal{C}[0, T]$, depending continuously on $v \in \mathcal{V}$, $K \in \mathcal{C}[0, T]$.

Proof: From semigroup theory we have a representation for the mild solution of (2.1):

$$\begin{aligned} x(t) &= \mathbf{S}(t)x_0 + \int_0^t \mathbf{S}(t-s)\alpha\, v(s)\, ds \\ &\quad + \int_0^t [\mathbf{A}\mathbf{S}(t-s)\beta] K(s)\, ds - K(t)\beta \end{aligned} \tag{2.2}$$

which gives the existence of the solution $x(\cdot) \in \mathcal{C}([0,T] \to \mathcal{W})$. Since, after applying the sensor functional Λ, one has

$$\begin{aligned} y &= y_0 + \varphi * v \qquad \text{with} \\ y_0 &= \langle \Lambda, \mathbf{S}(\cdot)x_0 \rangle + \psi * K - \langle \Lambda, \beta \rangle K(\cdot), \end{aligned} \tag{2.3}$$

one easily sees from **(H-2,3)** the continuity in t of y and the continuous dependence on v, K. ∎

Now we may ask: What space $\mathcal{U}$ would give continuity to $\mathcal{V}$ of $\boldsymbol{\Theta} : [y-u] \mapsto v$ for $y \in \mathcal{C}[0,T]$, $u \in \mathcal{U}$? We begin by noting the important property of *rate independence* for $\boldsymbol{\Theta}$: if $\mathbf{T}$ corresponds to a (continuous, isotone) transformation: $t \longleftrightarrow t'(t)$, then it is easy to see that $\boldsymbol{\Theta}[\mathbf{T}y] = \mathbf{T}\boldsymbol{\Theta}[y]$ — the relevant aspect of the input is the order of the ocurrences of (significant) local maxima and minima of $y-u$. Note that the topology chosen for $\mathcal{V}$ means that we will be concerned with a purely local analysis (in t) of the occurrence and timing of each switching point. It is precisely at this point that we note the significance of anomalous points: if $\bar{y} - \bar{u}$, however smooth, has an anomalous point at $\bar{t}$, then one can find smooth $y^\nu - u^\nu$, converging uniformly to it such that one has switching times $t^\nu \to \bar{t}$ but also, alternatively, such that there are no switching times t^ν at all near $\bar{t}$ — we expect a continuity result only if there are no anomalous points.

Starting with smooth functions u^ν, a pointwise analysis (as below) suggests the possibility of 'completion' to consider $u \in \mathcal{U}$ as having the form:

> Specify $0 = \tau_0 \le \tau_1 \le \ldots \le \tau_N < T = \tau_{N+1}$ and, for $j = 1, \ldots, N+1$, specify $\hat{u}_j \in \mathcal{C}(\mathcal{I}_j)$ where $\mathcal{I}_j$ is the closed 'interval' $[\tau_{j-1}, \tau_j]$; we also permit $\hat{u}_j \equiv \pm\infty$.
> Then $u \in \mathcal{U}$ is specified by giving N and $\{\tau_j\}$, $\{\hat{u}_j\}$.

Note that we consider the order to be significant, even in such a degenerate situation as $\mathcal{I}_j = \mathcal{I}_{j+1}$ ($\tau_{j-1} = \tau_j = \tau_{j+1}$). We topologize $\mathcal{U}$ by saying that $u^\nu \to u$ if (for large ν):

$$N = N^\nu \text{ is fixed and, for each } j,\ \tau_j^\nu \to \tau_j,\ \hat{u}_j^\nu \to \hat{u}_j. \tag{2.4}$$

Note that by "$\hat{u}_j^\nu \to \hat{u}_j$" we mean *uniform convergence*, e.g., after a 'normalization', mapping each $\mathcal{I}_j^\nu$ to $[0,1]$ with the obvious modifications for degenerate intervals and/or values $\pm\infty$.

In considering a thermostat nonlinearity Θ_u with jumps permitted for u, we have the obvious extended interpretation of the switching rules **(SR)** and of the notion of 'anomalous point' for inputs from $\mathcal{U}$. There is no change in the interpretation of **(SR)** in the interior of any of the intervals $\mathcal{I}_j$ and, noting the rate independence, we can 'expand' a jump at τ_j as a linear function with respect to a 'pseudotime' — the precise statement is rather complicated as we must consider, e.g., such possibilities as that t_* may be the common endpoint of degenerate intervals $\mathcal{I}_j$, $\mathcal{I}_{j+1}$ or that one may have infinite values. The key result, essentially the justification for our choice of $\mathcal{U}$, is the following lemma. (This really would require some minor modification of statement in the more complicated special cases indicated above, but then continues to be true.)

Lemma 2: Given $\bar{u} \in \mathcal{U}$, suppose $t_* \in (0, T)$ is *not* an anomalous point for Θ acting on $\bar{u}$; let $u^\nu \to \bar{u}$ in $\mathcal{U}$. Then,
(i) If t_* is not a switching time for $\Theta[\bar{u}]$, then there is $\varepsilon > 0$ such that there is no switching time for $\Theta[u^\nu]$ in $[t_* - \varepsilon, t_* + \varepsilon]$ for large enough ν.
(ii) If t_* is a switching time for $\Theta[\bar{u}]$ (say, from $v = 1$ to $v = 0$), then for large enough ν there are similar switching times t^ν for $\Theta[u^\nu]$, unique in some interval $[t_* - \varepsilon, t_* + \varepsilon]$, such that $t^\nu \to t_*$.

PROOF: The complete argument consists of the consideration of too many separate cases to present in full so we indicate only two of the simpler situations, leaving the others for the reader since the approaches are all much the same.

<u>Case 1:</u> Suppose $t_* = \tau_j$ (with $\mathcal{I}_j$, $\mathcal{I}_{j+1}$ nondegenerate) is not a switching time for $\Theta[\bar{u}]$. Suppose, e.g., $\bar{v} = \Theta[\bar{u}] \equiv 1$ near t_* so, since t_* is not an anomalous point, by assumption, we have $\hat{u}_j$ bounded away from 1 near τ_j (i.e., $\hat{u}_j(t) \leq 1 - \delta$ for $\tau_j - \varepsilon_1 \leq t \leq \tau_j$ with $\delta, \varepsilon_1 > 0$) and similarly for $\hat{u}_{j+1}$. For large ν one has, noting the uniform convergence, $|\hat{u}_j^\nu - \hat{u}_j| < \delta/2$ on $[\tau_j^\nu - \varepsilon_2, \tau_j^\nu]$ with a symmetric argument for $\hat{u}_{j+1}^\nu$. As $\tau_j^\nu \to \tau_j$, we cover $[t_* - \varepsilon, t_* + \varepsilon]$ by $[\tau_j^\nu - \varepsilon_2, \tau_j^\nu] \cup [\tau_j^\nu, \tau_j^\nu + \varepsilon_2]$ for each (large enough) ν so u^ν is bounded above by $1 - \delta/2$ on $[t_* - \varepsilon, t_* + \varepsilon]$, ensuring that there are no switching times there.

<u>Case 2:</u> Suppose t_* is interior to $\mathcal{I}_j$ and is a switching time for $\Theta[\bar{u}]$; e.g., $\bar{v} = \Theta[\bar{u}] = 1$ up to (near) t_* and then $\bar{v} = 0$. As t_* is not an anomalous point and we are in the interior of an interval of continuity for $\bar{u}$, we have $\bar{u}$ crossing 1 strictly from below. Thus, there exists $\varepsilon > 0$ with $\mathcal{I} = (t_* - \varepsilon, t_* + \varepsilon) \subset \mathcal{I}_j$

and $\delta = \delta(\varepsilon_1)$ [$\delta \searrow 0$ as $\varepsilon_1 \searrow 0$] such that $\frac{1}{2} < \bar{u} < 1 - \delta$ on $(t_* - \varepsilon,)$ on $(t_* - \varepsilon, t_* - \varepsilon_1)$ and $\bar{u} > 1 + \delta$ on $(t_* + \varepsilon_1, t_* + \varepsilon)$. Then uniform convergence for $\hat{u}_j^\nu$ gives $|u^\nu - \bar{u}| < \delta/2$ on $\mathcal{I}$ for large ν so u^ν (first) crosses 1 at some $t^\nu \in (t_* - \varepsilon_1, t_* + \varepsilon_1)$: let $\varepsilon_1 \searrow 0$. ■

Corollary 3: Suppose there are no anomalous points for the input $[\bar{y} - \bar{u}]$ to $\boldsymbol{\Theta}$. Then $y^\nu \to \bar{y}$ in $\mathcal{C}[0,T]$, $u^\nu \to \bar{u}$ in $\mathcal{U}$ implies that $v^\nu = \boldsymbol{\Theta}[y^\nu - u^\nu] \to \bar{v} = \boldsymbol{\Theta}[\bar{y} - \bar{u}]$ in $\mathcal{V}$.

PROOF: Immediate — we need only note that $y^\nu - u^\nu \to \bar{y} - \bar{u}$ in $\mathcal{U}$. ■

Remark 4: If there *are* (finitely many) anomalous points for $\bar{y} - \bar{u}$, then for any $y^\nu - u^\nu \to \bar{y} - \bar{u}$ one can find a subsequence for which $v^\nu = \boldsymbol{\Theta}[y^\nu - u^\nu] \to \bar{v}$ in $\mathcal{V}$ with $\bar{v} = \boldsymbol{\Theta}[\bar{y} - \bar{u}]$ provided we expand **(SR)** to admit *both* possibilities at an anomalous point. This even works consistently when $\bar{y}$ and $\bar{v}$ are dependent through the dynamics as (1.1)–(1.3); cf., [10].

Lemma 5: Assume **(H)** and suppose $k = \bar{K}'$ with $\bar{K} \in \mathcal{C}[0,T]$ (as in Lemma 1); fix $u \in \mathcal{U}$, $x_0 \in \mathcal{W}$, and a consistent initial state v_0 for $\boldsymbol{\Theta}$. Then there is an 'almost unique' solution $x = x(\cdot; k, u) \in \mathcal{C}([0,T] \to \mathcal{W})$ of (2.1) with

$$v = \boldsymbol{\Theta}[y - u], \qquad y = \langle \Lambda, x \rangle. \tag{2.5}$$

Further, with the definition:

> $\mathcal{S}_a(\bar{u})$ is the set of $K \in \mathcal{C}[0,T]$ with $K(0) = 0$ for which there is at least one anomalous point for y from (2.3) with $v = \boldsymbol{\Theta}[y - u]$,

one has continuous dependence of $x \in \mathcal{C}([0,T] \to \mathcal{W})$, $y \in \mathcal{C}[0,T]$, $v \in \mathcal{V}$ on $K \in \mathcal{C}[0,T]$, $u \in \mathcal{U}$ at $[\bar{K}, \bar{u}]$ subject only to the condition that $\bar{K} \notin \mathcal{S}_a(\bar{u})$.

PROOF: Existence is much as in the proof of Theorem 1 of [10] and, as in Remark 4 above, we require the extension of **(SR)** to admit the possibility of (ambiguous) anomalous switching times — which explains the 'almost uniqueness' of the result. One simply proceeds in t, using Lemma 1, switching v as appropriate (allowing for possible ambiguities). The only technical point of interest is to note that (2.3) gives y in a compact subset of $\mathcal{C}[0,T]$ (even with v ranging over all of $\mathcal{V}$ with the bound $|v| \le 1$), ensuring a bound on the variation of y and so, for fixed $u \in \mathcal{U}$, on the possible number of switching times.

The continuous dependence follows immediately from the above, noting the uniqueness when there are no anomalous points, by Lemma 1 and Corollary 3. ■

Theorem 6: Assume **(H)** and suppose $k = \bar{K}'$ with $\bar{K} \in \mathcal{C}[0,T]$; fix $x_0 \in \mathcal{W}$ and a consistent initial state v_0 for $\boldsymbol{\Theta}$. Let $\mathcal{J}$ be given as above — by (1.5), etc. — with $\mathcal{J}^2$ lower semicontinuous on $\mathcal{U}$ and assume, where there is ambiguity in the 'choice' of evolution along the lines of Remark 4 and Lemma 5, that this choice will be taken so as to minimize $\mathcal{J}$ for that u. Then the map: $u \mapsto \mathcal{J}$ (for the fixed $k = \bar{K}'$, etc.) is lower semicontinuous from $\mathcal{U}$ so there exists an optimal control $\bar{u} \in \mathcal{U}_{ad}$ for any $\mathcal{U}_{ad}$ for which a bound on $\mathcal{J}^2$ gives compactness in $\mathcal{U}$.

PROOF: The map: $u \mapsto [x, y, v]$ is well-defined and is actually continuous at any $\bar{u}$ for which $\bar{K} \notin \mathcal{S}_a(\bar{u})$ by Lemma 5, so $u \mapsto \mathcal{J}$ is lower semicontinuous then by the assumptions on $\mathcal{J}$ — and is lower semicontinuous in general along the lines of Remark 4. The usual argument by considering a minimizing sequence then gives existence of an optimal control. ■

3. Stochastics

For simplicity we are assuming that (the effect of) the external weather is characterized by a scalar function $k(\cdot)$ with a stochastically modeled uncertainty, $k = k_\omega$.

The first stochastic model to suggest itself is that we take k to have a known mean $k_0(\cdot)$ with additive white noise so

$$k\,dt = k_0\,dt + \sigma\,dw \tag{3.1}$$

where $w(\cdot)$ is standard Brownian motion and $\sigma > 0$ is given — constant, for simplicity. The problem (1.1), (1.2), (1.3) now (formally) becomes

$$dx = (\mathbf{A}x + \boldsymbol{\Theta}[\langle \Lambda, x\rangle - u]\alpha + k_0\beta)\,dt + (\sigma\beta)\,dw. \tag{3.2}$$

Because of the rather discontinuous and history dependent nature of the thermostat nonlinearity, this stochastic nonlinear partial differential equation does not quite satisfy the hypotheses of available treatments for stochastic equations (e.g., [4], [6], [5], [12]) but seems amenable to similar modes of analysis.

We will assume, as for Lemma 1, etc., that

$$k_0 = K_0' \text{ with } K_0 \in \mathcal{C}[0,T],\ K_0(0) = 0 \tag{3.3}$$

(distributional derivative) whence (3.1) gives simply: $k = K'$ with $K = K_0 + \sigma w$. Note that, almost surely, $K = K_\omega$ is continuous so, proceeding 'sample path'wise using Lemma 5, we almost surely have a solution (using any fixed $u \in \mathcal{U}$). For this to be a *stochastic solution*, we must show measurability in ω. From (2.2), it is sufficient to show the measurability of (the switching times for) $v = v_\omega$. Modulo some technical difficulties with anomalous points, this is a standard 'stopping time' argument. As of this writing some points remain unresolved — this is the principal gap in our argument — and we must *postulate* that

$$\boldsymbol{\mathcal{P}}\{K_\omega = [K_0 + \sigma w] \in \mathcal{S}_a(u)\} = 0 \text{ for all } u \in \mathcal{U}. \tag{3.4}$$

We then have:

Lemma 1: Assume **(H)**; fix $u \in \mathcal{U}$, $x_0 \in \mathcal{W}$ and a consistent v_0. Then there is a stochastic solution $x_\omega = x(\cdot; k_\omega, u)$ of (2.1), (2.5). ■

We might note a rather interesting line of argument for the measurability. Consider, for the moment, a genuine stochastic control problem (so the switching times are not given directly by a thermostat feedback but, rather, constitute a stochastic 'impulse control') with a modified cost functional penalizing the possibility that switching occurs other than as in **(SR)**. Using a smooth penalty term, one knows that this stochastic problem has a stochastic solution and the argument for our problem follows if one can obtain it in the limit of penalization. Again, this limit behavior is clear once one has (3.4) — which would itself follow from suitable uniform estimates for the approximating problems.

Note that, under our assumptions, Lemma 7 makes $\mathcal{J}_\omega$ a well-defined random variable. Setting $\mu = \mu_\omega = \|w\|_\infty$, we know all moments of μ are finite; from (2.2), (2.3) we have estimates for $|x(t)|_{\mathcal{W}}$ and the variation of y which grow linearly with μ. Thus, if we take g in (1.5) to be of, say, power growth in $|x|$, there is an integrable bound:

$$0 \le \mathcal{J}_\omega \le a + \mu_\omega^p + \mathcal{J}^2 \tag{3.5}$$

(with a, p independent of the choice of $u \in \mathcal{U}$) so $\bar{\mathcal{J}} = \bar{\mathcal{J}}(u)$ is well-defined.

Theorem 2: Assume **(H)**, the accumulated hypotheses on $\mathcal{J}$, and (3.4); fix the initial data and k_0 in (3.1) suitably. Then, under the assumptions on $\mathcal{U}_{ad}$ (or on $\mathcal{J}^2$) as in Theorem 6, there exists an optimal control $\bar{u} \in \mathcal{U}_{ad}$, minimizing the cost functional $\bar{\mathcal{J}}$ of (1.7).

PROOF: Consider a minimizing sequence $\{u^\nu\}$ in $\mathcal{U}_{ad}$ for $\bar{\mathcal{J}}$; by the compactness assumption we may assume $u^\nu \to \bar{u}$ in $\mathcal{U}_{ad}$. By (3.4) and Theorem 6, we have $\mathcal{J}_\omega(\bar{u}) \leq \underline{\lim}_\nu \mathcal{J}_\omega(u^\nu)$ almost surely (i.e., ω-ae) so, noting the integrable uniform bound (3.5), the Dominated Convergence Theorem gives

$$\begin{aligned} \bar{\mathcal{J}}(\bar{u}) &\leq \int \underline{\lim}_\nu \mathcal{J}_\omega(u^\nu)\, \mathcal{P}(d\omega) \\ &\leq \underline{\lim}_\nu \bar{\mathcal{J}}(u^\nu) = \inf\{\bar{\mathcal{J}}(u) : u \in \mathcal{U}_{ad}\} \end{aligned}$$

which makes $\bar{u}$ optimal. ∎

An interesting alternative to (3.1) as a stochastic model for the external input might be the following. Consider functions k_0, k_1 as in (3.3), which we interpret as giving (the effect of) the weather to each side of a 'front', and a constant k_*, the weather 'in' the front. We may take the motion of the front as given by a Brownian random variable $s = s_\omega$ with drift κ (so $ds = \kappa(t)\, dt + \sigma\, dw$ and, say, $s(0) = s_0$) and then define $k = k_\omega$ by

$$k = \begin{cases} k_0 & \text{if } s < 0 \\ k_* & \text{if } 0 \leq s \leq s_* \\ k_1 & \text{if } s_* < s \end{cases}$$

for $t \in [0, T]$. Note that, in this case, almost surely k_ω is piecewise as in (3.3) with a uniform bound for the integral $K = K_\omega$. Again we can use Lemma 5 to obtain $x_\omega = x(\cdot; k_\omega, u)$ for each ω (and any $u \in \mathcal{U}$). Again we postulate (3.4) and, essentially as above, again we would have the equivalents of the existence results above: Lemma 7 and Theorem 8.

References

[1] H. W. Alt, "On the thermostat problem", *Control and Cybernetics* **14**, pp. 171–193, 1985.

[2] M. Brokate, "Optimal control of systems with hysteresis", presented at the *4th International Conference on Control of Distributed Parameter Systems* (Vorau, 1988).

[3] M. Brokate and A. Friedman, "Optimal control for heat conduction problems with hysteresis", IMA preprint, 1988.

[4] R. J. Elliott, *Stochastic Calculus and Applications*, Springer-Verlag, New York, 1982.

[5] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, New York, 1986.

[6] E. Pardoux, (thèse) *Équations aux dérivées partielles stochastiques non linéaires monotones. Études de solutions fortes de type Ito*, Orsay, 1975.

[7] F. Preisach, "Über die magnetische Nachwirkung", *Zeitschrift für Physik* **94**, pp. 277–302, 1935.

[8] T.I. Seidman, "Optimal control for switching systems", in *Proceedings of the 21st Annual Conference on Information Science and Systems*, Johns Hopkins Univ., 1987, pp. 485–489.

[9] T. I. Seidman, "Switching systems and periodicity", in *Nonlinear Semigroups, PDE, and Attractors* (Proceedings of the 2nd Howard Univ. Symposium: 1987), T. Gill and W.W. Zachary, eds., Springer-Verlag Lecture Notes in Math., to appear.

[10] T.I. Seidman, "Switching systems, I", to appear.

[11] A. Visintin, "On the Preisach model for hysteresis", *Nonlinear Analysis* **8**, pp. 977–996, 1984.

[12] J. Zabczyk, *Topics in Stochastic Systems* (1-Z-4015-88), Flight Systems Research Laboratory, UCLA, 1988.