THE TRANSIENT SEMICONDUCTOR PROBLEM WITH GENERATION TERMS, II[1]

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ABSTRACT

We consider a modified time-dependent van Roosbroeck system with saturating velocities and a source term subject to assumptions designed to permit a standard generation term corresponding to impact ionization. This is then a nonlinear parabolic system with a right hand side of the form $S = S(., \mathbf{u}, \nabla\mathbf{u}, \mathcal{E})$, coupled with the map $\mathbf{L} : \mathbf{u} \mapsto \mathcal{E}$ defined by the Poisson equation at each t. The analysis leads to existence of solutions for all time as well as uniqueness (and continuous dependence on data) under stronger conditions.

KEYWORDS: Semiconductors, device modeling, transient, van Roosbroeck, impact ionization, nonlinear, partial differential equations, system, parabolic, existence, uniqueness.

AMS SUBJECT CLASSIFICATIONS: **35K60, 35Q20, 78A35.**

1 Introduction

We wish here to consider basic issues of existence and uniqueness for a standard model of semiconductor device physics. The 'drift/diffusion' model treated is a version of the van Roosbroeck model [15], [13] permitting the consideration of generation terms corresponding to impact ionization. From a mathematical viewpoint, the essential feature of this consideration is that it precludes the use of maximum principle arguments which formed an essential part of previous work (compare, e.g., the approaches of [6], [8], [12], [5], [3], involving more restricted models without generation terms). While specifically motivated by the semiconductor setting, the model is general enough to be applicable to a wide variety of physical situations involving reaction/diffusion systems of charged particles for which convection is determined by the electrostatic field (e.g., electrophoresis).

The results presented here represent an improved version of [10] in that we now have an existence proof without the geometric restriction imposed there and with the possibility of field-dependent diffusion. Additionally, under the same hypotheses

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as were used in [10] we can now show uniqueness by a somewhat more constructive version of the argument in [10].

The relevant variables are $\mathbf{u} = [u_1, \ldots, u_K]$ (*concentrations* for K species of charged particles) and $\mathcal{E} = \mathcal{E}_{tot} - \mathcal{E}_0$ (interactive contribution to the *electrostatic field*). The *drift/diffusion model* for semiconductor device physics consists of two parts: (1) the Poisson equation for the electrostatic potential ψ_{tot} (giving the field $\mathcal{E}_{tot}$ as $-\nabla\psi_{tot}$; we actually work with the *linear* map $\mathbf{L} : \mathbf{u} \mapsto \mathcal{E} = \mathcal{E}_{tot} - \mathcal{E}_0$ defined by the Poisson equation after subtracting the known inhomogeneities) and (2) the conservation laws governing the set $\mathbf{u}$ of concentrations.

The next section presents the basic formulation to be used and two sets of hypotheses **(H-1)**, **(H-2)**. In Section 3 we obtain existence of solutions for this system under the first (quite mild) set of hypotheses **(H-1)**, using an approach based on the Schauder Fixpoint Theorem. Then, in Section 4, it is shown under the somewhat stronger set of hypotheses **(H-2)** that a related Picard map is contractive with respect to a suitably constructed metric, giving uniqueness as well by the Banach Contractive Mapping Principle. The final section discusses briefly the relation of the present results to the physical setting and hopes for the future.

Notation: Spatially, we will be working with $\mathbf{u}$ on a fixed bounded region Ω in $\mathbb{R}^d$ whose boundary $\partial\Omega$ is partitioned into $\Gamma = \Gamma_D$ (at which we prescribe Dirichlet conditions) and $\Gamma_N = \partial\Omega\backslash\Gamma$ (at which we impose flux conditions). The evolution problem will be considered on a fixed bounded interval $[0, T]$ — but with $T > 0$ arbitrarily large so our results are actually global; we set $\mathcal{Q} = \mathcal{Q}_T := [0, T] \times \Omega$ and $\Sigma := [0, T] \times \partial\Omega$. We use $\|\cdot\|_p$ for the $L^p(\Omega)$-norm ($p = 2$ if the subscript is omitted). We abbreviate $L^\infty([0, T] \to L^p(\Omega))$, for example, by $L^\infty(\to L^p)$ and do not distinguish notationally between this as a space of $\mathbb{R}$-valued functions and as a space of $\mathbb{R}^K$-valued functions. We also permit the useful confusion between $\mathbf{L} : \mathbf{u} \mapsto \mathcal{E}$ as a map between functions on Ω (with t appearing only parametrically) and the induced map between functions on $\mathcal{Q}$. We assume that Ω is connected and Γ_D is a suitably large part of $\partial\Omega$ so we can simply take $\|\nabla u\|$ as the norm $\|u\|_{(1)}$ on the space $\mathcal{H}^1_\Gamma := \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ (i.e., the $H^1(\Omega)$-closure of $\{v \in C^\infty(\bar{\Omega}) : \operatorname{supp}\{v\} \cap \Gamma = \emptyset\}$). Finally, we introduce the space $\mathcal{X}_2 := L^\infty(\to L^2) \cap L^2(\to H^1)$ with the norm

$$\|v\|_{\mathcal{X}} := (\sup_{[0,T]}\{e^{-rt}\int_\Omega |v(t)|^2\} + \int_\mathcal{Q} e^{-rt}|\nabla v|^2)^{1/2} \tag{1.1}$$

where r is to be specified later (large enough).

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2 Formulation and Hypotheses

In this section we present the formalism within which we will be working and state two sets of hypotheses: a weaker set **(H-1)** which will suffice for the existence result in Section 3 and a stronger set **(H-2)** under which we can prove uniqueness in Section 4.

The electrostatic field $\mathcal{E}_{tot}$ is determined as $\mathcal{E}_{tot} = -\nabla\psi_{tot}$ where the potential ψ_{tot} satisfies the Poisson equation

$$-\nabla\cdot\underline{\epsilon}\nabla\psi_{tot} = N + \mathbf{q}\cdot\mathbf{u} = N + \sum q_k u_k \tag{2.1}$$

and the associated boundary conditions; here, N is the net charge distribution due to the 'doping' which defines the device. Note that it may be appropriate to consider (2.1) as holding for a spatial region Ω' other than the Ω considered for the concentrations. This region Ω' should then contain Ω and we assume that $\mathbf{u}$, as it appears on the right of (2.1), is given *a priori* on $\Omega'\backslash\Omega$. The boundary conditions for ψ are, of course, to be imposed at $\partial\Omega'$ and might include 'conditions at infinity'.

If we let ψ_0 be the solution of (2.1) corresponding to $\mathbf{u} = 0$ and set $\mathcal{E}_0 = -\nabla\psi_0$ then $\mathcal{E}_0$ contains all the relevant information given by N and the inhomogeneous boundary data. Setting $\mathcal{E} = \mathcal{E}_{tot} - \mathcal{E}_0$, we now have $\mathcal{E} = -\nabla\psi$ (restricted to Ω, if necessary) where ψ is the solution of the Poisson equation

$$-\nabla\cdot\underline{\epsilon}\nabla\psi = \mathbf{q}\cdot\mathbf{u} \tag{2.2}$$

with *homogeneous* boundary conditions of the same form as those used for (2.1). Clearly the map $\mathbf{L} : \mathbf{u} \mapsto \mathcal{E}$ defined in this way is *linear*.

For the concentration vector $\mathbf{u} = [u_1, \ldots, u_K]$, we assume a coupled system of conservation laws of the usual general form:

$$\dot{u}_k - \nabla\cdot[D_k\nabla u_k + u_k\mathcal{V}_k] = S_k \tag{2.3}$$

for $k = 1, \ldots, K$. Here, suppressing the index k as we will do in general, $\mathcal{J} := D\nabla u + u\mathcal{V}$ is the particle flux, S is the (net) *source*, D is the *diffusion coefficient*, and $\mathcal{V}$ is the *drift velocity*.

We also have *constitutive relations*

$$D = D_k(\cdot, \mathcal{E}), \qquad \mathcal{V} = \mathcal{V}_k(\cdot, \mathcal{E}), \qquad S = S_k(\cdot, \mathbf{u}, \nabla\mathbf{u}, \mathcal{E}). \tag{2.4}$$

We will not distinguish notationally between the *given* constitutive relation, $S = S_k$ as a function: $\mathcal{Q} \times \mathbb{R}^K \times \mathbb{R}^{Kd} \times \mathbb{R}^d \to \mathbb{R}$ and the derived 'substituted function' — $S(\cdot, \mathbf{u}(\cdot), \nabla\mathbf{u}(\cdot), \mathcal{E}(\cdot)) : \mathcal{Q} \to \mathbb{R}$. Note also that in writing the dependence of

D_k in terms of $\mathcal{E}$ (rather than $\mathcal{E}_{tot}$), etc., we are taking advantage of the fact that $\mathcal{E}_0$ is known *a priori*. The relevant hypotheses about N, Ω' (in relation to Ω), the boundary conditions and data for (2.1) are subsumed in our hypotheses on $\mathbf{L}$ and the functions of (2.4) (Recall that we omitted temperature coupling for expository simplicity — else $D_k, \mathcal{V}_k, S_k$ would also depend on the temperature).

We next state the set of hypotheses **(H-1)**, under which we will show existence in the next section. The generic constant γ need not be the same at each appearance.

(H-1.i) We have:
- Ω is bounded and connected in $\mathbb{R}^d$; $\partial\Omega$ is 'smooth enough' for the relevant Sobolev spaces and embeddings to make sense (e.g., satisfying a 'cone condition').
- All the functions $D_k(\cdots)$, $\mathcal{V}_k(\cdots)$, $S_k(\cdots)$ are measurable on $\mathcal{Q}$ and are continuous with respect to all other variables.

(H-1.ii) There is a constant $\delta > 0$ such that $\quad \delta \le D_k \le \gamma$.

(H-1.iii) $\mathcal{V}_k$ is bounded: $\quad |\mathcal{V}_k| \le \gamma$.

(H-1.iv) One has $|S_k(\cdot, r, \xi, \eta)| \le S^0(\cdot) + \gamma[|r| + |\xi|]$ for all η with some $S^0 \in L^2(\mathcal{Q})$.

(H-1.v) For fixed $\mathcal{E}(\cdot)$, S_k is (uniformly) Lipschitz continuous:
$$|S_k(\cdot, r, \xi, \eta) - S_k(\cdot, r', \xi', \eta)| \le \gamma[|r - r'| + |\xi - \xi'|].$$

(H-1.vi) The linear operator $\mathbf{L}$, defined by (2.2) with the homogeneous boundary conditions, is continuous from $L^2(\Omega)$ to $L^2(\Omega)$ (equivalently, from $L^2(\mathcal{Q})$ to $L^2(\mathcal{Q})$).

(H-1.vii) The data $\hat{u}_k$ are in $L^2(\to H^1) \cap H^1(\to H^{-1})$.

Under the stronger set of hypotheses **(H-2)** it will be possible to show, in Section 4, that a Picard iteration is contractive (with respect to a suitably chosen metric) giving uniqueness as well.

(H-2.i) = **(H-1.i)** with $d = 3$ so $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

(H-2.ii) D_k is independent of $\mathcal{E}$ and $0 < \delta \le D_k \le \gamma$.

(H-2.iii) $\mathcal{V}_k$ is Lipschitzian: $|\mathcal{V}_k(\cdot, \eta) - \mathcal{V}_k(\cdot, \eta')| \le \gamma|\eta - \eta'|$ as well as bounded: $|\mathcal{V}_k| \le \gamma$.

(H-2.iv) = **(H-1.iv)**.

(H-2.v) S_k satisfies a Lipschitz condition of the form:
$$|S_k(\cdot, r, \xi, \eta) - S_k(\cdot, r', \xi', \eta')| \le \gamma(|r - r'| + |\xi - \xi'| + (1 + |r| + |\xi|)|\eta - \eta'|).$$

(H-2.vi) For some $p \le 6$, $\mathbf{L}$ is compact from $L^p(\Omega)$ to $\mathcal{C}(\bar{\Omega})$.

(H-2.vii) With $2 \le p \le 6$ as in **(H-2.vi)**, one has $\hat{u}_k, \nabla\hat{u}_k \in L^p(\mathcal{Q})$ and $\dot{\hat{u}}_k \in L^p(\to L^{3p/(2p+1)})$.

We say that u is a (weak) solution of (2.3) if one has:

$$(2.5) \qquad (u - \hat{u}) \in \mathcal{Z}_0 := \{v \in L^2(\to \mathcal{H}^1_\Gamma) \cap H^1(\to H^{-1}) : \, v|_{t=0} = 0\},$$

$$\int_{\mathcal{Q}} \dot{u}v + \int_{\mathcal{Q}} D\nabla u \cdot \nabla v = \int_{\mathcal{Q}} Sv - \int_{\mathcal{Q}} u\mathcal{V} \cdot \nabla v \qquad \forall v \in \mathcal{Z}_0. \tag{2.6}$$

Implicit in (2.5) are the initial conditions: $u = \hat{u}$ at $t = 0$ and Dirichlet conditions: $u = \hat{u}$ at $\Gamma = \Gamma_D$; implicit in (2.6) are the homogeneous flux conditions: $\mathcal{J} \cdot \mathbf{n} = 0$ on $\partial\Omega\backslash\Gamma_D = \Gamma_N$.

3 Existence

In this section we prove, under **(H-1)**, the existence of a solution of the coupled system: (2.3) ($k = 1, \ldots, K$) with $\mathcal{E} = \mathbf{Lu}$ and (2.4). Our approach here is to define the map $\mathbf{T} : \mathcal{E} \mapsto \mathbf{u}$ by solving (2.3) with $\mathcal{E}$ taken as *given* and then to apply the Schauder Fixpoint Theorem to

$$\mathbf{L} \circ \mathbf{T} : \quad \mathcal{E} \overset{\mathbf{T}}{\longmapsto} \mathbf{u} \overset{\mathbf{L}}{\longmapsto} \mathcal{E} : \ L^2(\mathcal{Q}) \longrightarrow \mathcal{X} \longrightarrow L^2(\mathcal{Q}) \tag{3.1}$$

with $\mathcal{X} = \mathcal{X}_2 := L^\infty(\to L^2) \cap L^2(\to H^1)$.

Theorem 1: Under the hypotheses **(H-1)**, the system (2.3) for $k = 1, \ldots, K$ has a weak solution $[\mathcal{E}^*, \mathbf{u}^*]$ in $L^2(\mathcal{Q}) \times \mathcal{Z}$ with $\mathcal{E}^* = \mathbf{Lu}^*$ and the constitutive relations (2.4).

Proof: We divide the proof into the following five steps:

[1] $\mathbf{T}$ is well-defined; set $\mathcal{K}_1 := \{\mathbf{T}(\mathcal{E}) : \mathcal{E} \in L^2(\mathcal{Q})\} \subset \mathcal{X}$;
[2] $\mathcal{K}_1$ is bounded in $\mathcal{X}$;
[3] $\mathcal{K}_2 := \{\mathbf{Lu} : \mathbf{u} \in \mathcal{K}_1\}$ is precompact in $L^2(\mathcal{Q})$; set $\mathcal{K} := \overline{\text{co}}\mathcal{K}_2$;
[4] $\mathbf{T} : L^2(\mathcal{Q}) \to \mathcal{X}$ is continuous;
[5] $\mathbf{L} \circ \mathbf{T}$ has a fixpoint $\mathcal{E}^* \in \mathcal{K}$.

The desired solution pair is $[\mathcal{E}^*, \mathbf{u}^*]$ with $\mathbf{u}^* := \mathbf{T}(\mathcal{E}^*)$. Step **[1]** is fairly standard (detailed proof omitted); **[3]** and **[5]** are quick. The principal efforts are for **[2]** (the original estimate — exactly as in [10]) and the new argument for **[4]**.

[1] We are defining $\mathbf{T}(\bar{\mathcal{E}})$ as the (unique) weak solution $\mathbf{u} \in [\bar{\mathbf{u}} + \mathcal{Z}_0]$ of

$$\dot{u}_k - \nabla\cdot[\bar{D}_k \nabla u_k + u_k \bar{\mathcal{V}}_k] = \bar{S}_k(\cdot, \mathbf{u}, \nabla\mathbf{u}) \qquad (k = 1, \ldots, K) \tag{3.2}$$

where $\bar{D}$, $\bar{\mathcal{V}}$, $\bar{S}$ are defined by fixing $\mathcal{E} = \bar{\mathcal{E}}$ in (2.4). That such a solution exists is fairly standard in view of the assumed Lipschitz condition **(H-1.v)** with **(H-1.iv)** — one considers the Picard map: $\bar{\mathbf{u}} \mapsto \mathbf{u}$ defined by the *linear equation* obtained on replacing $\bar{S}(\cdot, \mathbf{u}, \nabla\mathbf{u})$ in (3.2) by $\bar{S}(\cdot, \bar{\mathbf{u}}, \nabla\bar{\mathbf{u}})$ (still with $\mathcal{E}$ fixed as $\bar{\mathcal{E}}$ so $\bar{D}, \bar{\mathcal{V}}$ are now fixed) and shows contractivity with respect to an exponentially weighted norm (1.1), using the estimate: $\quad \|\bar{S}(\cdot, \bar{\mathbf{u}}, \nabla\bar{\mathbf{u}}) - \bar{S}(\cdot, \bar{\mathbf{u}}', \nabla\bar{\mathbf{u}}')\| \leq M\|\bar{\mathbf{u}} - \bar{\mathbf{u}}'\|_{(1)}$.

[2] Set $z = z_k = u_k - \hat{u}_k$ so, formally,

$$\dot{z} = \nabla\cdot[D\nabla z + z\mathcal{V}] + (\bar{S} - \nabla\cdot[D\hat{u} + \hat{u}\mathcal{V}] + \hat{u}^{\cdot}) \tag{3.3}$$

with a weak formulation corresponding to (2.6). We take $v = 2e^{-rt}z|_{[0,\tau]}$ as test function in this weak formulation. Note that $v\dot{z} = ([e^{-rt}z^2]^{\cdot} + re^{-rt}z^2)$ on $(0,\tau)$ and that **(H-1.iv)** gives:

$$\|S(\cdot,\mathbf{u},\nabla\mathbf{u})\| \leq M[1 + \|\mathbf{z}\|_{(1)}]. \tag{3.4}$$

We will also estimate: $\|\mathbf{z}\|_{(1)}\|v\| \leq \frac{\delta}{MK}e^{-rt}\|\mathbf{z}\|_{(1)}^2 + \frac{MK}{\delta}e^{-rt}\|z\|^2$, etc. Here and henceforth, M, etc., are generic 'constants' which may depend only on Ω, norm bounds on $\hat{\mathbf{u}}$ as in **(H-1.vii)**, the modulus of ellipticity δ, and the generic 'constant' γ of **(H-1)**; specifically, M will *not* depend on r — nor, of course, on $\mathcal{E}$ or $\mathbf{u}$. It follows that (replacing τ by t):

$$\begin{aligned} e^{-rt}\|z(t)\|^2 + r\int_0^t e^{-rs}\|z\|^2 + 2\delta\int_0^t e^{-rs}\|z\|_{(1)}^2 \\ \leq \delta\int_0^t e^{-rs}\|z\|_{(1)}^2 + \tfrac{\delta}{2K}\int_0^t e^{-rs}\|\mathbf{z}\|_{(1)}^2 + \int_0^t e^{-rs}[M + M'\|z\|^2] \end{aligned}$$

for each k. Taking $r > M'$, this gives

$$e^{-rt}\|z_k(t)\|^2 + \delta\int_0^t e^{-rs}\|z_k\|_{(1)}^2 \leq M + \tfrac{\delta}{2K}\int_0^t e^{-rs}\|\mathbf{z}\|_{(1)}^2$$

whence, summing over k, we obtain the 'final' estimate

$$e^{-rt}\|\mathbf{z}(t)\|^2 + \tfrac{\delta}{2}\int_0^t e^{-rs}\|\mathbf{z}\|_{(1)}^2 \leq KM \tag{3.5}$$

which clearly bounds $\mathbf{z}$ (and so $\mathbf{u} = \mathbf{T}(\mathcal{E}) = [\mathbf{z} + \hat{\mathbf{u}}]$) in $\mathcal{X} = \mathcal{X}_2$. Note that the bound we have obtained is independent of $\mathcal{E} \in L^2(\mathcal{Q})$ so $\mathcal{K}_1$ is a bounded set in $\mathcal{X}$. Note that, while we have only obtained such a bound when $r > M'$, these $\mathcal{X}_2$-norms are equivalent so the *fact* that $\mathcal{K}_1$ is a bounded set is independent of r; we will be free to choose r independently in any subsequent calculation.

[3] Given the bound above on $\mathbf{u} = \mathbf{T}(\mathcal{E})$ in $L^2(\to H^1)$, we have bounds on each S_k and $\mathcal{J}_k$ in the space $L^2(\mathcal{Q})$ and so a bound on $\dot{u}_k = S_k + \nabla\cdot\mathcal{J}_k$ in $L^2(\to H^{-1})$. Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, application of the Aubin Compactness Theorem [1] gives precompactness of $\mathcal{K}_1$ in $L^2([0,T] \to L^2(\Omega)) = L^2(\mathcal{Q})$. Since, by **(H-1.vi)**, $\mathbf{L}$ is continuous from $L^2(\mathcal{Q})$ to $L^2(\mathcal{Q})$, it follows that the image $\mathcal{K}_2 = \mathbf{L}\mathcal{K}_1$ must be precompact so the set $\mathcal{K} = \overline{\mathrm{co}}\mathcal{K}_2$ is compact and convex in $L^2(\mathcal{Q})$. Note that $\mathbf{LT}(\mathcal{K}) \subset \mathcal{K}_2 \subset \mathcal{K}$ so $\mathcal{K}$ is invariant.

[4] Now suppose we have a sequence $\{\mathcal{E}^\nu\}$ in $L^2(\mathcal{Q})$ converging in $L^2(\mathcal{Q})$ to some $\mathcal{E}^0$; we write $\mathbf{u}^\nu := \mathbf{T}(\mathcal{E}^\nu)$, etc. We wish to show that $\mathbf{u}^\nu \to \mathbf{u}^0 = \mathbf{T}(\mathcal{E}^0)$ in the sense of $\mathcal{X}_2$.

The key to the argument is to assume, first, that $\mathcal{E}^\nu \to \mathcal{E}^0$ *pointwise* ae on $\mathcal{Q}$. This implies convergence ae on $\mathcal{Q}$ for $D, \mathcal{V}$ as well. It is not difficult to see that:

$\|[D(\cdot,\mathcal{E}^\nu) - D(\cdot,\mathcal{E}^0)]\nabla u^0\|^2 = \int_{\mathcal{Q}}|[D(\cdot,\mathcal{E}^\nu) - D(\cdot,\mathcal{E}^0)]\nabla u^0|^2 \to 0$

since we have convergence to 0 ae on $\mathcal{Q}$ for the integrand and, with $\nabla\mathbf{u}^0$ fixed in

$L^2(\mathcal{Q})$, **(H-1.ii)** gives the dominating bound $2\gamma^2|\nabla \mathbf{u}^0|^2$. Similarly, using **(H-1.iii)** and **(H-1.iv)** for domination, one obtains $L^2(\mathcal{Q})$ convergence:

$u^0\mathcal{V}(\cdot,\mathcal{E}^\nu) \to u^0\mathcal{V}(\cdot,\mathcal{E}^0), \quad S(\cdot,\mathbf{u}^0,\nabla\mathbf{u}^0,\mathcal{E}^\nu) \to S(\cdot,\mathbf{u}^0,\nabla\mathbf{u}^0,\mathcal{E}^0).$

We also recall that **(H-1.v)** gives:

$\|S(\cdot,\mathbf{u}^\nu,\nabla\mathbf{u}^\nu,\mathcal{E}^\nu) - S(\cdot,\mathbf{u}^0,\nabla\mathbf{u}^0,\mathcal{E}^\nu)\| \le M\|\mathbf{z}^\nu\|_{(1)}.$

Let M_ν be a bound for $\|[D(\mathcal{E}^\nu)-D(\mathcal{E}^0)]\nabla u^0\|^2$, etc.; note that we have just shown $M_\nu \to 0$.

Set $z = z^\nu = u^\nu - u^0$ much as in (3.3) and similarly take $v = 2e^{-rt}z|_{[0,\tau]}$ in the weak formulations (2.6) for u^ν, u^0; then subtract and estimate much as in **[2]** above. One obtains

$$\begin{aligned} &e^{-rt}\|z(t)\|^2 + r\textstyle\int_0^t e^{-rs}\|z\|^2 + 2\delta\int_0^t e^{-rs}\|z\|^2_{(1)} \\ (3.6)\qquad &\le \delta\textstyle\int_0^t e^{-rs}\|z\|^2_{(1)} + \frac{\delta}{2K}\int_0^t e^{-rs}\|\mathbf{z}\|^2_{(1)} + M_\nu + M'\int_0^t e^{-rs}\|z\|^2 \end{aligned}$$

from which, again choosing $r > M'$ and summing over k, one obtains a bound on $\|\mathbf{z}^\nu\|_{\mathcal{X}}$ which goes to 0 as $M_\nu \to 0$. Thus, $\mathcal{E}^\nu \to \mathcal{E}^0$ pointwise ae on $\mathcal{Q}$ implies $\mathbf{u}^\nu = \mathbf{T}(\mathcal{E}^\nu) \to \mathbf{u}^0 = \mathbf{T}(\mathcal{E}^0)$ in $\mathcal{X}$.

For arbitrary L^2 convergence: $\mathcal{E}^\nu \to \mathcal{E}^0$, we note that one can always find *subsequences* which converge ae on $\mathcal{Q}$. Since the limit is specified uniquely, one has the desired convergence for the full sequence: $\mathbf{T}(\mathcal{E}^\nu) \to \mathbf{T}(\mathcal{E}^0)$ in $\mathcal{X} = \mathcal{X}_2$, proving the continuity of $\mathbf{T} : L^2(\mathcal{Q}) \to \mathcal{X}_2$. *A fortiori*, noting **(H-1.vi)**, one has continuity of $\mathbf{L}\circ\mathbf{T}$ from $L^2(\mathcal{Q})$ to $\mathcal{K}_2 \subset L^2(\mathcal{Q})$.

Since $\mathcal{K}$ is compact and $\mathbf{T}$ is continuous, we note that $\mathcal{K}_3 := \{\mathbf{u} = \mathbf{T}(\mathcal{E}) : \mathcal{E} \in \mathcal{K}\}$ is compact in $\mathcal{X}_2$. For future reference, observe that $\{\nabla\mathbf{u} : \mathbf{u} \in \mathcal{K}_3\}$ is then compact in $L^2(\mathcal{Q})$ whence, setting $\sigma^2(t) = \sigma^2(t;\mathbf{u}) := 1 + \|\mathbf{u}(t,\cdot)\|^2_{(1)}$, we have σ in a compact subset of $L^2(0,T)$; thus, $\int_0^t e^{-\mu s}\sigma^2(s;\mathbf{u})\,ds \to 0$ uniformly in $[t,\mathbf{u}] \in [0,T]\times\mathcal{K}_3$ as $\mu \to \infty$. We have shown that, for any $\varepsilon > 0$,

$$(3.7)\qquad 0 \le \int_0^t e^{-\mu s}[1+\sigma^2(s;\mathbf{u})]\,ds < \varepsilon \qquad 0 \le t \le T,\ \mathbf{u} \in \mathcal{K}_3$$

for a suitably chosen $\mu = \mu(\varepsilon)$.

[5] Restricting, $\mathbf{L}\circ\mathbf{T}$ to the compact, convex, invariant set $\mathcal{K}$, the Schauder Theorem applies to give existence of a fixpoint $\mathcal{E}^* \in \mathcal{K}$. This completes the existence proof on setting $\mathbf{u}^* := \mathbf{T}(\mathcal{E}^*)$. □

Theorem 2: Let $\{\hat{\mathbf{u}}^\nu\}$ be as in **(H-1.vii)** with $\hat{\mathbf{u}}^\nu \to \hat{\mathbf{u}}^0$ and let $\{\mathbf{u}^\nu\}$ be corresponding solutions of the coupled system. Then (for a subsequence) we have $[\mathcal{E}^\nu,\mathbf{u}^\nu] \to [\mathcal{E}^0,\mathbf{u}^0]$ where $[\mathcal{E}^0,\mathbf{u}^0]$ is some solution corresponding to $\hat{\mathbf{u}}^0$.

Proof: From **[4]**, above, we know $\mathcal{K}\times\mathcal{K}_3$ is compact so we can find a convergent subsequence $[\mathcal{E}^\nu,\mathbf{u}^\nu] \to [\mathcal{E}^0,\mathbf{u}^0]$ (for some $[\mathcal{E}^0,\mathbf{u}^0]$) with, in addition, pointwise convergence ae on $\mathcal{Q}$ for $\mathcal{E}^\nu, \nabla\mathbf{u}^\nu$. Essentially the same argument as in Step **[4]** of the proof for Theorem 1 shows that $\mathbf{u}^0$ is a solution of (2.3) corresponding to $\mathcal{E}^0, \hat{\mathbf{u}}^0$. Clearly we also have $\mathcal{E}^0 = \lim \mathbf{L}\mathbf{u}^\nu = \mathbf{L}\mathbf{u}^0$, completing the proof. □

4 Uniqueness

In this section we prove, under essentially the same set of hypotheses **(H-2)** used in [10], that the solution of the coupled system whose existence was shown in Section 3 is actually unique. Answering a question raised by Hans Weinberger at the original conference presentation of [10], our present approach, somewhat more constructive than there, applies the Contractive Mapping Theorem to the map

$$\mathbf{L} \circ \mathbf{T}: \quad \mathcal{E} \overset{\mathbf{T}}{\longmapsto} \mathbf{u} \overset{\mathbf{L}}{\longmapsto} \mathcal{E} : \mathcal{C}(\bar{\mathcal{Q}}) \longrightarrow \mathcal{X}_p \longrightarrow \mathcal{C}(\bar{\mathcal{Q}})$$

where $\mathcal{X}_p := L^\infty(\to L^p) \cap \mathcal{X}_2$ with p as in **(H-2.vi)**. Note that the big difference in this setting from that of (3.1) is that we now wish to consider $\mathcal{E} \in \mathcal{C}(\bar{\mathcal{Q}})$ and so will need an estimate for $\mathbf{u}$ in $L^\infty(\to L^p(\Omega))$ to apply **(H-2.vi)**.

Theorem 3: Under the hypotheses **(H-2)**, the system (2.3) for $k = 1, \ldots, K$ has a unique weak solution $[\mathcal{E}^*, \mathbf{u}^*] \in L^2(\mathcal{Q}) \times \mathcal{Z}$ with $\mathcal{E}^* = \mathbf{L}\mathbf{u}^*$ and the constitutive relations (2.4). This solution is obtainable by iterating the map $\mathbf{L} \circ \mathbf{T}$, starting with any measurable $\mathcal{E}$.

Proof: We divide the proof into the following five steps, using notation from the proof of Theorem 1:

[1] $\mathcal{K}_1$ is bounded in $L^\infty(\to L^p)$, hence in $\mathcal{X}_p$;
[2] $\mathcal{K}$ is compact in $\mathcal{C}(\bar{\mathcal{Q}})$; after 2 iterations of $\mathbf{L} \circ \mathbf{T}$, starting with any measurable initial $\mathcal{E}$, the iterates are in $\mathcal{K}$;
[3] principal estimates for differences;
[4] $\mathbf{L} \circ \mathbf{T}$ is contractive on $\mathcal{K}$ with respect to a suitably chosen metric for $\mathcal{C}(\bar{\mathcal{Q}})$;
[5] There is a unique solution $[\mathcal{E}^*, \mathbf{u}^*]$, continuously dependent on the data $\hat{\mathbf{u}}$.

Here, **[2]** and **[5]** are quick; **[1]** is as in [10]; the interesting steps are **[3]** and **[4]**. For **[1]** and **[3]** we have somewhat similar estimation arguments, involving $\mathbf{z} = \mathbf{u} - \hat{\mathbf{u}}$ or $\mathbf{z} = \mathbf{u}' - \mathbf{u}''$, and will use a common notation. Setting

$$y = y_k := |z|^{p/2}, \qquad w := |z|^{p-1}\mathrm{sgn}\, z = z|z|^{p-2},$$

it follows that

$$\begin{aligned}
&(4.1) \qquad & zy \quad &= |z|^p = w^2, \qquad |z| = y^\vartheta, \\
&(4.2) \qquad & \nabla w \quad &= (2-\vartheta) y^{1-\vartheta} \nabla y, \qquad w\nabla z = \vartheta\, y \nabla y, \\
&(4.3) \qquad & p|\nabla w \cdot \nabla z| \quad &= \delta' |\nabla w|^2 \text{ with } \delta' = 2(2-\vartheta)\delta, \\
&(4.4) \qquad & v\dot{z} \quad &= [e^{-rt} w^2]^{\cdot} + r e^{-rt} w^2 \text{ for } v := p e^{-rt} w|_{[0,\tau]}
\end{aligned}$$

where, for convenience, we have set $\vartheta := 2/p \leq 1$.

[1] Let $\mathbf{z} = \mathbf{u} - \hat{\mathbf{u}}$ as in (3.3) and take v as in (4.1) for the weak formulation. This gives

$$\begin{aligned}
e^{-rt}\|y(t)\|^2 + r\int_0^t e^{-rs}\|y\|^2 + \delta' \int_0^t e^{-rs}\|y\|^2_{(1)} \\
\leq p\int_0^t e^{-rs} \int_\Omega [Sw + \gamma|z||\nabla w| + |\hat{\mathcal{J}}||\nabla w| + \hat{u}^{\cdot} w]
\end{aligned}$$

and, as in obtaining (3.5), one gets the desired bound on suitably estimating the terms on the right. For example, since (3.4), (3.5) bound S in $L^2(\mathcal{Q})$, we estimate w in L^2 by $\|w\| \leq \|y\|_p \|y\|^{1-\vartheta}$ and then observe that $H^1(\Omega) \hookrightarrow L^p(\Omega)$ (since we have assumed $d = 3$ and $p \leq 6$) so $\|y\|_p \leq M\|y\|_{(1)}$. The same estimate would have treated the term $\hat{u}^{\cdot} w$ had we assumed $\hat{u}^{\cdot} \in L^2(\mathcal{Q})$ in **(H-2.vii)**; for the hypothesis actually imposed we observe that $y^6 = |w|^{3p/(p-1)}$ so

$$\int_0^t e^{-rs} \int_\Omega \hat{u}^{\cdot} w \leq \varepsilon \int_0^t e^{-rs} \|y\|_{(1)}^2 + C_\varepsilon \int_0^t e^{-rs} \|\hat{u}^{\cdot}\|_{3p/(2p+1)}^p$$

(note that still another mode of estimation would work if we were to assume $\hat{u}^{\cdot} \in L^1(\to L^p)$ instead and these could be combined for alternate possibilities). One similarly treats the term $|\hat{\mathcal{J}}||\nabla w|$ by using **(H-2.vii)** to estimate $|\hat{\mathcal{J}}|y^{1-\vartheta}$ in L^2. As in (3.5) one thus bounds y in $\mathcal{X}_2$; clearly, $\|y(t)\|^2 = \|z\|_p^p$ so this bounds $\mathbf{z}$ in $L^\infty(\to L^p)$ and therefore, recalling the bound on $\|\mathbf{u}\|_{(1)}$ from the proof of Theorem 1, bounds $\mathbf{u} = \mathbf{T}(\mathcal{E})$ in $\mathcal{X}_p$, independently of $\mathcal{E}$.

[2] From **(H-2.vi)** it follows that one can introduce a space Y such that the embedding $L^p(\Omega) \hookrightarrow Y$ is compact and $\mathbf{L} : Y \to \mathcal{C}(\bar{\Omega})$ continuous; we may take $Y \hookrightarrow H^{-1}(\Omega)$. Since, for $\mathbf{u} \in \mathcal{K}_1$, we have bounds on $\mathbf{u}$ in $L^\infty(\to L^p)$ and on $\dot{\mathbf{u}}$ in $L^2(\to H^{-1})$, it follows that $\mathcal{K}_1$ is contained in a compact subset of $\mathcal{C}([0,T] \to Y)$ [14][2] so $\mathcal{E} = \mathbf{L}\mathbf{u} \in \mathcal{K}_2$ is in a compact subset of $\mathcal{C}(\to \mathcal{C}(\bar{\Omega})) = \mathcal{C}(\bar{\mathcal{Q}})$ whence $\mathcal{K} := \overline{\text{co}}\mathcal{K}_2$ is also compact in $\mathcal{C}(\bar{\mathcal{Q}})$.

[3] Suppose we have $\bar{\mathcal{E}}', \bar{\mathcal{E}}'' \in \mathcal{K}$ and, correspondingly, $\mathbf{u}' = \mathbf{T}(\bar{\mathcal{E}}')$, etc.; we set $\mathbf{z} := \mathbf{u}' - \mathbf{u}''$. It is also convenient to set

$$\begin{aligned}
\eta(t) = \eta_k(t) &= \|z_k\|_p = \|y_k\|^\vartheta \quad \text{so } \eta^p = \|y\|^2, \\
\zeta(t) &= \|\mathbf{z}\|_p = (\textstyle\sum_k \eta_k^p)^{1/p}, \qquad \bar{\zeta}(t) = \|\bar{\mathcal{E}}' - \bar{\mathcal{E}}''\|_\infty, \\
\hat{\zeta}(\tau) &= \sup\{\zeta(t) : 0 \leq t \leq \tau\}.
\end{aligned}$$

We also let $\tilde{\sigma} \geq 1$ bound pointwise *each* of the (finitely many functions $|u_k|, |\nabla u_k|$, etc.; then set $\sigma^2(t) = 1 + \|\tilde{\sigma}(t,\cdot)\|$. Note that the bound $\|\mathbf{L}\mathbf{u}\|_\infty \leq \gamma_* \|\mathbf{u}\|_p$ given by **(H-2.vi)** will give

$$\|\mathcal{E}' - \mathcal{E}''\|_\infty \leq \gamma_* \hat{\zeta}(t) \tag{4.5}$$

for each t. One then has

$$\begin{aligned}
|u'\mathcal{V}' - u''\mathcal{V}''| &\leq M[|z| + \sigma\bar{\zeta}] \\
|S' - S''| &\leq M[|\mathbf{z}, \nabla \mathbf{z}| + \sigma\bar{\zeta}|.
\end{aligned}$$

from the Lipschitz conditions **(H-2.iii,v)**.

[2] Consider the embedding map $\phi : \mathcal{Y} \hookrightarrow \mathcal{Y}'$ where $\mathcal{Y}$ is the Y-closure of the image of the M-ball of $L^p(\Omega)$ and $\mathcal{Y}'$ the corresponding image in $H^{-1}(\Omega)$. $\mathcal{Y}$ is compact and ϕ is a continuous bijection, so ϕ^{-1} is continuous. The bounds give $\mathbf{u}$ in a compact subset of $\mathcal{C}([0,T] \to H^{-1})$ by Arzela-Ascoli and 'lifting' by ϕ^{-1} gives compactness where desired.

We proceed by taking $v = 2e^{-rt}z|_{[0,\tau]}$ — as in obtaining the continuity of $\mathbf{T}$ via (3.6) but here with the advantage of the Lipschitz conditions which are now imposed. Thus, we get

$$\begin{aligned} e^{-rt}\|z(t)\|^2 &+ r\int_0^t e^{-rs}\|z\|^2 + 2\delta\int_0^t e^{-rs}\|z\|^2_{(1)} \\ &\leq 2\int_0^t e^{-rs}\int_\Omega (|u'\mathcal{V}' - u''\mathcal{V}''||\nabla z| + |S' - S''||z|) \\ &\leq \delta\int_0^t e^{-rs}\|z\|^2_{(1)} + \tfrac{\delta}{2K}\int_0^t e^{-rs}\|\mathbf{z}\|^2_{(1)} \\ &\qquad + M'\int_0^t e^{-rs}\|z\|^2 + M\int_0^t e^{-rs}\sigma^2\bar{\zeta}^2. \end{aligned}$$

Taking r large enough and summing over k, this gives the first principal estimate:

$$\int_0^t e^{-rs}\|\mathbf{z}\|^2_{(1)} \leq M\int_0^t e^{-rs}\sigma^2\bar{\zeta}^2. \tag{4.6}$$

Next take the test function $v = pe^{-rt}w|_{[0,\tau]}$ as in (4.1) and get

$$\begin{aligned} e^{-rt}\eta^p(t) &+ r\int_0^t e^{-rs}\eta^p + \delta'\int_0^t e^{-rs}\|y\|^2_{(1)} \\ &\leq 2\int_0^t e^{-rs}\int_\Omega (|u'\mathcal{V}' - u''\mathcal{V}''|y^{1-\vartheta}|\nabla y| + |S' - S''|y^{2-\vartheta}) \\ &\leq M\int_0^t e^{-rs}([\eta + \sigma\bar{\zeta}]\eta^{p/2-1}\|y\|_{(1)} + [\|\mathbf{z}\|_{(1)} + \sigma\bar{\zeta}]\|w\|); \\ e^{-rt}\zeta^p(t) &\leq M\int_0^t e^{-rs}(\sigma^2\bar{\zeta}^2 + \|\mathbf{z}\|^2_{(1)})\zeta^{p-2}; \\ \zeta^p(t) &\leq Me^{rt}\Big(\int_0^t e^{-rs}\sigma^2\bar{\zeta}^2\Big)\hat{\zeta}^{p-2}(\tau) \quad (t \leq \tau), \end{aligned}$$

using (4.6) for the last, in which we now take the sup over $[0,\tau]$ — first on the right, then on the left. Dividing by $\hat{\zeta}^{p-2}$, one then obtains the second principal estimate:

$$e^{-rt}\hat{\zeta}^2(\tau) \leq M_*\int_0^t e^{-rs}\sigma^2\bar{\zeta}^2. \tag{4.7}$$

[4] Now choose $\varepsilon > 0$ small enough: $\gamma_* M_* \varepsilon < 1$ with γ_* as in (4.5) and M_* as in (4.7); set $\mu = \mu(\varepsilon)$ as in (3.7), which still applies here. Then, multiplying (4.7) by $e^{-\mu t}$, we have

$$e^{-(r+\mu)t}\hat{\zeta}^2(t) \leq M_*\int_0^t e^{-\mu(t-s)}\sigma^2(s)\,e^{-(r+\mu)s}\bar{\zeta}^2(s)\,ds \tag{4.8}$$

$$\leq M_*\varepsilon\ \sup\{e^{-(r+\mu)s}\bar{\zeta}^2(s) : 0 \leq s \leq \tau\} \tag{4.9}$$

for $t \leq \tau \leq T$. Fixing r large enough to justify (4.6), (4.7) and $\mu = \mu(\varepsilon)$ as above, we now select the rather unusual norm:

$$\|f\|_* := \max_{0\leq\tau\leq T}\{e^{-(r+\mu)\tau}\max_{0\leq t\leq\tau}\max_{x\in\bar{\Omega}}\{|f(t,x)|\}\}$$

for $\mathcal{C}(\bar{Q})$. Then, from (4.5) and (4.8) we get

$$\|\mathcal{E}' - \mathcal{E}''\|_* \leq \gamma_* M_* \varepsilon \|\bar{\mathcal{E}}' - \bar{\mathcal{E}}''\|_* \tag{4.10}$$

— which is just the desired contractivity estimate.

[5] From (4.10) and the Contractive Mapping Theorem, we obtain existence and *uniqueness* of a fixpoint $\mathcal{E}^*$ of the map $\mathbf{L} \circ \mathbf{T}$, giving a unique solution of the system. The continuous dependence on the data $\hat{\mathbf{u}}$ follows from Theorem 2. Theorem 2 really gives continuity to $\mathcal{X}_2$ but a compactness argument lifts that to $\mathcal{X}_p$. Indeed, estimation along the lines of Step **[3]** shows this is actually *Lipschitz* continuity. □

5 Discussion

We conclude with a few comments on the physical relevance and some possible extensions of these results. Some of these will be treated more fully in a forthcoming paper [11].

• It has been customary in the literature to write the drift velocity $\mathcal{V}_k$ in the form $\mu_k \mathcal{E}$ initially taking the *mobility* μ_k constant but then $\mathcal{E}$-dependent to allow for the observed *velocity saturation* (boundedness of $\mathcal{V}$ even for large fields, as in **(H-1.iii)**). The difficulty is that this is a phenomenological approach to finding an acceptable simplified version, fixing on the approximate average velocity, of a fuller treatment involving distributions in velocity/energy as well as position; since the velocity average at t, x cannot really be simply a function of $\mathcal{E}(t,x)$, this leads to problems which have become important in a variety of problems of current technological interest. There are various models which tinker with the dependences of the μ_k in essentially *ad hoc* fashions, explicitly introduce averaged energies as dependent variables and take $D_k, \mathcal{V}_k, S_k$ to depend on these rather than directly on $\mathcal{E}$, etc. We fix on (2.4) as defining a class of problems of mathematical interest which covers most of the standard models. It seems likely that the methods here would handle the alternate models as well.

• Somewhat related to the consideration above is treatment of the *Einstein relation* connecting D_k to μ_k. For saturating velocities this would imply that one could not have uniform ellipticity as assumed in **(H-1.ii)** so the parabolic problem would become degenerate for large $|\mathcal{E}|$. (Indeed, there is some experimental indication of this for Si (but not for $GaAs$), although this is subject to interpretation.) It seems plausible that the present results might continue to be valid in a setting permitting certain such forms of degeneracy but, for the present, we are simply imposing **(H-1.ii)** as a general hypothesis.

• It is the condition **(H-2.vi)**, in particular, which is restrictive for applications since this amounts to a geometric condition on Ω, prohibiting consideration of regions with corners. Provided the boundary $\partial\Omega$ is smooth enough and one avoids 'change of type' for the boundary conditions (as here for Γ_D, Γ_N), this condition **(H-2.vi)** is a standard result — (2.2) gives $\psi \in W^{1,p}(\Omega)$ [7] for the Poisson equation and then $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\bar{\Omega})$ when $p > d$. Unfortunately, this generally fails otherwise [4]. Some of the corners occurring in interesting geometries involve no such problems (due to symmetry considerations) but others do and this is a genuine restriction. In comparison with the argument in [10], we note that here one has at least the existence proof under the condition **(H-1.vi)**, almost trivially satisfied for $\mathbf{L}$ given by (2.2), without needing the restrictive condition **(H-2.vi)**. It would also be desirable to permit dependence of D_k on $\mathcal{E}$.

Such extensions of the uniqueness result should be available if one would have greater regularity (specifically, better gradient estimates) for the solution $\mathbf{u}$ than is available from the arguments presented here. It seems unlikely that the indicated regularity here can be significantly improved without strengthening the hypotheses somehow. It *does* seem plausible that useful results could be obtained by noting (for the situations of applied interest) that the material properties expressed in the dependences on t, x are actually *piecewise smooth* in $x \in \Omega$.

• The homogeneous 'no flux' condition used here is typical, arising from insulation or by symmetry. For expository simplicity (avoiding boundary terms) we have considered only this case here but these methods could also apply to nonlinear boundary conditions on Γ_N of the form: $\mathcal{J}_k \cdot \mathbf{n} = \phi_k(\cdot, \mathbf{u})$ — or even $\phi_k(\cdot, \mathbf{u}, \mathcal{E})$ under the hypotheses **(H-2)**.

• The existence theorem could easily be 'improved' by permitting $D_k, \mathcal{V}_k$ to depend on $\mathbf{u}$ as well as on $\mathcal{E}$. This involves negligible modification of the arguments: one takes the argument of $\mathbf{T}$ now to be the *pair* $[\mathcal{E}, \mathbf{u}] \in L^2(\mathcal{Q})$, a modification which also permits imposing Lipschitz continuity only with respect to $\nabla\mathbf{u}$ in **(H-1.v)** (One could even permit a dependence on ∇u_k if this would satisfy a one-sided condition of monotonicity type). Including a dependence on $\mathbf{u}$ would almost 'automatically' include temperature coupling since the heat could be considered formally as just another 'species', satisfying much the same sort of equation. The only difference is that the source term for this, given by ohmic heating, is essentially $[\mathcal{E} \cdot \mathcal{J}]$ which does not satisfy **(H-1.iv)** and so requires a certain amount of special treatment.

• Physically, of course, it would be meaningless for the densities $\{u_k\}$ to be negative. Imposing appropriate structural assumptions on S_k, easy maximum principle arguments can be used to show that the $\{u_k\}$ do remain non-negative. While simple maximum principle arguments do not seem to apply here to provide $L^\infty(\mathcal{Q})$ bounds for $\mathbf{u}$, it seems plausible that (with some harmless modification of the hypotheses) these might be obtained for the present context by the techniques of [3]; one might then also expect $\mathcal{C}(\bar{\mathcal{Q}})$ regularity for $\mathbf{u}$. Unfortunately, neither those techniques, nor any others presently envisioned, seem likely to lead to the kind of bounds — *uniform as* $t \to \infty$ — which were obtained in [9] and which might lead to results for periodicity and/or steady states when one now includes the possibility of generation terms in S. It is not clear, physically, whether such results should reasonably be expected but we do note that the present analysis has made no use of such physically important properties as charge conservation and the relation of the direction of $\mathcal{V}_k$ to $q_k\mathcal{E}$ — perhaps there remains reason to hope that including these in the hypotheses could lead, as in [9] to such results.

• The contractivity argument in the proof of Theorem 3 worked with the map $\mathbf{T}$ which assumed an 'inner loop' as in Step **[1]** in the proof of Theorem 1. It would be interesting, instead, to iterate by starting with $\mathbf{u}, \nabla\mathbf{u}$, computing $\mathcal{E} = \mathbf{Lu}$, substituting in (2.4), and solving a linear equation to get the new $\mathbf{u}, \nabla\mathbf{u}$. It seems possible that contractivity for such an iteration might be demonstrated along the lines of the proof of Theorem 3.1 in [2]. Of much greater interest would be the adaptation of the arguments here to show convergence of appropriate discretizations. This is not quite a routine extension of the present results but the modifications do not seem to lead to any substantial new difficulties.

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