LARGE SHEARING MOTIONS OF NONLINEARLY VISCOELASTIC SLABS

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Abstract. We study the shearing motion of a slab consisting of an incompressible nonlinearly viscoelastic material of differential type. We describe reasonable general constitutive restrictions that ensure that the strain is bounded during any motion lasting a finite length of time. We briefly sketch how this simple property, which is easily verified, ensures that the governing initial-boundary-value problems have globally defined classical solutions.

1. Introduction

There is a very pretty collection of specific problems of nonlinear elastodynamics having accessible, explicit solutions (cf. Eringen and Şuhubi [1] and Truesdell and Noll [2]). E.g., there are several problems admitting periodic solutions with large amplitudes (cf. [3, 4]). But such well-behaved solutions are rare: Solutions can blow up because the material cannot sustain the loads placed upon it (cf. [5, 6, 7, 8]) or because shocks appear. For the latter reason the general existence theorems of [9, 10, 11, 12] and others give results only for a sufficiently short time interval.

We can prevent the appearance of shocks by introducing viscosity, so that the stress depends upon the strain and the strain rate, with the latter dependence ensuring the dissipation of energy. The pioneering papers of Greenberg, MacCamy, and Mizel [13], Kanel' [14], and MacCamy [15] on existence for scalar equations assumed that the stress depends linearly on the strain rate (as for a Newtonian fluid). Almost all subsequent analyses of problems of nonlinear viscoelasticity were based on this assumption. The notable exception to this rule is the work of Dafermos [16]. His particular constitutive assumptions are inappropriate for problems with compression; a suitable alternative set was given by Antman and Seidman [17]. Dafermos's assumptions are, however, perfectly suitable for scalar problems of shearing motion. Antman and Malek-Madani [18] were able to apply them to get an existence theory for a system of equations describing shearing in two directions, but only at the cost of imposing other unduly restrictive constitutive assumptions (which decouple the principal part of the differential operator).

In the present paper we exhibit a new set of constitutive restrictions for an incompressible viscoelastic body that are physically natural and have a character different from the restrictions of Dafermos. We then show how easy it is to use these restrictions to bound the strains in terms of the data. We finally give the barest hints showing how these bounds lead to an existence theory global in time, which suffers from none of the artificiality of the theory of Antman and Malek-Madani.

Notation. Vectors, which are elements of Euclidean 3-space, and vector-valued functions are denoted by lower-case, bold-face symbols. The (Gâteaux) differential of $\boldsymbol{u} \mapsto \boldsymbol{f}(\boldsymbol{u})$ at $\boldsymbol{v}$ in the direction $\boldsymbol{h}$ is $\frac{d}{dt}\boldsymbol{f}(\boldsymbol{v}+t\boldsymbol{h})\big|_{t=0}$. When it is linear in $\boldsymbol{h}$ we denote this differential by $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{v})\cdot\boldsymbol{h}$ or $\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{v})\cdot\boldsymbol{h}$. We often denote the function $\boldsymbol{u} \mapsto \boldsymbol{f}(\boldsymbol{u})$ by $\boldsymbol{f}(\cdot)$. The partial derivative of a function f with respect to to a scalar argument t is denoted by either f_t or $\partial_t f$. Ordinary derivatives are denoted by primes. Obvious analogs of these notations will also be used.

2. Formulation of the Governing Equations

Let $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{k}\}$ be a fixed orthonormal basis for Euclidean 3-space. We study the motion of an incompressible, isotropic, viscoelastic material of differential type of complexity 1 whose reference configuration is the slab consisting of all material points $x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + s\boldsymbol{k}$ with $0 \le s \le 1$. (The isotropy is not essential for our study, but it suggests the form of our basic constitutive assumption and it leads to a particularly beautiful and illuminating system of equations, special solutions of which were studied by Antman and Guo [3] and by Antman and Malek Madani [18].) We study shearing motions in which the displacement of the material point $x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + s\boldsymbol{k}$ has the form

$$\boldsymbol{u}(s,t) + s\boldsymbol{k} \equiv u_1(s,t)\boldsymbol{i}_1 + u_2(s,t)\boldsymbol{i}_2 + s\boldsymbol{k}. \tag{2.1}$$

Relative to the basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{k}\}$ the material strain tensor $\boldsymbol{E}$ and its time derivative $\boldsymbol{E}_t$ have matrices given by

$$\begin{aligned} 2[\boldsymbol{E}] &= \begin{bmatrix} 0 & 0 & \partial_s u_1 \\ 0 & 0 & \partial_s u_2 \\ \partial_s u_1 & \partial_s u_2 & \boldsymbol{u}_s\cdot\boldsymbol{u}_s \end{bmatrix}, \\ 2[\boldsymbol{E}_t] &= \begin{bmatrix} 0 & 0 & \partial_{st} u_1 \\ 0 & 0 & \partial_{st} u_2 \\ \partial_{st} u_1 & \partial_{st} u_2 & 2\boldsymbol{u}_s\cdot\boldsymbol{u}_{st} \end{bmatrix}. \end{aligned} \tag{2.2}$$

Note that $\det(\boldsymbol{I}+2\boldsymbol{E}) = 1$, so that (2.1) describes volume-preserving motions.

For our material, the second Piola-Kirchhoff stress tensor is the sum of $-p(\boldsymbol{I}+2\boldsymbol{E})^{-1}$ and a prescribed isotropic tensor-valued function of $\boldsymbol{E}, \boldsymbol{E}_t, s$; here p is the pressure (the Lagrange multiplier maintaining the constraint of incompressibility).

Let us assume that the material properties and the projection of the body force onto the $\{\boldsymbol{i}_1, \boldsymbol{i}_2\}$-plane are independent of the material coordinates x_1 and x_2. We can write the equations of motion in this plane in the vectorial form

$$\rho(s)\boldsymbol{u}_{tt} = \boldsymbol{n}_s + \boldsymbol{f} \tag{2.3}$$

where $\rho(s)$ is the given density at any material point with third coordinate s, where $\boldsymbol{n} = n_1\boldsymbol{i}_1 + n_2\boldsymbol{i}_2$ with n_1 and n_2 the shear components of the Piola-Kirchhoff stress tensor on the $\boldsymbol{k}$-face in the $\boldsymbol{i}_1$- and $\boldsymbol{i}_2$-directions, and where $\boldsymbol{f}$ is a given function with values in the $\{\boldsymbol{i}_1, \boldsymbol{i}_2\}$-plane containing contributions from the body force and from the prescribed pressure on the faces of the slab. ρ is assumed to be continuous and to have a positive lower bound. Since its mathematical treatment is routine, we assume that $\boldsymbol{f} = \boldsymbol{0}$. The third equation of motion, in the $\boldsymbol{k}$-direction, which we do not exhibit, is used to determine the pressure field.

Our constitutive equations have the general form

$$\boldsymbol{n}(s,t) = \hat{\boldsymbol{n}}(\boldsymbol{u}_s(s,t), \boldsymbol{u}_{st}(s,t), s) \tag{2.4}$$

where $\hat{\boldsymbol{n}}$ is a given well-behaved function. We limit our attention to isotropic materials. We introduce the invariants

$$\alpha := \sqrt{\boldsymbol{u}_s \cdot \boldsymbol{u}_s}, \quad \beta := \sqrt{\boldsymbol{u}_{st} \cdot \boldsymbol{u}_{st}}. \tag{2.5a}$$

Note that $|\alpha_t| \leq \beta$. By using a representation theorem for isotropic functions, versions of which are due to [19, 20, 21], we can write the constitutive function $\hat{\boldsymbol{n}}$ for our isotropic materials for the restricted class (2.1) of deformations in the form

$$\hat{\boldsymbol{n}}(\boldsymbol{u}_s, \boldsymbol{u}_{st}, s) = \mu(\alpha, \alpha_t, \beta, s)\boldsymbol{u}_s + \nu(\alpha, \alpha_t, \beta, s)\boldsymbol{u}_{st} \tag{2.5b}$$

where μ and ν are given constitutive functions, assumed to be continuously differentiable. For details of the derivation of (2.3)–(2.5), see [22, Sec. XVI.5] or [18].

For some of our work it is convenient to express our constitutive functions in a form that distinguishes elastic effects from dissipative effects. For this purpose we denote the first and second arguments of $\hat{\boldsymbol{n}}$ by $\boldsymbol{y} = y_1\boldsymbol{i}_1 + y_2\boldsymbol{i}_2$ and $\boldsymbol{z} = z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2$. We introduce the stored-energy function φ for the elastic part of the constitutive functions by

$$\mu(\alpha, 0, 0, s) = \frac{1}{\alpha}\frac{\partial\varphi}{\partial\alpha}(\alpha, s), \tag{2.6a}$$

so that

$$\mu(\sqrt{\boldsymbol{y}\cdot\boldsymbol{y}}, 0, 0, s)y_1 = \frac{\partial\varphi}{\partial y_1}(\sqrt{\boldsymbol{y}\cdot\boldsymbol{y}}, s), \quad \text{etc.} \tag{2.6b}$$

We introduce the dissipative part μ^{D} of μ by

$$\mu(\alpha, \alpha_t, \beta, s) \equiv \frac{1}{\alpha}\frac{\partial\varphi}{\partial\alpha}(\alpha, s) + \mu^{\mathrm{D}}(\alpha, \alpha_t, \beta, s). \tag{2.7}$$

We require that effects of internal friction grow with the strain rates:

2.8. Hypothesis. *Let the arguments of μ and ν be denoted $(\alpha, \boldsymbol{y}\cdot\boldsymbol{z}/\alpha, \sqrt{\boldsymbol{z}\cdot\boldsymbol{z}}, s)$. There is a positive number c such that*

$$\begin{aligned}\boldsymbol{\xi}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{z}}\cdot\boldsymbol{\xi} \equiv \frac{\partial}{\partial z_1}(\mu y_1+\nu z_1)\xi_1^2 + \left[\frac{\partial}{\partial z_2}(\mu y_1+\nu z_1)+\frac{\partial}{\partial z_1}(\mu y_2+\nu z_2)\right]\xi_1\xi_2 \\ + \frac{\partial}{\partial z_2}(\mu y_2+\nu z_2)\xi_2^2 \geq c(\xi_1^2+\xi_2^2) \quad \forall\,\boldsymbol{\xi}=\xi_1\boldsymbol{i}_1+\xi_2\boldsymbol{i}_2. \quad (2.8)\end{aligned}$$

This monotonicity condition ensures that the response is truly dissipative and that the governing equations of motion have a uniformly (hyperbolic-) parabolic character. Of course, we can replace μ with μ^{D} in (2.8).

Our novel constitutive restriction is

2.9. Hypothesis. *There is a continuously differentiable function ψ of α with $\psi(\alpha)\to\infty$ as $\alpha\to\infty$, there is a strictly increasing, convex function ω of β with $\omega(0)=0$, and there are non-negative numbers C and c such that*

$$\begin{aligned}\hat{\boldsymbol{n}}(\boldsymbol{u}_s,\boldsymbol{u}_{st},s)\cdot\frac{\boldsymbol{u}_s}{|\boldsymbol{u}_s|} &\equiv \mu(\alpha,\alpha_t,\beta,s)\alpha+\nu(\alpha,\alpha_t,\beta,s)\alpha_t \\ &\geq \psi'(\alpha)\alpha_t - C\psi(\alpha)+c\beta\alpha.\end{aligned} \quad (2.9)$$

The presence of ψ' in (2.9) implies that the viscosity depends mildly on the strain for large strain α. Indeed, ψ' can be a decreasing function of α, e.g., we could have $\psi'(\alpha)=1/\alpha$. The presence of the term $c\beta\alpha$, might suggest that the shear modulus μ grows at least linearly with the strain rate. Since $\psi(0)$ can have any sign, the presence of $-C\psi$ means that the behavior of ψ' for small α is immaterial.

We shall introduce a couple of other constitutive hypotheses below.

Boundary and initial conditions. We impose various complementary combinations of components of $\boldsymbol{u}_t$ and $\boldsymbol{n}$ at $s=0,1$. We impose initial conditions on $\boldsymbol{u},\boldsymbol{u}_t$ that we assume are compatible with the boundary conditions. Then our initial-boundary-value problem for a viscoelastic slab consists of the equations of motion (2.3) the constitutive equations (2.4), (2.5), the boundary conditions, and the initial conditions just described.

3. Energy Estimates

We let c and C denote typical positive constants that are supplied as data or that can be estimated in terms of data. Their meanings typically change with each appearance (even in the same equation or inequality. C may be regarded as increasing and c as decreasing with each appearance). Similarly $c(t)$ and $C(t)$ denote typical positive constants depending on the time t. Tacit in an inequality of the form $\|u\|\leq C$ is an assertion that there exists a positive number C such that this estimate holds.

We let T be any fixed but arbitrary time. Its arbitrariness enables us to study our initial-boundary-value problem for all time by restricting our attention to the times in the bounded interval $[0,T]$, while thereby avoiding difficulties that can arise on unbounded intervals.

Throughout our exposition we use without comment the Cauchy-Bunyakovskiĭ-Schwarz inequality and the elementary inequality $2|ab| \leq \varepsilon a^2 + b^2/\varepsilon$ for real a, b and for positive ε.

In this and the next section we assume that $\boldsymbol{u}_s$, $\boldsymbol{u}_t$, and $\boldsymbol{u}_{st}$ are continuous. The existence theory outlined in Section 5 shows that there exist such solutions.

We substitute (2.4), (2.5), and (2.7) into into (2.3), take the dot product of the resulting equation with $\boldsymbol{u}_t$, and integrate the resulting product by parts over $[0,1] \times [0,t]$ to obtain the energy equation

$$K(t) + \Phi(t) + W(t) = K(0) + \Phi(0) + \int_0^t \boldsymbol{n}(s,\tau) \cdot \boldsymbol{u}_t(s,\tau)\, d\tau \Big|_{s=0}^{s=1} \tag{3.1}$$

where

$$K(t) := \tfrac{1}{2} \int_0^1 \rho(s) \boldsymbol{u}_t(s,t) \cdot \boldsymbol{u}_t(s,t)\, ds, \tag{3.2a}$$

$$\Phi(t) := \int_0^1 \varphi(\alpha(s,t), s)\, ds, \tag{3.2b}$$

$$\begin{aligned} W(t) &:= \int_0^t \int_0^1 \left[\hat{\boldsymbol{n}}(\boldsymbol{u}_s, \boldsymbol{u}_{st}, s) - \hat{\boldsymbol{n}}(\boldsymbol{u}_s, 0, s)\right] \cdot \boldsymbol{u}_{st}\, ds\, d\tau \\ &\equiv \int_0^t \int_0^1 [\mu^{\mathrm{D}}(\alpha, \alpha_t, \beta, s)\alpha\alpha_t + \nu(\alpha, \alpha_t, \beta, s)\beta^2]\, ds\, d\tau \end{aligned} \tag{3.2c}$$

are the kinetic energy, the stored energy, and the work of the dissipative internal stresses at time t. We note that (2.8) implies that the integrand of (3.2c) exceeds $c\beta^2$.

We wish to bound the left-hand side of (3.1) in terms of the data. For this purpose we must estimate the integrals on the right-hand side of (3.1), which give the work done by the shear tractions on the faces. We first study the case in which

$$\boldsymbol{n}(0,t) \quad \text{and} \quad \boldsymbol{n}(1,t) \quad \text{are prescribed.} \tag{3.3}$$

We assume that these functions are locally square-integrable.

The "mass center" of the slab at time t is

$$\boldsymbol{c}(t) = \frac{\int_0^1 \rho(s)\boldsymbol{u}(s,t)\, ds}{\int_0^1 \rho(s)\, ds}. \tag{3.5}$$

We estimate it and its derivatives by integrating (2.3) with respect to s over $[0,1]$ to get

$$\left[\int_0^1 \rho(s)\, ds\right] \boldsymbol{c}_{tt}(t) = \boldsymbol{n}(1,t) - \boldsymbol{n}(0,t), \tag{3.6}$$

whence

$$|\boldsymbol{c}_t(t)| \leq C(t). \tag{3.7}$$

Let $\boldsymbol{e}$ be an arbitrary constant unit vector. Then for fixed t there is an $s^*(\boldsymbol{e},t) \in (0,1)$ such that $\boldsymbol{e} \cdot [\boldsymbol{u}_t(s^*,t) - \boldsymbol{c}_t(t)] = 0$, for if not, we would have $\boldsymbol{e} \cdot \boldsymbol{u}_t(s,t) > \boldsymbol{e} \cdot \boldsymbol{c}_t(t)$, say, for all s. Then the integration of the product of this inequality with ρ over $(0,1)$ would would yield an inequality that violates (3.5). We use this property of s^* to get

$$\boldsymbol{e}\cdot[\boldsymbol{u}_t(s,t)-\boldsymbol{c}_t(t)]=\boldsymbol{e}\cdot\int_{s^*}^{s}\boldsymbol{u}_{st}(\sigma,t)\,d\sigma \tag{3.8}$$

whence

$$|\boldsymbol{u}_t(s,t)-\boldsymbol{c}_t(t)|\le\sqrt{\int_0^1\boldsymbol{u}_{st}\cdot\boldsymbol{u}_{st}\,ds}. \tag{3.9}$$

Thus

$$|\boldsymbol{u}_t(s,t)|\le C(t)+\sqrt{\int_0^1\boldsymbol{u}_{st}\cdot\boldsymbol{u}_{st}\,ds}, \tag{3.13}$$

from which we obtain

$$\begin{aligned}\int_0^t\boldsymbol{n}(s,\tau)\cdot\boldsymbol{u}_t(s,\tau)\,d\tau\Big|_{s=0}^{s=1}&\le\int_0^t C(\tau)\,d\tau+\frac{1}{\varepsilon}\int_0^t|\boldsymbol{n}(1,\tau)|^2\,d\tau\\&\qquad+\varepsilon\int_0^t\int_0^1\boldsymbol{u}_{st}\cdot\boldsymbol{u}_{st}\,ds\,d\tau+\cdots\\&\le C(t)+\frac{2\varepsilon}{c}W(t).\end{aligned} \tag{3.14}$$

We now substitute (3.14) into (3.1) and choose ε sufficiently small to deduce that *if* (3.3) *holds, then*

$$K(t)+\Phi(t)+W(t)\le C(t). \tag{3.15}$$

Let us now replace (3.3) with

$$\boldsymbol{u}(0,t)\quad\text{and}\quad\boldsymbol{n}(1,t)\quad\text{are prescribed.} \tag{3.16}$$

We assume that $\boldsymbol{u}_{tt}(0,\cdot)$ and $\boldsymbol{n}(1,\cdot)$ are locally square integrable. We readily obtain (3.9) with $\boldsymbol{u}(0,t)$ replacing $\boldsymbol{c}_t(t)$, so that (3.13) holds, whence $\left|\int_0^t\boldsymbol{n}(1,\tau)\cdot\boldsymbol{u}_t(1,\tau)\,d\tau\right|$ is bounded by the right-hand side of (3.14). We must estimate the analogous work at $s=0$. From (2.3) we obtain

$$\begin{aligned}&\int_0^t\boldsymbol{n}(0,\tau)\cdot\boldsymbol{u}_t(0,\tau)\,d\tau\\&\quad=\int_0^t\boldsymbol{u}_t(0,\tau)\cdot\left[\boldsymbol{n}(1,\tau)-\int_0^1\rho(s)\boldsymbol{u}_{tt}(s,\tau)\,ds\right]\,d\tau\\&\quad=\int_0^t\boldsymbol{u}_t(0,\tau)\cdot\boldsymbol{n}(1,\tau)\,d\tau-\int_0^1\boldsymbol{u}_t(0,\tau)\cdot\rho(s)\boldsymbol{u}_t(s,\tau)\,ds\Big|_{\tau=0}^{\tau=t}\\&\qquad+\int_0^t\boldsymbol{u}_{tt}(0,\tau)\cdot\int_0^1\rho(s)\boldsymbol{u}_t(s,\tau)\,ds\,d\tau.\end{aligned} \tag{3.17}$$

By the techniques we have been using we easily find that the absolute value of (3.17) is dominated by

$$C(t)+\varepsilon K(t)+\int_0^t K(\tau)\,d\tau,$$

so that the left-hand side of (3.14) is here dominated by

$$C(t)+\varepsilon W(t)+\varepsilon K(t)+\int_0^t K(\tau)\,d\tau.$$

Thus for ε sufficiently small we easily get from (3.1) that

$$K(t)\le C(t)+C\int_0^t K(\tau)\,d\tau \quad \forall\, t. \tag{3.18}$$

We use the Gronwall inequality to estimate $K(t)$ and then obtain (3.15).

We now treat boundary conditions in which

$$\boldsymbol{u}_t(0,t) \quad \text{and} \quad \boldsymbol{u}_t(1,t) \quad \text{are prescribed} \tag{3.19}$$

(by introducing further constitutive restrictions below). We assume that the derivatives of these functions are locally square-integrable. We get a useful representation for the boundary values of $\boldsymbol{n}$, appearing on the right-hand side of (3.1) first by replacing s in (2.3) with σ, multiplying (2.3) by σ, and integrating the resulting equation with respect to σ over $[0,s]$. Next, we multiply (2.3) by $\sigma-1$ and integrate the resulting equation with respect to σ over $[s,1]$. We add the resulting equations (with $\boldsymbol{f}=\boldsymbol{0}$) to obtain

$$\boldsymbol{n}(s,t)=\int_0^1 \Gamma(s,\sigma)\rho(\sigma)\boldsymbol{u}_{tt}(\sigma,t)\,d\sigma+\int_0^1 \boldsymbol{n}(\sigma,t)\,d\sigma \tag{3.20}$$

where

$$\Gamma(s,\sigma)=\begin{cases}\sigma & \text{for} \quad \sigma<s,\\ \sigma-1 & \text{for} \quad s<\sigma.\end{cases}$$

We can now use (3.20) to express the last term of (3.1) as

$$\begin{aligned}&\int_0^t [\boldsymbol{u}_t(1,\tau)-\boldsymbol{u}_t(0,\tau)]\cdot\int_0^1 \boldsymbol{n}(s,\tau)\,ds\,d\tau\\ &+\int_0^t [\boldsymbol{u}_t(1,\tau)-\boldsymbol{u}_t(0,\tau)]\cdot\int_0^1 s\rho\boldsymbol{u}_{tt}\,ds\,d\tau+\int_0^t \boldsymbol{u}_t(0,\tau)\cdot\int_0^1 \rho\boldsymbol{u}_{tt}\,ds\,d\tau.\end{aligned} \tag{3.21}$$

We integrate the last two integrals by parts and use our standard inequalities to show that their absolute values are dominated by $C(t)+\varepsilon K(t)$. By virtue of (2.4), (2.5) the absolute value of the first integral is dominated by

$$\int_0^t |\boldsymbol{u}_t(1,\tau)-\boldsymbol{u}_t(0,\tau)|\int_0^1 |\varphi_{\boldsymbol{y}}(\alpha,s)+\mu^{\mathrm{D}}(\alpha,\alpha_t,\beta,s)\boldsymbol{u}_s+\nu(\alpha,\alpha_t,\beta,s)\boldsymbol{u}_{st}|\,ds\,d\tau. \tag{3.22}$$

Let us make the constitutive assumption that there is a constant C such that

$$\begin{aligned}|\varphi_{\boldsymbol{y}}+\mu^{\mathrm{D}}\boldsymbol{u}_s+\nu\boldsymbol{u}_{st}| &\equiv \sqrt{(\varphi_\alpha+\alpha\mu^{\mathrm{D}})^2+2(\varphi_\alpha+\alpha\mu^{\mathrm{D}})\nu\alpha_t+\nu^2\beta^2}\\ &\le C[1+\varphi+\mu^{\mathrm{D}}\alpha\alpha_t+\nu\beta^2].\end{aligned} \tag{3.23}$$

(We comment on the significance of (3.23) below.) Then using our estimates of the last two terms of (3.21) we deduce from (3.1) that

$$\Phi(t)+W(t)\le C(t)\left\{1+\int_0^t [\Phi(\tau)+W(\tau)]\,d\tau\right\}. \tag{3.24}$$

The Gronwall inequality implies that $\Phi(t)+W(t)\le C(t)$, so that (3.15) holds.

In summary, we have

3.25. Theorem. *If* (3.3) *or* (3.16) *holds, then* (3.15) *holds. If* (3.19) *and* (3.23) *hold, then* (3.15) *holds.*

We now discuss (3.23). Since $|\alpha_t| \le \beta$, the left-hand side of (3.23) is

$$\le \sqrt{3{\varphi_\alpha}^2 + 3(\alpha\mu^{\mathrm{D}})^2 + 3\nu^2\beta^2} \le \sqrt{3}[|\varphi_\alpha| + |\alpha\mu^{\mathrm{D}}| + |\nu\beta|]. \tag{3.26}$$

We look at circumstances in which (3.26) is less than the right-hand side of (3.23). If φ does not have wild oscillations for large α, then it is reasonable to assume that $\varphi_\alpha \le C(1+\varphi)$ (an hypothesis that is inappropriate for problems with compression). E.g., this inequality holds if $\varphi(\alpha, s) = C\alpha^p$ with $p > 1$ for α sufficiently large. Now suppose that $\nu > 0$. (This is not a consequence of (2.8), but is certainly consistent with it.) Then for $\beta \ge 1$ we have $\nu\beta \le \nu\beta^2$, which appears on the right-hand side of (3.23). For $\beta < 1$, $\nu\beta$ is dominated by a function of α and s; it is reasonable to assume that this function is dominated by $C(1+\varphi)$. Thus if we make these assumptions, together with the further assumption that the role of the mixed term $|\alpha\mu^{\mathrm{D}}|$ is subservient to those of the first and third terms on the right-hand side of (3.26), which essentially account for strain and for strain rate, then (3.23) holds. (This subservience can be made explicit, e.g., by using Young's inequality as below.)

4. Bounds on the Strains

Let us decompose the strain by

$$\boldsymbol{u}_s = \alpha \boldsymbol{e}_1 \tag{4.1}$$

with

$$\boldsymbol{e}_1 = \cos\theta \boldsymbol{i}_1 + \sin\theta \boldsymbol{i}_2, \quad \boldsymbol{e}_2 = \boldsymbol{k} \times \boldsymbol{e}_1 = -\sin\theta \boldsymbol{i}_1 + \cos\theta \boldsymbol{i}_2, \quad \partial_t \boldsymbol{e}_1 = \theta_t \boldsymbol{e}_2 \tag{4.2}$$

wherever $\boldsymbol{u}_s \ne \boldsymbol{0}$. Thus

$$\boldsymbol{u}_{st} = \alpha_t \boldsymbol{e}_1 + \alpha\theta_t \boldsymbol{e}_2, \quad \beta = \sqrt{\alpha_t^2 + \alpha^2\theta_t^2}, \tag{4.3a,b}$$

$$\hat{\boldsymbol{n}} = (\mu\alpha + \nu\alpha_t)\boldsymbol{e}_1 + \nu\alpha\theta_t \boldsymbol{e}_2. \tag{4.4}$$

Let us suppose that $\boldsymbol{n}(0,t)$ is prescribed. We define

$$g_\gamma(s,t) := \boldsymbol{e}_\gamma(s,t) \cdot \int_0^s \rho(\sigma)\boldsymbol{u}_t(\sigma,t)\,d\sigma, \quad \gamma = 1, 2. \tag{4.5}$$

Equation (2.3) yields

$$\boldsymbol{n}(s,t) = \boldsymbol{n}(0,t) + \int_0^s \rho \boldsymbol{u}_{tt}\,d\sigma, \tag{4.6}$$

from which we obtain

$$\boldsymbol{n}(s,t) \cdot \boldsymbol{e}_1(s,t) = \boldsymbol{n}(0,t) \cdot \boldsymbol{e}_1(s,t) + \partial_t g_1(s,t) - \theta_t(s,t) g_2(s,t), \tag{4.7}$$

Thus (2.9) and (4.7) yield

$$\psi'(\alpha)\alpha_t - C\psi(\alpha) \leq C(t) + \partial_t g_1(s,t) + |\theta_t(s,t)g_2(s,t)| - c\beta\alpha. \tag{4.8}$$

The methods we used in Section 3 immediately show that there are functions C_γ such that

$$|g_\gamma(s,t)| \leq C_\gamma(t), \quad \gamma = 1, 2. \tag{4.9}$$

Thus (4.3b) implies that

$$|\theta_t(s,t)g_2(s,t)| - c\beta\alpha \leq |\theta_t(s,t)|[C_2(t) - c\alpha^2]. \tag{4.10}$$

Consequently, wherever $\alpha \geq \sqrt{C_2(t)/c}$, (4.8) reduces to

$$\psi'(\alpha)\alpha_t - C\psi(\alpha) \leq C(t) + \partial_t g_1(s,t). \tag{4.11}$$

We now show that

$$\alpha(s,t) \leq C(t) \quad \text{for all } s. \tag{4.12}$$

Let $C_3(t)$ be a continuous function bounded below by $\sqrt{C_2(t)/c} + \max_s \alpha(s,0)$. ($\alpha(s,0)$ is assumed to be continuous.) Suppose that there is a ξ in $(0,1]$ and a $\tau_2 > 0$ such that $\alpha(\xi,\tau_2) > C_3(\tau_2)$. (Were there no such (ξ,τ_2), there would be nothing to prove.) Since α and C_3 are continuous, there is a last time $\tau_1 < \tau_2$ at which $\alpha(\xi,\tau_1) = C_3(\tau_2)$. We now integrate (4.11) with respect to t from τ_1 to t ($\leq \tau_2$) for $s = \xi$ to obtain

$$\psi(\alpha(\xi,t)) - \psi(C_3(\tau_2)) - C\int_{\tau_1}^{t} \psi(\alpha(\xi,\tau))\,d\tau \leq C(t) + g_1(\xi,t) - g_1(\xi,\tau_1), \tag{4.13a}$$

i.e.,

$$\psi(\alpha(\xi,t)) \leq C(t) + C\int_{\tau_1}^{t} \psi(\alpha(\xi,\tau))\,d\tau, \tag{4.13b}$$

as a consequence of (4.9). The Gronwall inequality implies that $\psi(\alpha(\xi,t)) \leq C(t)$, and the properties of ψ then yield (4.12).

The proof of (4.12) when (3.19) holds is more complicated. We must impose (3.23) to get our energy estimate. We can combine the methods used above with a much simplified version of those of [17] to deduce (4.12). We omit the details.

5. Comments on the Existence Theory

We wish to supplement (4.12) with further estimates, based on the supposition that our initial-boundary-value problem has solutions. These estimates then tell us where to seek solutions. Accordingly, we formally differentiate (2.3) with respect to t, take the dot product of this derivative with $\boldsymbol{u}_{tt}$, and then integrate the resulting equation by parts the with respect to s over $[0,1]$ to obtain

$$\frac{d}{dt}\int_0^1 \rho|\boldsymbol{u}_{tt}|^2\,ds + 2\int_0^1 \boldsymbol{u}_{stt}\cdot\boldsymbol{n}_t\,ds = 2\boldsymbol{u}_{tt}\cdot\boldsymbol{n}_t\Big|_{s=0}^{s=1}. \tag{5.1}$$

We focus on the second integral of (5.1) into which we substitute (2.4) to convert it to

$$2\int_0^1 \left[\boldsymbol{u}_{stt}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{z}}\cdot\boldsymbol{u}_{stt}+\boldsymbol{u}_{stt}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{y}}\cdot\boldsymbol{u}_{st}\right] ds. \tag{5.2}$$

The first term in the integrand of (5.2) is nonnegative, by (2.8). To control the second term we introduce the further constitutive hypothesis: There is a continuous function λ such that

$$\left|\boldsymbol{\xi}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{y}}(\boldsymbol{y},\boldsymbol{z},s)\right|^2 \le \lambda(M)\,\boldsymbol{\xi}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{z}}(\boldsymbol{y},\boldsymbol{z},s)\cdot\boldsymbol{\xi} \qquad \forall\,\boldsymbol{\xi} \quad \text{when} \quad |\boldsymbol{y}|\le M. \tag{5.3}$$

This inequality, which strengthens (2.8), says that when the strain is bounded, the "modulus of elasticity" $\hat{\boldsymbol{n}}_{\boldsymbol{y}}$ is suitably dominated by the "modulus of viscosity" $\hat{\boldsymbol{n}}_{\boldsymbol{z}}$, i.e., the dependence of the modulus of elasticity on the strain rate is weaker in a precise sense than the dependence of the modulus of viscosity on the strain rate. (A direct generalization of the inequality of Dafermos [16] to vector-valued constitutive functions would give (5.3) valid for all $\boldsymbol{y}$ in which λ is a pure constant. The availability of (4.12) enables us to weaken these restrictions considerably.)

Using (5.3) we estimate the second term in (5.2) by

$$\left|\int_0^1 \boldsymbol{u}_{stt}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{y}}\cdot\boldsymbol{u}_{st}\,ds\right| \le \int_0^1 \boldsymbol{u}_{stt}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{z}}\cdot\boldsymbol{u}_{stt}\,ds+\lambda(C(T))\int_0^1|\boldsymbol{u}_{st}|^2\,ds \tag{5.4}$$

for $t\le T$. Thus we obtain from (5.1) and (5.2) that

$$\frac{d}{dt}\int_0^1 \rho|\boldsymbol{u}_{tt}|^2\,ds+\int_0^1 \boldsymbol{u}_{stt}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial\boldsymbol{z}}\cdot\boldsymbol{u}_{stt}\,ds \le \lambda(C(T))\int_0^1|\boldsymbol{u}_{st}|^2\,ds+2\boldsymbol{u}_{tt}\cdot\boldsymbol{n}_t\Big|_{s=0}^{s=1}. \tag{5.5}$$

We integrate (5.5) with respect to t, and use the energy estimate (3.15), the Gronwall inequality, and embedding theorems in a lengthy computation to deduce that

$$|\boldsymbol{u}_{st}(s,t)|\equiv\beta\le C(t). \tag{5.6}$$

We omit the details of proof of (5.6) because they follow the pattern of Section 8 of [17].

We now exploit the bounds (4.12) and (5.6). For fixed T and for $C(T)$ exceeding the $C(T)$'s from both (4.12) and (5.6) we define the cut-off function

$$[[\boldsymbol{x}]] := \begin{cases} \boldsymbol{x} & \text{if} \quad |\boldsymbol{x}|\le 2C(T), \\ 2C(T)\frac{\boldsymbol{x}}{|\boldsymbol{x}|} & \text{if} \quad |\boldsymbol{x}|\ge 2C(T) \end{cases} \tag{5.7}$$

To motivate the crucial definition that follows, we note that (2.6) allows us to write

$$\hat{\boldsymbol{n}}(\boldsymbol{y},\boldsymbol{z},s)=\varphi_{\boldsymbol{y}}(|\boldsymbol{y}|,s)+\left[\int_0^1 \hat{\boldsymbol{n}}_{\boldsymbol{z}}(\boldsymbol{y},\zeta\boldsymbol{z},s)\,d\zeta\right]\cdot\boldsymbol{z}. \tag{5.8}$$

We introduced the modified constitutive function

$$\boldsymbol{n}^\sharp(\boldsymbol{y},\boldsymbol{z},s)=\varphi_{\boldsymbol{y}}(|\,[[\boldsymbol{y}]]\,|,s)+\left[\int_0^1\hat{\boldsymbol{n}}_{\boldsymbol{z}}([[\boldsymbol{y}]],\zeta[[\boldsymbol{z}]],s)\,d\zeta\right]\cdot\boldsymbol{z}. \tag{5.9}$$

The function $\boldsymbol{n}^\sharp$, which is Lipschitz continuous, agrees with $\hat{\boldsymbol{n}}$ for $|\boldsymbol{y}|,|\boldsymbol{z}|\leq C(T)$, and satisfies the monotonicity condition (2.8). Thus a solution of the initial-boundary-value problem for $\boldsymbol{n}^\sharp$ must agree with the corresponding solution for $\hat{\boldsymbol{n}}$. The virtue of $\boldsymbol{n}^\sharp$ is that it behaves nicely for large values of its arguments, whereas $\hat{\boldsymbol{n}}$ need not.

The differential equation obtained by substituting $\boldsymbol{n}^\sharp$ for $\boldsymbol{n}$ in (2.3) is

$$\rho(s)\boldsymbol{v}_t=\boldsymbol{n}^\sharp(\boldsymbol{u}_s,\boldsymbol{v}_s,s)_s \tag{5.10}$$

where $\boldsymbol{v}:=\boldsymbol{u}_t$. If we regard $\boldsymbol{u}_s$ as frozen, then the properties of $\boldsymbol{n}^\sharp$ ensure that (5.10) is a well-behaved parabolic system, which can be analyzed by exploiting (2.8) in Galerkin's method (cf. [23]). To handle the actual (5.10) we write it as a system for $\boldsymbol{u}$ and $\boldsymbol{v}$ by supplementing (5.10) with

$$u_s(s,t)=u_s(s,0)+\int_0^t v_s(s,\tau)\,d\tau. \tag{5.11}$$

For full details of this approach, see [17]. (Since $\hat{\boldsymbol{n}}$ is vector-valued, we cannot use tools, such as the Maximum Principle, developed for parabolic equations for scalar-valued functions, as is done in [16], e.g.)

6. Comments

Reference [17] showed how to treat a scalar problem of nonlinear viscoelasticity involving compression. Here we show how to treat a vector problem without compression. It seems reasonable to expect that these methods can be effectively combined to handle general problems of nonlinear viscoelasticity in which there is but one independent spatial variable. The chief difficulty for problems in which there are more than one independent spatial variables is to determine a natural generalization of (2.8). One could argue on physical grounds that such a generalization should be that the stress as a function of the strain rate must satisfy a strong ellipticity condition. In this case the mathematical tools are not yet available for the treatment of such problems.

For a scalar version of our problem obtained by constraining $u_2=0$, we do not encounter θ_t in (4.3) and (4.4). Hence we can get (4.12) and its consequences merely by adopting the weakened form of (2.9) in which $c=0$.

We could replace (2.9) with the alternative condition: There are numbers $M>0$, $C\geq 0$, and there is a continuously differentiable function ψ on (M,∞) with ψ convex and with $\psi(z)\to\infty$ as $z\to\infty$ such that

$$\begin{aligned}
n_1(\boldsymbol{y},\boldsymbol{z},s)&\geq \psi'(z_1)z_1-C\psi(z_1,s) &&\text{for}\quad z_1\geq M,\\
n_2(\boldsymbol{y},\boldsymbol{z},s)&\geq \psi'(z_2)z_2-C\psi(z_2,s) &&\text{for}\quad z_2\geq M,\\
n_1(\boldsymbol{y},\boldsymbol{z},s)&\leq -\psi'(z_1)z_1+C\psi(z_1,s) &&\text{for}\quad z_1\leq -M,\\
n_2(\boldsymbol{y},\boldsymbol{z},s)&\leq -\psi'(z_2)z_2+C\psi(z_2,s) &&\text{for}\quad z_2\leq -M.
\end{aligned} \tag{6.1}$$

These conditions, which also yield (4.12) seem to presume an unrealistic level of uncoupling.

We can handle problems for anisotropic materials simply by adopting our critical assumptions (2.8) and (2.9), but omitting the intermediate terms introduced by the identity signs. Clearly these forms of the assumptions are motivated by those for isotropic materials.

If the reasonable assumption (3.23) does not hold, we could nevertheless obtain the conclusion of Theorem 3.25 by increasing the dependence of the dissipative terms on the strains, along the lines of [17].

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