SWITCHING SYSTEMS, I[1]

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ABSTRACT:

A model is introduced for a *switching system* consisting of a number of *modes* (e.g., differential equations) together with a set of *switching rules.* The rules permit occasional nonuniqueness but are shown to preserve a continuous dependence property: the limit of solutions is a solution.

KEY WORDS:

switching, multimodal system, existence, global, differential equation

[1]This research has been partially supported by AFOSR and NSF under grants AFOSR-82-0271 and CDR-85-00108, respectively.

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1.Introduction

By a *switching system* we mean a (finite) set of *modes* together with a set of *switching rules* of a special form. A more detailed definition will be provided later but, for orientation, let us consider a prototypical example.

Example 1: Let X be the plane $\mathbb{R}^2$ and suppose one has available two modes given by the differential equations

$$\dot{x} = f_j(x) \qquad\qquad (j = 1, 2;\ x \in X)$$

(Actually, we suppose that there are (closed) sets $\mathcal{U}_j \subset X$ for which each mode is available so it is only in $\mathcal{U}_1 \cap \mathcal{U}_2$ that *both* $j = 1$ *and* $j = 2$ would be available.) The switching rule is a modified form of the heuristic: Don't switch unless you must! Introducing the complementary *forbidden sets* $\mathcal{R}_j := X \backslash \mathcal{U}_j$, we consider the interesting case in which $\mathcal{R}_1$ contains a global attractor for the mode $\dot{x} = f_1(x)$ and similarly for $\mathcal{R}_2$; we assume, here, that $\mathcal{R}_1, \mathcal{R}_2$ are disjoint so at least one mode is always available. Suppose we start the system with state $x(0)$ in (the interior of) $\mathcal{U}_1 \cap \mathcal{U}_2$ and in mode 1. Our solution coincides with the solution of $\dot{x} = f_1(x)$ until this hits the boundary $\partial\mathcal{U}_1 = \partial\mathcal{R}_1$; this is inevitable since we have assumed $\mathcal{R}_1$ contains a global attractor. Since it is forbidden to enter $\mathcal{R}_1$ in mode 1 (i.e., for a state $x(t) \in \mathcal{R}_1$ the mode $j = 1$ is unavailable), we switch to mode 2. With this change in mode the trajectory may or may not enter $\mathcal{R}_1$ but coincides with a solution of $\dot{x} = f_2(x)$ until hitting $\partial\mathcal{U}_2 = \partial\mathcal{R}_2$. We expect, then, to follow the individual differential equations alternately, switching from $j = 1$ to $j = 2$ at $\partial\mathcal{R}_1$ and back from $j = 2$ to $j = 1$ at $\partial\mathcal{R}_2$. $\square$

An apparently minor quibble arises: Suppose there would be a point $\xi \in \partial\mathcal{R}_1$ for which the equation $\dot{x} = f_1(x)$ has a trajectory τ passing through ξ but staying locally in $\mathcal{U}_1$ (see Figure 1), i.e., tangential[4] to $\partial\mathcal{R}_1$ at ξ. Do we or do we not switch? This figure shows a neighboring trajectory τ entering $\mathcal{R}_1$ at ξ_1 so, for the switching system, the policy is clear: switch modes at ξ_1. On the other hand, the figure shows another neighboring trajectory τ_2 which never hits $\partial\mathcal{R}_1$ until ξ_2 so the policy for the switching system is again clear: switch modes at ξ_2. Taking limits through trajectories like τ_1 we would expect switching at ξ for a switching system trajectory coinciding (in part) with the earlier part of τ while taking limits through trajectories like τ_2 would suggest, for the switching system, that one continue to coincide with τ and defer switching until ξ'. In order to have any chance of preserving the principle that a limit of solutions should be a solution, we accept *both* possibilities: switching at ξ *and* deferring switching until ξ'. This means, of course, that we must accept the consequence that a solution for the switching system which initially coincides with the early part of τ must then have a nonunique continuation. This possibility of nonuniqueness is a significant characteristic[5] of the theory of switching systems — although for the considerations of this paper it does not materially affect the results. Even apart from this possibility, we will see from Examples 3 and 4 that care is needed in deducing properties of a switching system from corresponding properties of the modes used to define it.

In the next section we will (very briefly) indicate some possible applications/ examples of the formulation although a more detailed exploration of these will be deferred to a later presentation. Here, these are only intended as suggestive and to motivate the precise formulation of the switching rules in Section 3.

Section 4 contains our principal result, on the continuous dependence of regular solutions, together with a global existence result. Section 5 discusses an important special case: linear switching systems. Finally, the last section will note some open problems and directions for generalization. A principal topic for further discussion is the existence of periodic solutions but this will

[4]Although shown as such in Figure 1, this need not mean tangency in the usual geometric sense since we do not impose enough regularity on $\partial\mathcal{R}_j$ for this to be necessarily meaningful; it might be better to say that τ fails to be transversal at ξ. All we really mean, here, in using the term 'tangential', is that τ contains a point ξ of $\partial\mathcal{R}_1$, but does not enter $\mathcal{R}_1$ at ξ.

[5]Note that the geometry of Figure 1 is *generic* in that *some* (nearby) point of tangency must occur for any small perturbation of the direction field and/or the boundary $\partial\mathcal{R}_1$.

be deferred to a separate paper [8]; it is in this context that the possibility of nonuniqueness described above becomes overwhelmingly significant.

2.Motivation

The original motivation for formulating a notion of 'switching systems' came from an attempt to model thermostats. The two modes, in this case, correspond to FURNACE OFF and FURNACE ON and the thermostat is a device to switch between these. It consists of a sensor (measuring the temperature θ at a particular position) and a pair of set points θ_1, θ_2 (Typically, the gap $\theta_2 - \theta_1 > 0$ is fixed and the mean $(\theta_1 + \theta_2)/2$ is adjustable.) with two internal states corresponding to the two modes. If the furnace is OFF, then it will be switched ON when θ crosses θ_1 from above; in particular, the furnace will always be on when $\theta < \theta_1$. When, eventually, the temperature rises to have θ cross the upper set point θ_2 (from below), then the furnace will be switched OFF; no recrossing of the lower set point affects the state until this occurs; the furnace is always OFF for $\theta > \theta_2$.

This, together with the partial differential equation governing the evolving state (spatial temperature distribution), seems to describe the physics quite well except for the same minor quibble noted in the Introduction: What happens if, e.g., with the furnace ON, the sensed temperature $\theta(t)$ rises to θ_2 without (immediately) crossing? (Since the evolution is given by a pde, one can find initial conditions for which this is would actually occur.) It was the analysis of this situation which led to the present model. A completely different model is discussed, for example, in [5]; the discussion in [1] is closer in its concerns with the present analysis. More detailed analysis of switching system thermostat models will be deferred to [9].

An entirely different setting leading to switching system models is the Hamilton – Jacobi – Bellman formulation of optimal feedback control of multimodal (variable structure) systems with switching costs. Suppose one has a system which can operate in any of J modes, e.g., corresponding to differential equations

$$\dot{x} = f_j(x) \qquad (j = 1, \ldots, J) \tag{2.1}$$

with attendant (x, j)-dependent running costs but with the possibility of switching at any time from the j-th to the k-th mode with cost $c_{j,k}(x)$; there may also be other control possibilities implicit in (2.1). We suppose $V_j(x)$ is the optimal infinite horizon (discounted) cost if one is at the state x in the mode j. Clearly, we would switch to the k-th mode if $V_j(x) > V_k(x) + c_{j,k}(x)$ but would not switch if the reverse inequality would hold, i.e., we would always choose

$$k = argmin\{V_k(x) + c_{j,k}(x) \ : \ k = 1, \dots, J\} \tag{2.2}$$

where (x, j) is the current state and k is to be the 'new' mode (set $c_{j,j}(x) := 0$ for completeness). For a discussion of this approach, see [4]. The effort, then has gone into the construction of the value functions $\{V_j(\cdot)\}$ with (2.1, 2.2) taken as defining the controlled dynamics. We observe that the possibility of nonuniqueness in (2.2) means that a further, more detailed, analysis of the dynamics is needed precisely at the switching surfaces

$$\mathcal{S}_{j,k} := \{x : \ V_j(x) = V_k(x) + c_{j,k}(x)\}.$$

If we let $\mathcal{R}_j$ be the open set

$$\mathcal{R}_j := \{x : \ V_j(x) > V_k(x) + c_{j,k}(x) \ \text{ for some } \ k \neq j\},$$

then we will obtain a switching system model for the optimally controlled dynamics. Note that the possibility of nonunique continuation if a trajectory τ of $\dot{x} = f_j(x)$ is tangential to the switching surface simply means that the optimal cost is attained nonuniquely: either continuing without switching or paying the switching cost and continuing in a new mode give the same (optimal) cost.

There appear to be connections between the theory of switching systems presented here and 'viability theory', c.f., [2]. Of particular interest, in this connection, is the notion of a 'heavy trajectory', cf. [3], corresponding to the reluctance to change modes implicit in our switching rules. Our considerations are, however, almost disjoint from those of [2] since we can also write, e.g., (2.1) as a differential inclusion $\dot{x} \in F(x)$ by setting $F(x) := \{f_j(x) : j = 1, \dots, J\}$ but we are obviously emphasizing the case: $F(x)$ finite whereas [2] emphasizes the quite distinct case: $F(x)$ convex.

A final motivating setting comes from singular perturbation theory. Consider, for example, a system

$$\dot{x} = f(x, y), \qquad \varepsilon \dot{y} = g(x, y) \tag{2.3}$$

for very small $\varepsilon > 0$. A principal concern of singular perturbation theory is to initiate an analysis of (2.3) by comparison with the reduced order (implicit) model

$$\dot{x} = f(x, y) \ , \qquad\qquad g(x, y) = 0. \tag{2.4}$$

If we can solve $g(x, y) = 0$ to obtain $y = Y(x)$, then (2.5) becomes simply

$$\dot{x} = f_*(x) := f(x, Y(x)). \tag{2.5}$$

Suppose, however, the graph $\{(x, y) : g(x, y) = 0\}$ would look like that of Figure 2. As shown, we observe

that $A = \partial g/\partial y$ is negative, corresponding to stability of the perturbation equation: $\varepsilon\dot{y} = Ay$, along the branches $y = Y_j(x)$ for $j = 1, 2$ but we have $A > 0$ (instability) on $y = Y_*(x)$. Thus, local analysis shows that once we have $y \approx Y_i(x)$ with $a < x$ we would expect to stay close to the solution of

$$\dot{x} = f_1(x) := f(x, Y_1(x))$$

(for very small $\varepsilon > 0$) unless/until this solution would reach $x = a$. Assuming $f_1(a) < 0$, this trajectory would (try to) enter the forbidden region $\mathcal{R}_1 := (-\infty, a)$ for which Y_1 does not exist. One would have to make a more detailed analysis of (2.3) to see just what would happen then but, assuming $Y_2(\bar{x})$ would be globally attractive for $\dot{\zeta} = g(\bar{x}, \zeta)$ for $\bar{x} < a$, we would expect a rapid transient behavior 'switching' to the branch $y = Y_2(x)$. The solution would then stay close to this branch unless/until it reached $x = b$, with a similar 'switching' there if $f_2(b) > 0$. This behavior is, of course, the standard *hysteresis loop* and is essentially similar to the description of the thermostat above. Other interesting varieties of behavior become possible when x, y may 'live' in higher-dimensional spaces. We defer to [8] any further discussion of this and of the extent to which the switching system model may provide a suitable reduced order model with which to initiate the analysis of (2.3). The considerations seem complementary to those of [6].

3.Definitions

Motivated by the examples of the previous section, we introduce a formal definition of 'switching system'. This may not be the most general possible notion (c.f., Section 6 for further comment) but more than covers the motivating settings.

First, by a **mode** we mean a semidynamical system with *state space* X:

$$\pi : \mathbb{R}^+ \times X \longrightarrow X \text{ continuous, with } \pi(0, \xi) = \xi \text{ and} \tag{3.1}$$

$$\pi(t + s, \xi) = \pi(t, \pi(s, \xi)) \text{ for } t, s \geq 0.$$

Note that π may be interpreted as the solution map: $\xi =: x(0) \longmapsto x(t)$ for an autonomous differential equation[6] $\dot{x} = f(x)$ but the exposition is somewhat simpler this way and more general in some respects. We will consider a variable structure system corresponding to a set $\{\pi_j : j \in \mathcal{J}\}$ of such modes where $\mathcal{J}$ is suitable *index set* (usually finite: $j = 1, \ldots, J$).

For each $j \in \mathcal{J}$ we assume we are given a *forbidden set* $\mathcal{R}_j \subset X$ while for each pair $(j, k) \in \mathcal{J} \times \mathcal{J}$ there is an *admissible switching set* $\mathcal{S}_{j,k} \subset X$. (Any of these sets may possibly be empty.) Our concern with $\mathcal{R}_j$ is that the mode

[6]This restriction to autonomous equations is not very significant in the sense that one can formally include t as a component of the state. Somewhat more restrictive is the implicit assumption that a unique global solution (i.e., for all $t \in \mathbb{R}^+$) exists for each initial $\xi \in X$, see Section 6.

π_j may possibly *enter* $\mathcal{R}_j$ at some $\xi \in \partial\mathcal{R}_j$, by which we mean

(3.2)
(i) for some ξ_0 we have $\pi_j(\cdot, \xi_0) \in X \setminus \mathcal{R}_j$ on $(0, \varepsilon)$ with $\varepsilon > 0$ and $\pi_j(\varepsilon, \xi_0) = \xi$;
(ii) $\pi_j(\varepsilon, \xi) \in \mathcal{R}_j$ for arbitrarily small $\varepsilon > 0$.

We impose the set of geometric hypotheses:[7]

(3.3)
(i) $\mathcal{S}_{j,k} \cap \bar{\mathcal{R}}_k = \emptyset$, $\mathcal{R}_j = X \setminus \mathcal{S}_{j.j}$;
(ii) each $\mathcal{S}_{j,k}$ is closed in X (so each $\mathcal{R}_j$ is open);
(iii) for each $\xi \in \partial\mathcal{R}_j$ there is some $k \in \mathcal{J}$ such that $\xi \in \mathcal{S}_{j,k}$;
(iv) for each $j \in \mathcal{J}$: for each $\xi \in X$ there is a neighborhood $\mathcal{N}_j(\xi)$ which nontrivially intersects only finitely many of $\{\mathcal{S}_{j,k}\}$.

By a **switching system** we mean a specification:

$$\Sigma := [\mathcal{J}, \{\pi_j : j \in \mathcal{J}\}, \{\mathcal{S}_{j,k} :\ j, k \in \mathcal{J}\}]$$

subject to (3.1) and (3.3).

For a function: $t \longmapsto [x(t), j(t)] : \mathbb{R}^+ \to X \times \mathcal{J}$ to be considered as a possible solution of such a switching system Σ we first require that

(3.4)
(i) $j(\cdot)$ is piecewise constant with isolated[8] 'jumps';
(ii) whenever $j(\cdot)$ is constant $(= k)$ on an interval (s, t) we have $x(t) = \pi_k(t - s,\ x(s))$.

Note that this automatically forces $x(\cdot)$ to be continuous: $\mathbb{R}^+ \longrightarrow X$ since each mode π_k is continuous in t. The requirement that the 'jumps' in $j(\cdot)$ be isolated means that $j(\cdot)$ will have both left- and right-handed 'limits' at a jump:

(3.5) $$j(t-) = j\ , j(t+) = k\ \ (\text{with}\ \ j \neq k).$$

[7] This set might be weakened slightly if we did not prefer to keep it independent of any knowledge of $\{\pi_j\}$. For example, (iii) is really needed only if π_j would enter $\mathcal{R}_j$ at ξ and even then we might permit $\xi \in \partial\mathcal{R}_k$ if π_k does not enter $\mathcal{R}_k$ at ξ; compose the reformulation in Section 6.

[8] This requirement could be relaxed somewhat and we could refer to the notion here as defining a *regular solution* of Σ.

We refer to such a time t as a **switching time** and to the situation as a *transition* (from mode π_j to mode π_k) or, more succinctly, as a *switch*: $j \rightsquigarrow k$. If t' is a switching time (and there are any subsequent switches) then there is a unique *next* switching time t''. We refer to (t', t'') as an *interswitching interval* [9] We impose the *switching rules*:

(3.6)
(i) a switch: $j \rightsquigarrow k$ is permitted at time t only if $x(t) \in \mathcal{S}_{j,k}$
(ii) if $x(t) \in \mathcal{R}_j$, then $j(t) = j$ is forbidden — hence t must be a switching time if π_j enters $\mathcal{R}_j$ at $x(t)$ when one 'arrives' at $x(t) = \xi \in \partial\mathcal{R}_j$ with $j(t-) = j$.

Thus, by a **solution** of the switching system Σ on the time interval $[0, T)$ we mean a function pair

$$[x(\cdot),\ j(\cdot)] : [0, T) \longrightarrow X \times \mathcal{J}$$

subject to (3.4) and (3.6). For definiteness we take $j(\cdot)$ to be left-continuous. The values $[x(0), j(0)]$ are called the *initial data* for the solution; we have x continuous at 0 so $x(0+) = x(0)$ but we do permit (subject to (3.6) (i)) an immediate switch for $j(\cdot)$.

Remark 1: Usually we only consider solutions with $T = \infty$, i.e., $[0, T) := \mathbb{R}^+$. It should be clear that there is nothing 'magic' about 0 and *any* interval $\mathcal{I} := [a, b)$ would be equally appropriate but, as we consider only autonomous systems here, there is no loss of generality in translating in time by a to consider $[0, T)$ with $T := b - a$; initial data at $t = a$ then becomes initial data at $t = 0$. The other interesting possibility would be to consider solutions 'for all time' : $\mathcal{I} = \mathbb{R}$; in this case it is somewhat irrelevant to consider 'initial data'. This possibility will be of interest to us in the context of stable linear switching systems in Section 5: note Theorem 6.

Example 2: We present here, partly for its own sake[10] and partly in contrast with the flavor of Example 1, an example with an infinite dimensional state space X.

[9] We have $j(\cdot)$ constant (say, $= k$) on such an interswitching interval and (3.4) (ii) requires the solution simply to follow the mode π_k during it.

[10] The specific significance of this particular example lies in its canonical relation to general linear switching systems as discussed later in Theorem 6.

We take this space to be[11]

$$X^o := \{\xi(\cdot): \text{ measurable from } \mathbb{R}^+ \ to \ [0,1]\}. \tag{3.7}$$

The metric topology we consider on X^o will be that induced by a weighted L^1-norm:

$$d(\xi,\xi') := \int_0^\infty \varphi(s)|\xi(s)-\xi'(s)|ds, \tag{3.8}$$

where we assume that:

$$\text{the weight } \varphi \text{ is positive and, nonincreasing with} \tag{3.9}$$

$$\int_0^\infty \varphi(s)ds =: M < \infty,$$

e.g., $\varphi(s) = Ce^{-\alpha s}$. Next, set $\mathcal{J} := \{1,2\}$ and define the modes $\pi_j (j = 1,2)$ by

$$[\pi_j(t,\xi)](s) := \begin{cases} \xi(s-t) & \text{for } s > t, \\ j-1 & \text{on } [0,t] \end{cases} \tag{3.10}$$

for $t \in \mathbb{R}^+$ and $\xi \in X^o$. One easily verifies that each π_j satisfies (3.1). Even before introducing the sets $\{\mathcal{S}_{j,k}\}$ we note a representation formula

$$[x(t)](s) := \begin{cases} \xi(s-t) & \text{for } s > t, \\ j(t-s)-1 & \text{on } [0,t] \end{cases} \tag{3.11}$$

for any measurable switching function $j(\cdot)$, not necessarily satisfying (3.4) (i). Note that this automatically makes $[x(\cdot), j(\cdot)]$ satisfy (3.4) (ii) whenever (3.4) (i) really does hold.[12]

Next, suppose we have a specified function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that:

$$\begin{array}{rl} \text{(i)} & \psi(0) = 0, \quad \int_0^\infty \psi(s)ds = 1; \\ \text{(ii)} & sup\{|\psi(s)| : s \geq t\} \leq \varphi(t); \\ \text{(iii)} & |\psi(t)-\psi(s)| \leq K\varphi(s)(t-s) \text{ for } t > s \geq 0. \end{array} \tag{3.12}$$

[11]The particularly interesting subset of X^o is

$$\mathcal{X}_o := \{\xi(\cdot): \text{ measurable from } \mathbb{R}^+ \text{ to } \{0,1\}\}$$

and we could really have used any X suitably embedding $\mathcal{X}_o$.

[12]Observe also that $\mathcal{X}_0$ is invariant: if $\xi = x(0) \in \mathcal{X}_0$, then $x(t) \in \mathcal{X}_0$ for each $t \in \mathbb{R}^+$ for any $j(\cdot)$.

We then introduce the *sensor functional*

$$\xi \longmapsto \theta[\xi] := \int_0^\infty \psi(s)\,\xi(s)ds \tag{3.13}$$

and, for any X^o-valued state function $x(\cdot)$, the associated *sensor function*

$$t \longmapsto \hat{\theta}(t) = \hat{\theta}(t;x) := \theta[x(t)] := \int_0^\infty \psi(s)[x(t)](s)\,ds. \tag{3.14}$$

Note that θ is continuous on X^o and that if $x(\cdot)$ is given by (3.11) we have

$$\hat{\theta}(t) = \int_0^\infty \psi(t+s)\xi(s)ds + \int_0^t \psi(t-s)[j(s)-1]ds. \tag{3.15}$$

so that, by (3.9) and (3.12) (ii), $\hat{\theta}(t)$ is almost independent of ξ for large t: the ξ-dependent part of $\hat{\theta}(t)$ is the first integral in (3.15) which is bounded by $\int_t^\infty \psi$ which goes to 0 as $t \to \infty$. We have

$$\begin{aligned}
\hat{\theta}(t) - \hat{\theta}(t') &= \int_0^\infty [\psi(t+s) - \psi(t'+s)]\xi(s)ds \\
&\quad + \int_0^{t'} [\psi(t-s) - \psi(t'-s)][j(s)-1]ds \\
&\quad + \int_{t'}^t \psi(t-s)[j(s)-1]ds
\end{aligned}$$

for $t > t' \geq 0$ so, using (3.12) (ii, iii), we obtain the uniform Lipschitz condition:

$$\begin{aligned}
|\hat{\theta}(t) - \hat{\theta}(t')| &\leq \int_0^\infty K\varphi(t'+s)(t-t')ds \\
&\quad + \int_0^{t'} K\varphi(t'-s)(t-t')ds + \int_0^{t-t'} \psi(s)ds \\
&\leq [KM + \varphi(0)](t-t').
\end{aligned} \tag{3.16}$$

In terms of $\theta[\cdot]$ we define, finally,

$$\begin{aligned}
&\mathcal{R}_1 = \{\xi \in X : \theta[\xi] < C_1\}, && \mathcal{R}_2 := \{\xi \in X : \theta[\xi] > C_2\}, \\
&\mathcal{S}_{1,1} := \{\xi \in X : \theta[\xi] \geq C_1\}, && \mathcal{S}_{2,2} := \{\xi \in X : \theta[\xi] \leq C_2\}, \\
&\mathcal{S}_{1,2} := \partial\mathcal{R}_1 = \{\xi \in X : \theta[\xi] = C_1\}, && \mathcal{S}_{2,2} := \partial\mathcal{R}_2 = \{\xi \in X : \theta[\xi] = C_2\}
\end{aligned} \tag{3.17}$$

where C_1, C_2 are specified constants with $0 < C_1 < C_2 < 1$. One immediately verifies (3.3) since $\theta[\cdot]$ is continuous. Note that for a (possible) solution

$[x(\cdot), j(\cdot)]$ we can have a switch: $j \rightsquigarrow k$ at time t only if $\hat{\theta}(t) = C_j$. Thus, if $(t_\nu, t_{\nu+1})$ is an interswitching interval we have $|\hat{\theta}(t_{\nu+1}) - \hat{\theta}(t_\nu)| = C_2 - C_1$ so, by (3.16),

$$t_{\nu+1} - t_\nu \geq (C_2 - C_1)/[KM + \varphi(0)] =: \tau_{min} \tag{3.18}$$

which certainly ensures (3.4) (i).

It is interesting to note that if, e.g., we were to proceed in the mode π_2 then

$$x(\tau + t) = \pi_2(t, \xi) \text{ with } \xi := x(\tau),$$

$$\begin{aligned} \hat{\theta}(\tau + t) &= \theta[\pi_2(t, \xi)] \\ &= \int_0^\infty \psi(t+s)\xi(s)ds + \int_0^t \psi(t-s)[2-1]ds \\ &\geq \int_0^t \psi(s)ds \longrightarrow 1 \quad \text{by (3.12) (i)} \end{aligned}$$

so $\mathcal{R}_2$ is a global attractor for π_2 and, indeed, one reaches $\mathcal{R}_2$ (from any starting point $\xi \in X^o$, proceeding by π_2 in time not greater than $\tau = \tau_2$ where

$$\int_0^{\tau_2} \psi(s)ds = C_2.$$

Similarly

$$\begin{aligned} \theta[\pi_1(t, \xi)] &= \int_0^\infty \psi(t+s)\xi(s)ds \\ &\leq \int_t^\infty \psi(s)ds = 1 - \int_0^t \psi(s)ds \to 0 \end{aligned}$$

so $\mathcal{R}_1$ is a global attractor for π_1 and one has an upper bound τ_1 for the time to reach $\mathcal{R}_1$.

For the switching system $\Sigma := [\mathcal{J}, \{\pi_j\}, \{\mathcal{S}_{j.k}\}]$, this ensures that, for every consistent starting point $[x(0), j(0)]$ (i.e., $x(0) \notin \mathcal{R}_{j(0)}$) one has at least one solution of Σ — e.g., corresponding to switching whenever this is permitted by (3.6) (i) — and every solution switches infinitely often with the uniform bounds

$$\tau_{min} \leq t_{\nu+1} - t_\nu \leq \tau_{max} := max\{\tau_1, \tau_2\}$$

for interswitching intervals. Clearly, the state function $x(\cdot)$ is always related to the switching function $j(\cdot)$ through the representation formula (3.11) and one relates $j(\cdot)$ to $x(\cdot)$ through $\hat{\theta}(\cdot)$ and (3.6); one always has (3.4) in this case. □

We next consider the appropriate notion of convergence for switching systems. We say $\Sigma^n \longrightarrow \Sigma^0$ providing

(i) $\mathcal{J}^n, \mathcal{J}^0 = \mathcal{J}$ (slightly more generally, $\mathcal{J}^0 = limsup\mathcal{J}^n$);

(ii) if $\xi \in \mathcal{R}_j^0$ then $\xi \in \mathcal{R}_j^n$ for all $n \geq \bar{n}_j(\xi)$;

(iii) if ξ^0 is the limit of a subsequence $\xi^{n(m)} \to \xi^0$ with each $\xi^{n(m)} \in \mathcal{S}_{j,k}^{n(m)}$, then $\xi^0 \in \mathcal{S}_{j,k}$;

(3.19) (iv) (3.3) (iv) holds uniformly in n, i.e., for each j, ξ there are a neighborhood $\mathcal{N} = \mathcal{N}_j(\xi)$, and a finite subset $\mathcal{K} = \mathcal{K}_j(\xi) \subset \mathcal{J}$ such that $\mathcal{S}_{j,k}^n \cap \mathcal{N} = \phi$ for $k \notin \mathcal{K}$ and each n;

(v) $\pi_j^n(t,\xi) \to \pi_j^0(t,\xi)$ locally uniformly, i.e., uniformly on some $[0,T] \times X$-neighborhood of each (t,ξ), for each j.

(This is actually a form of 'upper convergence' for switching systems. It would be plausible to adjoin the condition

$$\xi^0 \in \mathcal{S}_{j,k}^0 \ \textit{only} \text{ if there is a sequence } \ \xi^n \to \xi^0 \text{with } \ \xi^n \in \mathcal{S}_{j,k}^n$$

as a complement to (iii) above. Since we will only seek to prove an upper semi-continuity result for the solution set, the definition (3.19) is adequate for our purposes.)

4.Basic Results

In this section we formulate and prove our basic general results. The first two results give a kind of continuous dependence of the solution set on the initial data and the system: we show (under suitable hypotheses) that *the limit of solutions is a solution* when such a limit exists and that such limits always exist for subsequences corresponding to the possible choices when nonunique continuations are permitted by the switching rules (3.6). We also provide an existence result in a somewhat more restricted setting.

For the first results we must introduce a suitable notion of convergence for sequences $\{[x^n, j^n]\}$. For the *state trajectories* $\{x^n(\cdot)\}$ we use, simply, uniform[13] convergence on $[0,T]$ — uniform on finite intervals $[0,T']$ if $T = \infty$.

[13] We will always take X to be a metric space (or a topological vector space) so such uniformity is meaningful.

For the *index functions* $\{j^n(\cdot)\}$ the relevant notion is again the natural one but is a bit more difficult to describe. It is convenient, here, to think of an index function $j(\cdot)$ as being specified by its sequence[14] of switching times

$$0 =: t_0 < t_1 < \ldots < t_{\bar\nu} := T \tag{4.1}$$

and the assumed interswitching indices $\{j_\nu : \nu = 0, 1, \ldots, \bar\nu\}$ where $j(0) = j_0$ and, for $\nu = 1, \ldots, \bar\nu$,

$$j(t) = j_\nu \ \text{ for } \ t_{\nu-1} < t \leq t_\nu. \tag{4.2}$$

Suppose we have an index function $j^0(\cdot)$ with isolated switching times $\{t_\nu^0 : \nu = 0, \ldots, \bar\nu^0\}$ as in (4.1). If $\bar\nu^0 = \infty$ we say that $j^n(\cdot) \to j^0(\cdot)$ if $\bar\nu^n \to \infty = \bar\nu^0$ (e.g., $\bar\nu^n = \infty$ for n large) and, for each $\nu = 1, 2, \ldots$, one has:

$$\begin{array}{rl} \text{(i)} & t_\nu^n \longrightarrow t_\nu^0 \ \text{ as } \ n \to \infty; \\ \text{(ii)} & j_\nu^n = j_\nu^0 \ \text{ for } \ n \geq n_\nu. \end{array} \tag{4.3}$$

If $\bar\nu^0 < \infty$ we ask instead that $\bar\nu^n \geq \bar\nu^0$ for every (large enough) n and that we have (4.3) for each $\nu = 1, \ldots, \bar\nu^0$. (We will also require (4.3) (ii) for $\nu = 0$ although the initial index j_0 has no direct effect on the state trajectory since the condition (3.4) (ii) is effectively applicable only for nontrivial intervals.)

Theorem 1: Suppose $\{\Sigma^n\}$ is a sequence of switching systems on X converging to a switching system Σ^0 in the sense of (3.19) and suppose $[x^n, j^n]$ is a solution of Σ^n on $[0, T]$ for each $n = 1, 2, \ldots$. Let $[x^0, j^0] : \mathbb{R}^+ \to X \times \mathcal{J}$ with $j^0(\cdot)$ satisfying (3.4) (i). Then, if $[x^n, j^n] \longrightarrow [x^0, j^0]$ as above, $[x^0, j^0]$ is a solution of Σ^0 on $[0, T]$.

PROOF : This is just the assertion that *the limit of solutions is a solution* (modulo independent verification of the regularity condition that switching times are isolated for the limit). The proof is quite straightforward.

We have already defined the switching times $\{t_\nu^n : \nu = 0, \ldots, \bar\nu^n; n = 0, 1, \ldots\}$ and now define the corresponding *switching points* $\xi_\nu^n := x^n(t_\nu^n)$. By the uniform convergence, we have $x^0(\cdot)$ continuous and, using (4.3), we see that $\xi_\nu^n \to \xi_\nu^0$ as $n \to \infty$ for each fixed $\nu = 1, \ldots, \bar\nu^0$. We must verify the

[14] Note that (3.4) (i) prohibits having $\bar\nu = \infty$ for finite T but when $T = \infty$ we can have either $\bar\nu < \infty$ or $t_\nu \to \infty$ as $\nu \to \infty =: \bar\nu$.

switching rules (3.6) for $[x^0, j^0]$ and also (3.4) (ii), with (3.4) (i) given by assumption.

Fixing ν, we set $j := j^0_{\nu-1} (= j^n_{\nu-1}$ for large n by (4.3)) and $k := j^0_\nu (= j^n_\nu)$ so at $\bar{t} := t^0_\nu$ one has a switch: $j \rightsquigarrow k$ for $j^0(\cdot)$. Since $\xi^n_\nu \in \mathcal{S}^n_{j,k}$ for each $n = 1, 2, \ldots$ and $\xi^n_\nu \to \xi^0_\nu$, we have $\xi^0_\nu \in \mathcal{S}^0_{j,k}$ by (3.19) (iii). Next, suppose one were to have $\bar{\xi} := x^0(\bar{t}) \in \mathcal{R}^0_j$ for some $\bar{t}$ which is *not* a switching time and had $j^0(\bar{t}) = j$. We would then have $j^n(\bar{t}) = j$ for large n since $\bar{t}$ must be in the interior of the interswitching interval $(t^n_{\nu-1}, t^n_\nu)$ for each large n with ν fixed. Hence $\bar{\xi}^n := x^n(\bar{t}) \to \bar{\xi}$ would imply $\bar{\xi}^n \in \mathcal{R}^0_j$ for large enough n since, by (3.3) (ii), $\mathcal{R}^0_j$ is open. But (3.19) (ii) would then give $\bar{\xi}^n \in \mathcal{R}^n_j$ which is a contradiction. Thus (3.6) (ii) holds at non-switching times. Since $x^0(\cdot)$ is continuous, $x^0(t^0_\nu) \in \mathcal{R}^0_j$ would also give $x^0(\bar{t}) \in \mathcal{R}^0_j$ for nearby times since $\mathcal{R}^0_j$ is open; that is impossible. Thus (3.6) (ii) always holds.

The verification of (3.4) (ii) for t, s in the interior of an interswitching interval $(t^0_{\nu-1}, t^0_\nu)$ is an immediate consequence of (3.19) (iv) since

$$\begin{aligned} x^0(t) &= lim\ x^n(t) = lim\ \pi_k(t-s, x^n(s)) \\ &= \pi_k(t-s, lim\ x^n(s)) = \pi_k(t-s, x^0(s)) \end{aligned}$$

and this extends to the endpoints by continuity. $\square$

Theorem 1 shows that a limit, when it exists, must be a solution. To complement this, we next show that, in some sense, such a limit always exists. A certain additional condition is needed and we say that a set of index functions (or the corresponding solutions) is *uniformly regular* if, for each finite T' (we consider only $T' = T$ if $T < \infty$), there is a minimum length $\tau = \tau(T')$ for interswitching intervals in $[0, T']$ — more formally, if

$$t_\nu \leq T' \Longrightarrow t_{\nu-1} + \tau(T') \leq t_\nu \tag{4.4}$$

for each $(t_{\nu-1}, t_\nu)$ associated with $j(\cdot)$ in the set.

Theorem 2: Suppose $\{\Sigma^n\}$ is a sequence of switching systems converging to Σ^o in the sense of (3.19) and suppose $[x^n, j^n]$ is a solution of Σ^n on $[0, T]$ for each $n = 1, 2, \ldots$. Assume $\{j^n\}$ is uniformly regular on $[0, T]$ and $x^n(0) =: \xi^n_0 \to \xi^0_0$, $j^n(0) = j_0$. Then there is a subsequence $n(m)$ such that $[x^{n(m)}, j^{n(m)}]$ converges to a solution $[x^0, j^0]$ of Σ^0 with $x^0(0) = \xi^0_0$.

Proof : We will proceed by repeated extraction of subsequences, followed by a Cantorial diagonal argument. Abusing notation slightly, we continue to write $\{[x^n, j^n]\}$, etc., for the subsequences at each stage. For exposition we treat only the case $T = \infty$ (so 'solution' means a global solution), leaving the trivial modification for $T < \infty$ to the reader; we also, similarly, assume $lim\ \bar{\nu}^n = \infty$ although it may or may not turn out that $\bar{\nu}^0 = \infty$. As above we denote the switching points by $\xi_\nu^n := x^n(t_\nu^n)$.

The construction proceeds by recursively (in ν) obtaining j_ν^0 and then t_ν^0, ξ_ν^0. This will determine $[x^0(\cdot), j^0(\cdot)]$. The construction will immediately give the convergence $[x^n, j^n] \to [x^0, j^0]$ with (3.4) (i) following from the assumed uniform regularity. Application of Theorem (T1) then shows $[x^0, j^0]$ is a solution of Σ^0 as asserted.

We have $j_0^n = j_0^0$ and $\xi_0^n \to \xi_0^0$ by assumption and, of course, $t_0^n = 0 = t_0^0$. It may be that $j_1^n = j_0^n = j_0^0$ (infinitely often) but it is also possible that t_0^n is (infinitely often) a switching time for $j^n(\cdot)$. In the latter case, having a switch: $j := j_0^0 \rightsquigarrow k(n) := j_1^n$ we must have $\xi_0^n \in \mathcal{S}_{j,k(n)}^n$. By (3.19) (iv) and the fact that $\xi_0^n \to \xi_0^0$, this means $k(n) \in \mathcal{K} = \mathcal{K}_j(\xi_0^0)$ for all but finitely many n. Since $\mathcal{K}$ is finite, this means there must be at least one index $\bar{k} \in \mathcal{K}$ such that $k(n) = \bar{k}$ for infinitely many n. Choose $j_1^0 = \bar{k}$ and extract the sequence for which $k(n) = \bar{k} = j_1^0$.

Working now with this subsequence (still denoted by $[x^n, j^n]$), consider $\{t_1^n\}$ which necessarily has a convergent subsequence (noting that we will accept convergence to ∞); we take t_1^0 to be the limit so, extracting this subsequence, we now have $t_1^n \to t_1^0$. Note that the uniform regularity condition ensures that

$$t_1^n = t_1^n - t_0^n \geq min\,\{1, \tau(1)\} =: \tau_1 > 0$$

so, in the limit, $t_1^0 - t_0^0 \geq \tau_1$. For $t_0^0 < s' < t' < t_1^0$ we have (with $j = j_1^n = j_1^0$)

$$x^n(t) = \pi_j^n(t - t_0^n, \xi_0^n) \to \pi_j^0(t - t_0^0, \xi_0^0) \tag{4.5}$$

uniformly on $[s', t']$ by (3.4) (ii), (3.1), and (3.19) (v). It follows that this can be taken to define $x^0(\cdot)$ as a continuous function on $[t_0^0, t_1^0]$ (on $[t_0^0, \infty)$ if $t_1^0 = \infty$) with $\xi_1^0 := x^0(t_1^0)$ well-defined if $t_1^0 < \infty$. We obtain (3.6) (ii) for $x^0(\cdot)$ on $[t_0^0, t_1^0]$ exactly as in the proof of Theorem 1.

Assuming $t_1^0 < \infty$, it is clear from (4.5) and (3.19) (v) that $\xi_1^n := x^n(t_1^n) \to x^0(t_1^0) =: \xi_1^0$. As before, since each t_1^n is a switching time we have $\xi_1^n \in \mathcal{S}_{j,k(n)}^n$ for each n with $j = j_1^n = j_1^0$ and $k(n) = j_2^n = j^n(t_1^n+)$. As before we can select $\bar{k} \in \mathcal{K}_j(\xi_1^0)$ and extract a subsequence such that each $j_2^n = \bar{k} =: j_2^0$. As before, we extract a subsequence for which $t_2^n \to t_2^0$ with $t_2^n - t_1^n \geq min\{1, \tau(t_1^0+1)\} =: \tau_2$ so $t_2^0 - t_1^0 \geq \tau_2$. As before we have

$$x^n(t) = \pi_j^n(t - t_1^n, \xi_1^n) \longrightarrow \pi_j^0(t - t_1^0, \xi_1^0) =: x^0(t) \tag{4.6}$$

uniformly on any $[s', t'] \subset [t_1^0, t_2^0]$, defining $x^0(\cdot)$.

Note that the switching times cannot 'bunch up' since, for any $T' < \infty$, we must have $t_\nu^0 \geq \nu\tau(T')$ if $t_\nu^0 \leq T'$ so $\nu \leq T'/\tau(T')$. The same argument given above for construction of $j_1^0, t_1^0, x^0(\cdot)$ on $[t_0^0, t_1^0]$, and ξ_1^0 shows that we can proceed recursively with $t_\nu^0 \to \infty$ unless we have $t_{\bar\nu}^0 = \infty$ for some $\bar\nu = \bar\nu^0 < \infty$. The argument also shows that $[x^n, j^n] \longrightarrow [x^0, j^0]$ on $[0, T_\nu^0]$ for each ν (up to $\nu = \bar\nu$ if $\bar\nu < \infty$) where, of course, this refers to the resulting subsequence remaining after all the extractions to this point. Clearly this convergence carries over to the subsequence (of the orginal sequence) obtained by a Cantorial diagonal construction since, from some ν-dependent point on, this coincides with the subsequence directly associated with arriving at t_ν^0. The definition of convergence on $[0, \infty)$ shows that we must necessarily have $[x^{n(m)}, j^{n(m)}] \to [x^0, j^0]$ on $[0, \infty)$ for the diagonal subsequence, as asserted and, by Theorem 1, we then also know that $[x^0, j^0]$ is a solution of Σ^0. □

We turn next to an existence theorem for global solutions, i.e., on $[0, \infty)$. We first provide a cautionary example, showing that even in a case with $\mathcal{J} = \{1, 2\}$ with each π_j asymptotically stable it is (surprisingly?) possible for the switching system to support 'blowup' — solutions which 'escape to infinity' in finite time.

Example 3: Take $X = \mathbb{R}^2$ and let π_1, π_2 be the solution operators associated with the ordinary differential equations:

$$\begin{aligned} &\text{(i)} \quad \dot x = (1 + |x|^2)\left[\begin{pmatrix} 1 & 5 \\ -5 & -2 \end{pmatrix} x + \begin{pmatrix} 10 \\ -4 \end{pmatrix}\right]; \\ &\text{(ii)} \quad \dot x = (1 + |x|^2)\left[\begin{pmatrix} 1 & -5 \\ 5 & -2 \end{pmatrix} x + \begin{pmatrix} 10 \\ 4 \end{pmatrix}\right], \end{aligned} \tag{4.7}$$

respectively. Clearly the solution paths in $\mathbb{R}^2$ are the same as for the *linear* equations:

$$
(4.8) \qquad \begin{array}{rl} \text{(i)} & \dot{u} = u + 5v + 10, \\ & \dot{v} = -5u - 2v - 4; \\ \text{(ii)} & \dot{u} = u - 5v + 10, \\ & \dot{v} = 5u - 2v + 4 \end{array}
$$

which are easily seen to be exponentially stable: the characteristic exponents for (4.8) are $[-1 \pm 4i]/2$. For any starting point, then, (4.7) has bounded solutions so the velocities are also bounded and the solutions go exponentially to the same attractors: $(0, -2), (0, 2)$.

Now take

$$\mathcal{R}_1 := \{(u, v) : v < -1\}, \mathcal{R}_2 := \{(u, v) : v > 1\}$$

$$\mathcal{S}_{1,2} := \{(u, -1)\}, \mathcal{S}_{2,1} := \{(u, 1)\}.$$

One easily sees that every solution of the resulting switching system Σ will alternate modes with switching points alternately on $v = \pm 1$ so (eventually) one keeps $|v| \leq 1$. If one then has $u(\bar{t}) \geq 0$, we subsequently have $\dot{u} > 5(1 + u^2)$, giving blowup before $\bar{t} + \pi/20$. (Similarly, $u(\bar{t}) \leq -16$ gives $\dot{u} < -(1 + u^2)$.)

Note that if we had used the linear equations (4.8), instead of (4.7), to define the modes π_1, π_2 then we could not get blowup in finite time but nevertheless would obtain instability (unbounded solutions) for a switching system comprised of exponentially stable modes. □

Theorem 3: Let $\Sigma := [\mathcal{J}, \{\pi_j\}, \{\mathcal{S}_{j,k}\}]$ be a switching system. Suppose the index set $\mathcal{J}$ is finite and each $\mathcal{S}_{j,k} (j \neq k)$ is compact. Then for any consistent initial data $(\bar{\xi}, \bar{\jmath}) \in X \times \mathcal{J}$ (i.e., $\bar{\xi} \notin \mathcal{R}_{\bar{\jmath}}$) there is global regular solution[15] $[x(\cdot), j(\cdot)]$ with $x(0) = \bar{\xi}, j(0) = \bar{\jmath}$.

Proof : Simply follow the condition (3.4) (ii) in developing the evolution of $x(\cdot)$ with switching as permitted/required by (3.6), choosing almost arbitrarily when nonunique choices may occur. We show that switching times will 'almost automatically' be isolated and that the construction cannot stop (i.e., one can neither have finite escape time nor an impasse for which no admissible continuation exists).

[15]Indeed, we see that (almost) every solution constructed following the switching rules (3.6) is regular (satisfying (3.4) (i)) and global.

We set $\xi_0 := \bar{\xi}, j_0 = j(0) := \bar{\jmath}$. This is permissible since, by assumption, $\bar{\xi} \notin \mathcal{R}_{\bar{\jmath}}$. It is possible that $\bar{\xi} \in \mathcal{S}_{\bar{\jmath},k}$ for some $k \neq \bar{\jmath}$, in which case we can choose to switch immediately, making $0 = t_0$ a switching time and $j_1 = k \neq j_0 = \bar{\jmath}$. If one had $\bar{\xi} \in \partial\mathcal{R}_{\bar{\jmath}}$ with $\pi_{\bar{\jmath}}$ entering $\mathcal{R}_{\bar{\jmath}}$ at $\bar{\xi}$ then an immediate switch is mandatory by (3.6) (ii) and some switch is permissible by (3.3) (iii); else one could permissibly choose $j_1 = j_0 = \bar{\jmath}$.

Now proceed in the mode $\pi_j (j = j_1)$ until at time $\bar{t}$ one arrives at some $\mathcal{S}_{j,k} (k \neq j)$. Either $\xi_1 := x(\bar{t}) := \pi_j(\bar{t} - t_0, \xi_0)$ is in $\partial\mathcal{R}_j$ and π_j enters $\mathcal{R}_j$ at ξ_1 so, as above, a switch is mandatory by (3.6) (ii) and available by (3.3) (iii) or an optional choice is available and one can choose either to continue in π_j or to switch, making $\bar{t}$ a switching time t_1.

Next, we observe that there is an absolute minimum interswitching time $\bar{\tau} > 0$ before *mandatory* switching[16]. To see this, note that if we make a switch: $j \rightsquigarrow k$ it must be at some $\bar{\xi} \in \mathcal{S}_{j,k}$. Then the evolution proceeds in mode π_k and a subsequent switch cannot be mandatory unless/until $\pi_k(\hat{t}, \bar{\xi}) =: \hat{\xi}$ would be in $\partial\mathcal{R}_k$ for some $\hat{t}$ (and π_k enters $\mathcal{R}_k$ at $\hat{\xi}$). By (3.3) (i), the closed sets $\mathcal{S}_{j,k}$ and $\bar{\mathcal{R}}_k$ are disjoint so, by the continuity of π_k, there exists $\tau = \tau_k(\bar{\xi}) > 0$ and $\mathcal{N} = \mathcal{N}(\bar{\xi})$ such that $\pi_k(t, \xi) \in \bar{\mathcal{R}}_k$ for $(t, \xi) \in [0, \tau] \times \mathcal{N}$. By the assumed compactness of $\mathcal{S}_{j,k}$, we can cover $\mathcal{S}_{j,k}$ by a finite number of these neighborhoods and let $\tau_{j,k}$ be the smallest of the associated $\tau' s$. The minimum of these (over the finite set $\{(j, k) \in \mathcal{J} \times \mathcal{J} : j \neq k\}$) is then $\bar{\tau}$.

The assumptions made for this theorem do not imply disjointness of $\mathcal{S}_{j,k}$ and $\mathcal{S}_{k,i}$ $(k \neq i, j)$, etc., so it would be conceivable that the geometry would permit such *voluntary* switching as to violate the condition (3.4) (i). This is the point of the use of "almost" in the first paragraph of this proof: we must[17] restrain our voluntary choices so as to avoid violation of (3.4) (i). Certainly the minimum time $\bar{\tau}$ for mandatory switching means that it is always *possible* to avoid violation of (3.4) (i) — e.g., one could choose to switch only when this is mandatory.

Clearly the construction of $[x(\cdot), j(\cdot)]$ then proceeds for $t \to \infty$. Note that the number of switching times may be infinite, with $x(\cdot)$ defined in-

[16]Note that in many contexts it is reasonable to have: $\mathcal{S}_{j,k} \subset \partial\mathcal{R}_j$ for $k \neq j$. In this case one would have the same minimum $\bar{\tau}$ for voluntary switching as is obtained here for mandatory switching.

[17]Alternatively, we could work with a more general notion of solution but this leads to technical complications and we do not discuss this here; see Section 6, however.

ductively on the interswitching intervals by the appropriate mode: $x(t) = \pi_j(t - t_{\nu-1},\ \xi_{\nu-1})$ for $t_{\nu-1} \leq t \leq t_\nu$ with ξ_ν then given as $x(t_\nu)$. On the other hand, there may be a last switching time $\bar{t}$ with $x(t) = \pi_j(t - \bar{t}, \bar{\xi})$ for $\bar{t} \leq t < \infty$ (assuming this never would give $x(t) \in \mathcal{R}_j$, triggering a mandatory switch). In either case, this defines $[x(\cdot), j(\cdot)] : \mathbb{R}^+ \to X \times \mathcal{J}$ and it is clear from the construction that (assuming such restraint in voluntary switching as may be needed to avoid violating (3.4) (i)) any such $[x, j]$ is, indeed, a solution of Σ as desired. We may note, also, that all solutions are obtainable in this way. $\square$

One might feel that the restriction to finite $\mathcal{J}$ and compact $\mathcal{S}_{j,k}$ for $j \neq k$ avoids the difficulties observed in connection with Example 3. Indeed, Theorem 2 shows that 'escape to infinity in finite time' is then impossible. It is tempting to conjecture that, with these restrictions, if each component mode has the *boundedness property*:

(4.9) $\qquad$ for each solution the set $\{x(t) : t \in \mathbb{R}^+\}$ is bounded

then the same would be true for the switching system. We will see that this need not be true even if the state space X is taken to be locally compact: we provide a counterexample with $X = \mathbb{R}^2$ and two modes.

Example 4: Begin by defining a mode π_1 as the solution operator for a differential equation : $\dot{x} = f(x)$ for $x = (u, v) \in \mathbb{R}^2 =: X$. The direction of (the 2-vector) $f(x)$ will be specified by describing the integral curves; the speed $|f(x)|$ along the curves can be specified independently. We first set

$$\mathcal{S} := \{u \leq -2\} \cup \{|v| \geq 1\} \cup \{(u, 0) : |u| \leq 2\}.$$

This will be the set of stationary points for the flow ($f(x) = 0$) and elsewhere we set

$$|f(x)| := dist\,(x, \mathcal{S}).$$

The direction of f is irrelevant (undefined) on $\mathcal{S}$, particularly for $\{u \leq -2\}$ and $\{|v| \geq 1\}$, and need only be determined on

$$\{(u, v) : u > -2,\ |v| < 1\} \setminus \{(u, 0) : u > -2\}.$$

For $u \leq 2$ we take $f(u, v)$ to be of the form: $(\alpha, 0)$ with $sgn\,\alpha = -sgn\,v$. Finally, we fill out the half-strip $\{(u, v) : u > 2; |v| < 1\}$ with the family of integral curves $\{\mathcal{C}(\lambda) : 0 < \lambda < 1\}$ given by

$$\mathcal{C}(\lambda) := \{(2 + [\sqrt{\lambda} - v][\lambda + v]/[1 - \lambda], v) : -\lambda < v < \sqrt{\lambda}\}.$$

Each point $x = (u, v)$ in the (open) half-strip is on a unique integral curve $\mathcal{C}(\lambda)$ and we take the direction of $f(x)$ to be tangent to that curve, oriented so that motion along the curve is counterclockwise; see Figure 3. This defines π_1.

Now define another mode π_2 by setting $\pi_2(t, \xi) := -\pi_1(t, -\xi)$ on $\mathbb{R}^+ \times \mathbb{R}^2$. Clearly we have (3.1) for each $\pi = \pi_j$ ($j \in \mathcal{J} := \{1, 2\}$)and it is easy to see that (4.9) holds for each mode since, for π_1, the set $\{x(t) : t \in \mathbb{R}^+\}$ for a solution consists either of a single point in $\mathcal{S}$ or of (part of) a segment $\{(u, \sqrt{\lambda}) : -2 < u \leq 0\}$ with $0 < \lambda < 1$ together with (possibly) (part of) the curve $\mathcal{C}(\lambda)$ together with (possibly) (part of) the segment $\{(u, -\lambda) : -2 < u \leq 0\}$. We set

$$\mathcal{R}_1 := \{(u, v) : -2 < u < -1;\ 0 < v < 1\}\ ,\ \mathcal{S}_{1,2} := \partial \mathcal{R}_1\ ;$$

$$\mathcal{R}_2 := \{(u, v) : 1 < u < 2,\ -1 < v < 0\}\ ,\ \mathcal{S}_{2,1} := \partial \mathcal{R}_2$$

to complete the specification of Σ.

Now consider the (unique) solution of Σ starting in mode π_1 (i.e., $j(0) = j_o = j_1 = 1$) at $x(0) = \xi_o = (0, -\lambda_1)$ with $0 < \lambda_1 < 1$. The state moves (in mode π_1) to the right until reaching $(2, -\lambda_1)$, then loops around on $\mathcal{C}(\lambda_1)$ until reaching $(2, \lambda_2)$ with $\lambda_2 := \sqrt{\lambda_1}$, then moves left until reaching the switching point $\xi_1 := (-1, \lambda_2)$. All of this takes finite time so the speed is bounded away from 0 (since the path is bounded away from $\mathcal{S}$). Now the state continues moving left (in mode π_2) until reaching $(-2, \lambda_2)$, then follows $-\mathcal{C}(\lambda_2)$ around until reaching $(-2, -\lambda_3)$ with $\lambda_3 := \sqrt{\lambda_2}$, then moves right until reaching the switching point $\xi_2 := (1, -\lambda_3)$. Etc. The complete path of this solution is an expanding 'spiral' which is composed, alternately, of curves $\mathcal{C}(\lambda_{2n-1})$ and $-\mathcal{C}(\lambda_{2n})$, connected by straight segments along $v = \pm\lambda_\nu$. Note that $\lambda_{\nu+1} = \sqrt{\lambda_\nu}$ ($\nu = 1, 2, \ldots$) so $\lambda_\nu \to 1$. Since each curve $\mathcal{C}(\lambda_\nu)$ includes the point $(2 + \lambda_\nu^{3/2}/[1 - \lambda_\nu], 0)$, this shows that the complete path is unbounded: (4.9) fails for Σ. $\square$

5. Linear Switching Systems

As a particularly interesting and important case, we consider certain bimodal systems ($\mathcal{J} = \{1, 2\}$) with linear dynamics.

Let X be, e.g., a real Banach space and suppose $\mathbf{A}$ is the infinitesimal generator of a C_0 semigroup $\mathbf{S}$ of linear operators on X:

$$\begin{array}{rl} \text{(i)} & \mathbf{S}(t+s) = \mathbf{S}(t)\mathbf{S}(s), \quad \mathbf{S}(0) = \mathbf{1}; \\ \text{(ii)} & t \longmapsto \mathbf{S}(t)\xi \text{ continuous in } t \text{ for each } \xi \in X \,; \\ \text{(iii)} & \|\mathbf{S}(t)\| \leq Me^{\omega t} \text{ for } t \in \mathbb{R}^+. \end{array} \tag{5.1}$$

We now assume the two modes are given by

$$\dot{x} = \mathbf{A}x + u_j \qquad (j = 1, 2) \tag{5.2}$$

where u_1, u_2 are specified (constant) elements of X so the standard variation of parameters formula gives

$$\pi_j(t, \xi) = \mathbf{S}(t)\xi + [\int_0^t \mathbf{S}(s)ds]u_j \tag{5.3}$$

We also suppose $\lambda \in X^*$ is specified (with $\lambda \neq 0$) and set

$$\mathcal{R}_1 := \{\xi \in X : \langle\lambda, \xi\rangle \; < \alpha_1\}, \tag{5.4}$$

$$\mathcal{R}_2 := \{\xi \in X : \langle \lambda, \xi \rangle \ > \alpha_2\},$$

$$\mathcal{S}_{1,2} := \{\langle \lambda, \xi \rangle = \alpha_1\} = \partial \mathcal{R}_1 \ , \mathcal{S}_{2,1} = \{\langle \lambda, \xi \rangle = \alpha_2\} = \partial \mathcal{R}_2$$

where $\alpha_1 < \alpha_2$ are specified scalar values. A switching system Σ specified as in (5.3, 5.4) will be called a **linear switching system**. What characterizes linear switching systems is the representation for solutions:

$$\begin{aligned} (5.5) \quad x(t) &= \mathbf{S}(t)\xi + \int_0^t \mathbf{S}(t = s) u_{j(s)} ds \\ &= x_0(t) + \int_0^t \sigma_1(s)[\mathbf{S}(t-s)u_1] ds + \int_0^t \sigma_2(s)[\mathbf{S}(t-s)u_2] ds \end{aligned}$$

where $x_0(t) := \mathbf{S}(t)\xi$ with $\xi := x(0)$ and

$$\sigma_k(t) := \{1 \ \text{ if } \ j(t) = k; \ \text{ else } \ 0\} \qquad (k = 1, 2)$$

Although the switching surfaces $\mathcal{S}_{j,k}$ are not compact, as assumed for Theorem 3, we see that one still has global existence:

Theorem 4: Let Σ be a linear switching system. Then for any consistent initial data $(\bar{\xi}, \bar{\jmath})$ the set of global solutions is non-empty and equicontinuous with a lower bound $\tau(T')$ on lengths of interswitching times.

Proof : The argument for existence is exactly as in the proof of Theorem 3 except for the discussion of the minimum for interswitching intervals. Since the switching surfaces $\mathcal{S}_{1,2}, \mathcal{S}_{2,1}$ are parallel hyperplanes, we see that any interswitching interval must permit time for a transit across the gap.

From the representation (5.5) we can see that *any* such function (i.e., whether or not one obeys the switching rules (3.6)) satisfies

$$\begin{aligned} (5.6) \qquad |x(\bar{t}) - x(t)| &\le |x_0(\bar{t}) - x_0(t)| + \int_t^{\bar{t}} |\mathbf{S}(t-s) u_{j(s)}| ds \\ &\le |x_0(\bar{t}) - x_0(t)| + \overline{M}|\bar{t} - t| \end{aligned}$$

where

$$\overline{M} = \overline{M}(T') := M max \ \{|u_1|, |u_2|\} max \ \{1, e^{\omega T'}\}.$$

If $(t, \bar{t})$ is an interswitching interval, then (5.4) gives

$$\alpha_2 - \alpha_1 = |\ \langle \lambda, x(\bar{t}) - x(t) \rangle \ | \le |\lambda| \ \ |x(\bar{t}) - x(t)|.$$

Given any $T' < \infty$, $x_0(\cdot)$ is uniformly continuous on $[0, T']$ so there is some $\tau_0 = \tau_0(T')$ such that

$$|x_0(\bar{t}) - x_0(t)| < (\beta - \alpha)/2\ |\lambda| \quad \text{if } (\bar{t} - t) < \tau_o$$

for $0 \le t < \bar{t} \le T'$; we aslo set

$$\tau_1 = \tau_1(T') := (\alpha_2 - \alpha_1)/w\ |\lambda|\ \overline{M}.$$

Then (5.6) gives the lower bound

$$(\bar{t} - t) \ge min\{\tau_o, \tau_1\} =: \tau(T') \tag{5.7}$$

as well as the equicontinuity. □

Remark 2: Suppose $\varphi(s)$ is a nondecreasing bound for $|\mathbf{S}(s)\xi - \xi|$ so (5.1) (ii) gives $\varphi(s) \to 0$ as $s \to 0$; let ψ be the inverse function of φ. Since

$$|x_0(\bar{t}) - x_0(t)| \le \|\mathbf{S}(t)\|\ |\mathbf{S}(\bar{t} - t)\,\xi - \xi| \le Me^{\omega t}\varphi(\bar{t} - t),$$

we can use $\psi([\alpha_2 - \alpha_1]/2\ |\lambda|\ Mmax\{1, e^{\omega T'}\})$ as $\tau_0(T')$. In particular, if $\xi \in \mathcal{D}(\mathbf{A})$ we can take $\varphi(s) := \widetilde{M}|\mathbf{A}\xi|s$ for small s (any $\widetilde{M}$ bounding $Me^{\omega s}$) and this gives $\tau(\cdot)$ decaying at worst like $e^{-\omega T'}$ if $\omega > 0$ and fixed (independent of T') in the stable case: $\omega < 0$. Since what we are really estimating is

$$< \lambda,\, x_0(\bar{t}) - x_0(t) > = < \mathbf{S}^*(\bar{t} - t)\lambda - \lambda,\ \mathbf{S}(t)\xi >,$$

we could proceed rather similarly if we bounded $|\mathcal{S}^*(\bar{t} - t)\lambda - \lambda|$ instead. In particular, if $\mathbf{A}^*$ generates an adjoint semigroup $\mathbf{S}^*(\cdot)$ on X^* and if $\lambda \in \mathcal{D}(\mathbf{A}^*)$, then we obtain the same 'at worst exponential' decay rate for $\tau(\cdot)$, now for arbitrary $\xi \in X$. □

We call a linear switching system Σ *stable* if the defining semigroup $\mathbf{S}(\cdot)$ is (exponentially) stable: $\omega = -\alpha < 0$ in (5.1) (iii). Note that in this case

$$|\xi|_* := sup\{e^{\alpha t}|\mathbf{S}(t)\xi| : t \in \mathbb{R}^+\}$$

defines an equivalent norm[18] on $X(|\xi| \le |\xi|_* \le M|\xi|)$ and use of this norm makes $M = 1$ in (5.1) (iii). Without loss of generality we henceforth assume

[18]Alternatively, we could take

$$|\xi|_* := [\int_0^\infty e^{2\alpha t}|\mathbf{S}(t)\xi|^2 dt]^{1/2}$$

to retain a Hilbert space structure for X if it is originally Hilbertian.

that $M = 1$ so

$$\|\mathbf{S}(t)\| \leq e^{-\alpha t}$$

with $\alpha = -\omega > 0$. If, in addition, one has

$$\text{(5.8)} \qquad \mathbf{S}(\sigma) \text{ is compact [so } \mathbf{S}(t) \text{ is compact for } t \geq \sigma]$$

for some $\sigma > 0$, then we call Σ *compact* but we note that stability alone, without (5.8), already gives certain compactness.

Lemma 1: Let $\mathbf{S}(\cdot)$ be stable and, for $\mathcal{U} \subset X$, set

$$\mathcal{S}_0 = \mathcal{S}_0(\mathcal{U}) := \{0\} \cup \{\mathbf{S}(t)u : t \in \mathbb{R}^+, u \in \mathcal{U}\},$$

$$\mathcal{I}_T = \mathcal{I}_T(\mathcal{U}) := \{\int_0^T \mathbf{S}(t)u(t)dt : u(\cdot) \text{ measurable, } \mathcal{U}-\text{valued}\}.$$

Suppose $\mathcal{U}$ is compact. Then $\mathcal{S}_0$ is compact and, for each $T > 0$ (including $T = \infty$), $\mathcal{I}_T(\mathcal{U})$ is precompact in X.

Proof : Note that (5.1) gives continuity of $\mathbf{S}(t)u$ jointly in (t, u) so the image of $[0, T] \times \mathcal{U}$ is compact for any $T < \infty$. On the other hand, the tail is covered by any ε -ball at 0 if $T > [log\, \mu/\varepsilon]/\alpha$ where $\mu := max\{|u| : u \in \mathcal{U}\} < \infty$. This makes $\mathcal{S}_0$ precompact and one easily sees that it is closed in X, hence compact.

For $\mathcal{I}_T$ we consider $T < \infty$ first. Then one easily sees that $\mathcal{I}_T(\mathcal{U}) \subset T\mathcal{S}(\mathcal{U})$, hence is precompact. For $T = \infty$ we show $\mathcal{I}_\infty$ totally bounded. For any $\varepsilon > 0$ we cover $\mathcal{I}_{T'}$ by $\varepsilon/2$- balls, taking $T' > [log\,(2\mu/\varepsilon\alpha)]/\alpha$ so, uniformly,

$$|\int_{T'}^\infty \mathbf{S}(t)u(t)dt| \leq \int_{T'}^\infty e^{-\alpha t}\mu dt \leq \varepsilon/2$$

Then $\mathcal{I}_\infty$ is contained in the ε-ball cover with the same centers: each integral in $\mathcal{I}_\infty$ is the sum of one in $\mathcal{I}_{T'}$ and a small tail. □

Theorem 5: Let Σ be a stable linear switching system. Then there is a bounded invariant set $\mathcal{B} \subset X$ such that (the state component of) every solution of Σ eventually enters and stays in $\mathcal{B}$. For any compact set $\mathcal{U} \subset X$ there is a compact convex set $\mathcal{S}(\mathcal{U})$ such that (the state component of) every solution which starts in $\mathcal{U}$ will lie entirely in $\mathcal{S}(\mathcal{U})$. Finally, if Σ is compact then $\mathcal{B}$, above, can be taken to be compact.

PROOF : From (5.5), any solution of Σ has

$$x(t) = \mathbf{S}(t)x(0) + \int_0^\infty \mathbf{S}(s)u(s)ds$$

with

$$u(s) := \{u_{j(t-s)} \text{ for } s \le t;\ 0 \text{ for } s > t\}.$$

Thus, $x(t)$ will be in $\mathcal{S}(\mathcal{U})$ for all such $x(\cdot)$ with $x(0) \in \mathcal{U}$ and for each $t \in \mathbb{R}^+$ if we set

$$\mathcal{S}(\mathcal{U}) := \overline{co}[\mathcal{S}_0(\mathcal{U}) + \mathcal{I}_*]$$

where $\mathcal{I}_* := \mathcal{I}_\infty(\{u_1, u_2, 0\})$. Both $\mathcal{S}_0(\mathcal{U})$ and $\mathcal{I}_*$ are compact by Lemma (L1) so this $\mathcal{S}(\mathcal{U})$ is compact and convex as desired.

The same analysis gives $x(t) \in [\mathcal{B}_\varepsilon + \mathcal{I}_*](\mathcal{B}_\varepsilon := \{\xi \in X : |\xi| \le \varepsilon\})$ when $t \ge [log\,(|x(0)|/\varepsilon)]/\alpha$ so, e.g., $[\mathcal{B}_1 + \mathcal{I}_*] =: \hat{\mathcal{B}}$ is globally attractive. This $\hat{\mathcal{B}}$ will not, in general, be invariant so we take

$$\mathcal{B} := \{\mathbf{S}(t)\hat{\xi} + \mathcal{I}_t(\{u_1, u_2\}) : \hat{\xi} \in \hat{\mathcal{B}}, t \in \mathbb{R}^+\}. \tag{5.9}$$

Taking $t = 0$, we see that $\hat{\mathcal{B}} \subset \mathcal{B}$ so $\mathcal{B}$ is globally attractive; $\mathcal{B}$ is obviously bounded and we need only show invariance. For $\xi \in \mathcal{B}$ we have, from the definition (5.9), that

$$\xi = \mathbf{S}(t)\hat{\xi} + \int_0^t \mathbf{S}(t-s)u_{\hat{\jmath}(s)}ds$$

for some $\hat{\xi} \in \hat{\mathcal{B}}$, $t \in \mathbb{R}^+$, measurable $\hat{\jmath}(\cdot) : [0,t] \longrightarrow \mathcal{J}$. Any solution $[x(\cdot), j(\cdot)]$ with $x(0) = \xi$ will have, by (5.5),

$$\begin{aligned} x(r) &= \mathbf{S}(r)[\mathbf{S}(t)\hat{\xi} + \int_0^t \mathbf{S}(t-s)u_{\hat{\jmath}(s)}ds] + \int_0^r \mathbf{S}(r-\varrho)u_{j(\varrho)}d\varrho \\ &= \mathbf{S}(t+r)\hat{\xi} + \int_0^{t+r} \mathbf{S}(t+r-s)u_{\tilde{\jmath}(s)}ds \end{aligned}$$

where we have set

$$\tilde{\jmath}(s) := \begin{cases} \hat{\jmath}(s) & \text{on } [0,t], \\ j(s-t) & \text{on } (t, t+r]. \end{cases}$$

This shows $x(r) \in \mathcal{B}$ for any such $x(\cdot)$, any $r \in \mathbb{R}^+$, i.e., $\mathcal{B}$ is invariant.

Finally, suppose Σ is compact, i.e., (5.8). Clearly, if $x(t) \in \hat{\mathcal{B}}$ for any solution $[x(\cdot), j(\cdot)]$ and some $t \in \mathbb{R}^+$, then $x(t+\sigma)$ is in $\Gamma_0 := [\mathbf{S}(\sigma)\hat{\mathcal{B}} + \mathcal{I}_*]$ which is precompact by (5.8) and Lemma 1; hence, the compact convex set $\Gamma_2 := \overline{co}\Gamma_1$ is a global attractor. Again, this may not be invariant but we can introduce

$$\Gamma_3 := \{\mathbf{S}(t)\hat{\xi} + \mathcal{I}_t(co\{u_1, u_2\}) : \hat{\xi} \in \Gamma_2,\ t \in \mathbb{R}^+\}$$

which, as for $\mathcal{B}$ above, is an invariant global attractor. It is not difficult to verify from its form, noting the definition of Γ_2, that Γ_3 is convex. Since $\Gamma_3 \subset [\mathcal{S}_0(\Gamma_2) + \mathcal{I}_*]$, it is precompact. We set $\Gamma := \bar{\Gamma}_3$ and note that Γ is obviously convex, compact, and globally attractive. To see its invariance, note that if $[x(\cdot), j(\cdot)]$ is any solution of Σ starting at $\xi \in \Gamma$ (so one has $\xi_k \to \xi$ with $\xi_k \in \Gamma_3$), then for any $t \in \mathbb{R}^+$

$$\begin{aligned} x_k(t) &:= \mathbf{S}(t)\xi_k + \int_0^t \mathbf{S}(t-s)u_{j(s)}ds \\ &\to \mathbf{S}(t)\xi + \int_0^t \mathbf{S}(t-s)u_{j(s)}ds = x(t) \end{aligned}$$

by the continuity of $\mathbf{S}(t)$. On the other hand, the proof of invariance of Γ_3 shows $x_k(t) \in \Gamma_3$ so $x(t) \in \bar{\Gamma}_3 = \Gamma$. $\square$

We conclude this section by considering solutions on *all* of $\mathbb{R}$ for a stable linear switching system Σ. For such a solution $[x(\cdot), j(\cdot)] : \mathbb{R} \to X \times \mathcal{J}$, we refer to the restriction of $j(\cdot)$ to $(-\infty, \bar{t}]$ as the *switching history* (at time $\bar{t}$) of the solution; it is convenient to represent this by a $\{0,1\}$-valued function $\eta(\bar{t})$ on $\mathbb{R}^+$:

(5.10) $$[\eta(\bar{t})](s) = \eta(\bar{t}; s) := j(\bar{t} - s) - 1.$$

At least for the class of *bounded solutions* (i.e., with $x(\cdot) \in L^\infty(\mathbb{R} \to X)$), we will see that the restriction of $x(\cdot)$ to $(-\infty, \bar{t}]$ can be recovered from $\eta_{\bar{t}}$. This permits a canonical representation in terms of a system as in Example 2; abstractly, the dynamics and switching are characterized by the function ψ appearing in (3.13) and the values of C_1, C_2 in (3.17).

Theorem 6: Let Σ be a stable linear switching system as in (5.3), (5.4) such that each $\mathcal{R}_j$ is globally attractive for π_j. Then, if $[x(\cdot), j(\cdot)]$ is a bounded solution of Σ on $(-\infty, \bar{t}]$ and $\eta(\bar{t}; s)$ is given by (5.10), we have

(5.11) $$x(\bar{t}) = v_1 + \int_0^\infty \eta(\bar{t}; s)(\mathbf{S}(s)[u_2 - u_1])ds$$

$$\text{with } v_j := \int_0^\infty \mathbf{S}(s)u_j ds.$$

Next, let Σ^0 be, as in Example 2, defined by (3.10), (3.17) with

$$\psi(t) := c < \lambda, \mathbf{S}(t)[u_2 - u_1] > \qquad t \in \mathbb{R}^+ \tag{5.12}$$

with $c := 1/ < \lambda, v_2 - v_1 >$ and using suitable constants C_1, C_2 in (3.17). Let $[x(\cdot), j(\cdot)] : \mathbb{R} \to X \times \mathcal{J}$ be a bounded solution on $(-\infty, 0] =: \mathbb{R}^-$ of Σ; define $y(\cdot) : \mathbb{R}^+ \to X^o$ by $y(t) := \eta(t; \cdot)$ as in (5.10) and let $\hat{\jmath}$ be the restriction of $j(\cdot)$ to $\mathbb{R}^+$. Then $[x(\cdot), j(\cdot)]$ is a solution on $\mathbb{R}^+$ of Σ^0 and $x(\cdot)$ is given on $\mathbb{R}^+$ by (5.11).

Proof : Define $\tilde{x}(t)$ on $(-\infty, \bar{t}]$ by the right hand side of (5.11) with t (variable) replacing $\bar{t}$. Using (5.5) we have

$$x(\bar{t}) = \mathbf{S}(\bar{t} - t)x(t) + \int_t^{\bar{t}} \mathbf{S}(\bar{t} - s)\, u_{j(s)} ds$$

for $t < \bar{t}$ as the restriction of $[x(\cdot), j(\cdot)]$ is a solution on $[t, \bar{t}]$ of Σ. Since

$$u_{j(s)} = u_1 + [j(s) - 1][u_2 - u_1] = u_1 + \eta(\bar{t}; \bar{t} - s)[u_2 - u_1],$$

a bit of manipulation gives

$$x(\bar{t}) - \tilde{x}(t) = \mathbf{S}(\bar{t} - t)[x(t) - \tilde{x}(t)]$$

for arbitrary $t \le \bar{t}$. As $x(\cdot)$ is bounded by assumption and one easily verifies that $\tilde{x}(\cdot)$ is also bounded, this gives

$$|x(\bar{t}) - \tilde{x}(\bar{t})| \le e^{-\alpha(\bar{t} - t)} M$$

for some M and arbitrarily large $(\bar{t} - t)$, whence $x(\bar{t}) = \tilde{x}(\bar{t})$ as desired.

Our choice of c for (5.12) gives the normalization (3.12) (i); if we take $\varphi(t) := c\,|\lambda| e^{-\alpha t}$ in (3.8), then (3.9) and (3.12) (ii) hold. The condition (3.12) (iii) can be obtained along the lines of Remark 2 if either $[u_2 - u_1] \in \mathcal{D}(\mathbf{A})$ or $\lambda \in \mathcal{D}(\mathbf{A}^*)$ but we always have

$$|\psi(t) - \psi(s)| \le \varphi(s)\gamma(t - s) \tag{5.13}$$

$$\text{where } \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \text{ with } \gamma(r) \to 0 \text{ as } r \to 0,$$

taking φ as above and $\gamma(r) := |[\mathbf{S}(r) - \mathbf{1}][u_2 - u_1]|$. Going over Example 2 we see that (3.12) (iii) was used only to obtain a uniform lower bound on the interswitching intervals through Lipschitz continuity of $\hat{\theta}$. We would now have (3.16) replaced by a uniform continuity estimate

$$|\hat{\theta}(t) - \hat{\theta}(t')| \leq |\lambda|[\gamma(t-t') + c|t-t'|]$$

and this again gives $t_{\nu+1} - t_\nu \geq \tau_{min}$ as in (3.18). We will be taking

$$C_j := c[\alpha_j - <\lambda, v_1>] \qquad (j = 1, 2) \tag{5.14}$$

for (3.17). One easily sees that v_j is globally attractive for π_j $(j = 1, 2)$ so the assumption that $\mathcal{R}_j$, as given by (5.4), is attractive for π_j implies, with our definitions, that $0 < C_1 < C_2 < 0$. We see from (5.11) that

$$<\lambda, x(\bar{t})> = <\lambda, v_1> + \int_0^\infty \eta(\bar{t}; s)\psi(s)ds/c,$$

$$\hat{\theta}(\bar{t}) = \int_0^\infty \eta(\bar{t}; s)\psi(s)ds = c[<\lambda, x(\bar{t})> - <\lambda, v_1>]$$

so (5.14) gives

$$\hat{\theta}(\bar{t}) = c_j \Longleftrightarrow <\lambda, x(\bar{t})> = \alpha_j \tag{5.15}$$

with $y(t) = \eta(t, \cdot\cdot)$ entering the region $\mathcal{R}_j^0$ under π_j^0 (i.e., in the senses of (3.17), (3.10)) precisely if $x(t)$ were entering $\mathcal{R}_j$ under π_j (in the senses of (5.3), (5.4)). Thus *the switching rules for* Σ *and* Σ^0 *exactly correspond*, specifying precisely the same permissible and mandatory transitions. The validity of (5.11) and the dynamics for the switching components then give the desired correspondence[19] of solutions. □

6.Comments and Discussion

It was already mentioned above that more detailed consideration of certain directions for further investigation will be presented in [8], [9]. We note here, quite briefly, some other possible extensions of the notions presented here and possible areas for further investigation.

[19]Note that Σ may have unbounded solutions on $\mathbb{R}$ for which (5.11) does not apply and, of course, Σ^0 has solutions on $\mathbb{R}^+$ which do not correspond to any solution on $\mathbb{R}^+$ of Σ. The correspondence is: { solutions of Σ^0 on $\mathbb{R}$} $\leftrightarrow$ { bounded solutions of Σ on $\mathbb{R}$ }.

The most obvious of these would be consideration of *time-dependent* problems. This could involve either time-dependent modes or time-dependent sets $\{\mathcal{S}_{j,k}\}$ or both. As noted, the usual trick of absorbing the time-dependence into an augmented state handles the local theory but unduly complicates the consideration of global (asymptotic) questions. No essentially new ideas seem to be involved here.

Slightly more delicate would be the treatment of solutions for which one admits the possibility of a limit of switching times, a possibility we have eschewed here in restricting consideration to 'regular solutions'. If $\bar{t}$ would be a limit: $t_\nu \to \bar{t}-$ with each t_ν an isolated switching time for a switch: $j_\nu \rightsquigarrow j_{\nu+1}$, consider the set of pairs

$$\bar{\mathcal{P}} := \cap_{\bar{\nu}}\{(j_\nu, j_{\nu+1}) : \nu \geq \bar{\nu}\}$$

and note that this must be nonempty if we have (3.3) (iv). Then, without attempting to assign a value to $j(\bar{t})$, it would be plausible to accept a solution with $j(\cdot) = \bar{k}$ on an interval $(\bar{t}, ?)$ for any $\bar{k}$ such that some $(\bar{\jmath}, \bar{k})$ is in $\bar{\mathcal{P}}$ since $j(\cdot)$ must take the value $\bar{\jmath}$ infinitely often as $t_\nu \to \bar{t}-$ and $x(\bar{t})$ must be in $\mathcal{S}_{\bar{\jmath},\bar{k}}$. Similar considerations would apply to consideration of $j(\cdot)$ constant on an interval $(?, \bar{t})$ and then switching infinitely often to the right of $\bar{t}$. These, however, would be only the simplest possibilities and it is not entirely clear how to define 'solution' so as to omit or significantly weaken the condition (3.4) (i).

An interesting possibility, also, would be to weaken the implicit assumption that X is specified[20] before introducing the modes $\{\pi_j : j \in \mathcal{J}\}$. One could, alternatively, associate a state space X_j with each π_j and then 'glue these together at the switching sets $\mathcal{S}_{j,k}$, i.e., one would have (continuous) functions $T_{j,k} : X_j \supset \mathcal{S}_{j,k} \longrightarrow X_k$ and would have distinct left- and right-hand limits for $x(\cdot)$ across a switch: $j \rightsquigarrow k$ so $x(t-) = \xi_- \in \mathcal{S}_{j,k} \subset X_j$ and $x(t+) = \xi_+ := T_{j,k}(\xi_-) \in X_k$. In (3.4) (ii) we would use ξ_- for $x(t)$ or ξ_+ for $x(s)$ in the obvious way. Note that each π_j would only be defined on X_j (which might be thought of as manifold with boundary $\partial\mathcal{R}_j$) and 'π_j enters $\mathcal{R}_j$ at ξ' would mean, simply, that $\xi \in \partial\mathcal{R}_j :=$ (bdry X_j) is not a possible

[20]In connection with this, one could also consider j-dependent index sets: $\mathcal{J}_j := \{k :$ one has (nonempty) sets $\mathcal{S}_{j,k} \subset X_j\}$. It seems likely that one could always construct a 'universal' $\mathcal{J}$ and a universal $X = \cup\{X_j : j \in \mathcal{J}\}$ in such a way as to reduce this more general notion to the original.

initial value for π_j; this involves, also, a modification of the notion (3.1) of 'mode'. One can generalize the notion of switching system by not requiring each π_j to be defined on all of $\mathbb{R}^+ \times X$. Let us weaken (3.1) to:

(6.1)

- (i) for each $\xi \in X$ there exists an interval $\mathcal{I}(\xi) \subset \mathbb{R}^+$ with $\xi \longmapsto \mathcal{I}(\xi)$ upper semicontinuous: if $\bar{t} \in \mathcal{I}(\xi_n)$ for any $\xi_n \to \bar{\xi}$ in X, then $[0, \bar{t}] \subset \mathcal{I}(\bar{\xi})$;
- (ii) $\pi(t, \xi) \in X$ is defined for $\xi \in X, t \in \mathcal{I}(\xi)$ with $\pi(0, \xi) = \xi$;
- (iii) $t \in \mathcal{I}(\pi(s, \xi)) \iff (t+s)\imath\mathcal{I}(\xi)$ and then $\pi(t+s,\, \xi) = \pi(t, \pi(s, \xi))$;
- (iv) if $\xi_n \to \bar{\xi} \in X$ and $\bar{t} \in \mathcal{I}(\xi_n)$, then $\pi(\cdot, \xi_n) \to \pi(\cdot, \bar{\xi})$ uniformly on $[0, \bar{t}]$

Now suppose we are given an index set $\mathcal{J}$, a family of state spaces $\{X_j : j \in \mathcal{J}\}$ and a family of modes $\{\pi_j : j \in \mathcal{J}$ each acting on the corresponding X_j as in (6.1) with the time intervals now denoted by $\mathcal{I}_j(\xi)$ for $\xi \in X_j,\ j \in \mathcal{J}$. Next, suppose we are given sets $\mathcal{S}_{j,k} \subset X_j$ (possibly empty) for $k \neq j$ and maps $T_{j,k} : \mathcal{S}_{j,k} \longrightarrow X_k$. We assume:

(6.2)

- (i) each $\mathcal{S}_{j,k}$ is closed in X_j; each $T_{j,k}$ is continuous from $\mathcal{S}_{j,k}$ to $X_k \setminus \mathcal{E}_k$;
- (ii) for each $\xi \in X_j$ there is a neighborhood $\mathcal{N}$ of ξ for which $\{k \in \mathcal{J} : [\mathcal{S}_{j,k} \cap \mathcal{N}] \neq \emptyset\}$ is finite;
- (iii) $\mathcal{E}_j := \{\xi \in X_j : \mathcal{I}_j(\xi) = \{0\}\} \subset \mathcal{S}_j := \cup\{\mathcal{S}_{j,k} :\ k \neq j\}$.

The specification

$$\Sigma := \{\mathcal{J}, \{(X_j, \pi_j) : j \in \mathcal{J}\}, \{(\mathcal{S}_{j,k}, T_{j,k}) : j, k \in \mathcal{J}, k \neq j\}\},$$

as above, is then a *switching system*.

It is now convenient to define a *solution* of Σ (on an interval $[0, T]$ with[21] $T < \infty$) as given in terms of a finite[22] partition

$$0 = t_o < t_1 < \ldots < t_{\bar{\nu}} = T.$$

[21]We now define a solution on $\mathbb{R}^+$ by requiring that the restrictions be solutions on $[0, T]$ for arbitrarily large T.

[22]This subsumes the regularity condition (3.4) (i) from the original definition.

We assume we give, for $\nu = 1, \ldots, \bar{\nu}$,

$$j = j_\nu \in \mathcal{J} \text{ and } x_\nu(\cdot) : [t_{\nu-1}, t_\nu] \longrightarrow X_j \tag{6.3}$$

satisfying, for $\nu = 1, \ldots, \bar{\nu}$:

$$\begin{array}{rl} \text{(i)} & x_\nu(t) = \pi_j(t - t_{\nu-1}, \xi_\nu^1) \text{ on } [t_{\nu-1}, t_\nu] \\ & \text{with } j = j_\nu \text{ and } \xi_\nu^1 := x_\nu(t_{\nu-1}) \in X_j \\ & \text{(implicitly this requires } [0, t_\nu - t_{\nu-1}] \subset \mathcal{I}_j(\xi_\nu^1)); \\ \text{(ii)} & x_\nu(t_\nu) =: \xi_\nu^R \in \mathcal{S}_{j,k} \text{ and } \xi_{\nu+1}^L = T_{j,k}(\xi_\nu^R) \\ & \text{with } j = j_\nu, k = j_{\nu+1} \text{ for } 1 \le \nu < \bar{\nu} \text{ (and for } \nu = 0 \\ & \text{unless } j_1 = j_o \text{ and } \xi_1^L = \xi_o). \end{array} \tag{6.4}$$

Note that we have $[x(\cdot), j(\cdot)]$ well-defined on $[0, t]$ except at $\{t_\nu : \nu = 0, \ldots, \bar{\nu} - 1\}$, with $j(\cdot)\, \mathcal{J}$-valued and $x(\cdot)$ taking values in the appropriate state space for each subinterval of the partition. It is then sightly awkward to define $x(t_\nu)$ but we have the limits $x(t_\nu-) = \xi_{\nu-1}^R$ and $x(t_\nu+) = \xi_\nu^L$.

It is not difficult to see that any switching system in the earlier sense becomes a switching system in the sense just defined on taking $X_j := X \setminus \mathcal{R}_j$, each $T_{j,k}$ to be the (suitably restricted) identity map, and each mode π_j 'maximally defined' on $\mathbb{R}^+ \times X_j$ to give (3.1) from (3.3). The requirement that $T_{j,k}$ not take values in $\mathcal{E}_k$ is just a weaker form of the earlier requirement in (3.3) (i) that $\mathcal{S}_{j,k} \cap \bar{\mathcal{R}}_k = \emptyset$ since we recognize $\mathcal{E}_k$ as corresponding, for the original definition, to the set of points in $\partial\mathcal{R}_k$ at which π_k would enter $\mathcal{R}_k$. The solution set for the re-interpreted switching system (new definitions) corresponds precisely to the original solution set.

One advantage of this reformulation of the notion of switching system is to make it easy to define an extension of a switching system. We will call $\Sigma' := \{\mathcal{J}', \ldots\}$ an **extension** of Σ and write $\Sigma' \supset \Sigma$ (equivalently, we call Σ a **restriction** of Σ') if each is a switching system (in the sense of our reformulated definition) with

$$\mathcal{J} \subset \mathcal{J}', X_j \text{ closed in } X_j'(j \in \mathcal{J} \subset \mathcal{J}'),\ \mathcal{S}_{j,k} \subset \mathcal{S}_{j,k}',$$

and if π_j, $T_{j,k}$ are the appropriate restrictions of π_j', $T_{j,k}'$. It is easy to see that every solution of Σ is also a solution of Σ'. Conversely, if we take $\mathcal{S}_{j,k}$ to be $\mathcal{S}_{j,k}' \cap X_j$ whenever $j, k \in \mathcal{J} \subset \mathcal{J}'$, then any solution of Σ' which stays in $\{X_j : j \in \mathcal{J}\}$ will also be a solution of Σ. In particular, whenever we would have an invariant set (as in Theorem 3) we could cut down the state space(s) in the obvious way to obtain a restriction of Σ – indeed, this is essentially what it means to have an invariant set.

Finally, we note the interesting possibility of considering two-point boundary value problems and/or (optimal) control problems in the context of switching systems as a generalization of the usual problems for differential equations.

Acknowledgments: The original stimulus to this work came from discussions with K. Glashoff and J. Sprekels regarding the thermostat model in [5] particularly with regard to the existence of (nontrivial) periodic solutions. The author would like to acknowledge the value of discussions with them and with N. Bhatia, I. Capuzzo-Dolcetta, and S. Saperstone. In particular, several lengthy discussions with Capuzzo-Dolcetta were critical to the development of these ideas.

Acknowledgment is due to the Air Force Office of Scientific Research for support of this research under #AFOSR-82-0271 and, more recently, also to the NSF for additional support under CDR-85-00108 during the writing of the current version.

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