Optimal control of a nonlinearly viscoelastic rod

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Abstract

We consider some typical optimal control problems for a nonlinear model of longitudinal vibrations in a viscoelastic rod. In trying to follow the usual pattern of showing that every infimizing sequence of controls contains a subsequence suitably converging to an optimal control, we confront the severe technical difficulty that the constitutive function cannot be uniformly Lipschitzian in its arguments — e.g., it blows up at a 'total compression.' One needs to make careful use of the structure of the system to overcome this difficulty.

1. Introduction

We consider a PDE model for the longitudinal motion of a uniform[1] viscoelastic rod:

$$w_{tt} = \nu_s + f, \qquad \nu = n\,(w_s, w_{st}) \tag{1.1}$$

[1]This uniformity is purely for expository simplicity. There would be no difficulty in permitting the density ρ, here normalized to 1, and the constitutive function $n(\cdot\cdot)$ to depend explicitly (piecewise continuously) on s as well.

holding on $\mathcal{Q} := (0,\ell)\times(0,T)$. Here $w = w(s,t)$ is the position at time $t \in [0,T]$ of the material point with reference position $s \in (0,\ell)$ so w_s gives the *strain* and w_{st} is the *strain rate*; $\nu = \nu(s,t)$ is then the *contact force* (given by the constitutive function n) and $f = f(s,t)$ is the external *body force*. One natural set of boundary conditions for this problem consists of the specification of the contact forces at the endpoints. It is plausible to consider either the body force f as a distributed control or the contact force $\hat{\nu}$ at $s = \ell$ as a boundary control. We take homogeneous boundary conditions at one end, for simplicity, so that

$$\nu\Big|_{s=0} \equiv 0, \qquad \nu\Big|_{s=\ell} = \hat{\nu}(t). \tag{1.2}$$

It is convenient to introduce $u := w_s,\ v := w_t$ and then to rewrite (1.1) in the form

$$\begin{array}{lrl} i) & u_t & = v_s, \\ ii) & \langle \omega, v_t\rangle + \langle \omega_s, n(u,u_t)\rangle & = \omega(\ell)\hat{\nu} + \langle \omega, f\rangle, \end{array} \tag{1.3}$$

with (1.3-*ii*) holding for $t \in (0,T)$ and for all suitable test functions $\omega \in H^1(0,\ell)$. Here and below we use $\langle\cdot,\cdot\rangle$ to denote the $L^2(0,\ell)$ inner product and comparable duality products.

This model was considered in [1] and [3] (and is generalized in the forthcoming paper [2] to a full vector model which considers transverse motion, shear, and torsion as well as longitudinal motion). Suitable assumptions there on the *constitutive function* $n(\cdot\cdot)$ (permitting fully nonlinear dependence on the strain rate) ensure, for suitable data, both well-posedness and the preclusion of 'total compression', i.e., $u = w_s$ is pointwise bounded away from 0. In Section 3 we adapt those hypotheses, with particular attention to weakening the conditions imposed on the data $f, \hat{\nu}$ which we take as possible controls.

Our primary objective in this paper is to prove the existence of optimal controls for three closely related and reasonably typical optimal control problems:

PROBLEM 1: Taking the boundary condition $\hat{\nu}$ (contact force at: $s = \ell$) as control, track a target trajectory $\bar{w}(\cdot\cdot)$ over $[0,T]$,

PROBLEM 2: Taking the external force f as control, approximate a target state $[\bar{w}, \bar{w}_t]$ at the fixed time $T > 0$,

PROBLEM 3: Again with boundary control $\hat{\nu}$, attain a target state in minimum time, subject to constraints.

These problems are closely related here in that essentially the same *a priori* estimates suffice to provide the compactness needed for the arguments. Note that in order to take limits through the nonlinearity of the constitutive relation, the compactness obtained must necessarily involve pointwise bounds and convergence for u, u_t, since we impose no growth condition restricting the dependence of $n(y,z)$ on z and since $n(y,z)$ must blow up as $y \to 0$ to penalize 'total compression'. This lack of regularity in the constitutive function is a principal technical difficulty for our analysis.

2. Compactness and optimal control

Perhaps the principal technical point of this paper is that the same estimates and compactness needed to obtain existence (and well-posedness) for the direct problem (with $f, \hat{\nu}$ specified) also suffice to give existence of optimal controls for the particular problems we consider.

We do not attempt to seek especially weak conditions on the initial data but we impose, as sufficient for our arguments, the requirement that the initial data satisfy:

$$\overset{\circ}{w}, \overset{\circ}{w}_t \in H^2(0,\ell) \quad \text{with} \quad \overset{\circ}{w}(s) > \bar{c} > 0. \tag{2.1}$$

On the other hand, we weaken slightly the assumption (used in [1], e.g.,) that $f_t, \hat{\nu}_t$ have L^2 bounds. Let us define

$$\mathcal{U} := W^{1,p}\left([0,T] \to L^2(0,\ell)\right), \quad \mathcal{V} := W^{1,4/3}(0,T). \tag{2.2}$$

where p is an arbitrary fixed number with $p > 1$. [Note that $\mathcal{U} \subset C\left([0,T] \to L^2(0,\ell)\right)$ with a fixed modulus of continuity (depending on the choice of p) and that $\mathcal{V} \subset C[0,T]$ with compact embedding.] Our key hypothesis on the cost functionals considered for optimization is an appropriate coercivity condition:

> *f is bounded in $\mathcal{U}$ and $\hat{\nu}$ is bounded in $\mathcal{V}$ (at least when the cost functional is bounded)*

While deferring the detailed proofs, which constitute the next two sections, we now assert two lemmas which will be fundamental to our arguments:

LEMMA 1: *Under the hypotheses of Section 3, the equation (1.1), equivalently (1.3), has a unique solution corresponding to any choice of data $[f, \hat{\nu}]$ in $\mathcal{U} \times \mathcal{V}$ and any choice of initial data satisfying (2.1).*

LEMMA 2: *Under the hypotheses of Section 3, the solutions of (1.1), (1.2) corresponding to $[f, \hat{\nu}]$ bounded in $\mathcal{U} \times \mathcal{V}$ for fixed or suitably bounded initial data as in (2.1) all lie in a fixed compact set: In particular, there is a compact $\mathcal{K}$ such that*

$$u_t = v_s \in \mathcal{K} \subset C(\mathcal{Q}) \tag{2.3}$$

and, of course, w, $u = w_s$, and $v = w_t$ also lie similarly in compact subsets of $C(\mathcal{Q})$. Further, u is uniformly bounded away from 0 and $v_t = w_{tt}$ is bounded in $L^\infty((0,T) \to L^2(0,\ell))$.

We now show how these lemmas may be used to prove the existence of optimal controls for the problems under consideration, beginning with the following corollary.

COROLLARY 3: *Suppose that $\{f^k, \hat{\nu}^k\}$ are bounded in the sense of the coercivity condition and converge weakly in $L^2(\mathcal{Q}), \times L^2(0,T)$: $f^k \rightharpoonup \bar{f}, \hat{\nu}^k \rightharpoonup \bar{\nu}$.*

Then the corresponding solutions converge on $\mathcal{Q}$ (i.e., $w^k, u^k = [w^k]_s, v^k = w_t^k, u_t^k = v_s^k = w_{st}^k$ converge uniformly and $v_t^k = w_{tt}^k$ converges weakly in $L^2(\mathcal{Q})$) to the solution corresponding to the limit data $\bar{f}, \bar{\nu}$.

PROOF: By Lemma 1 there are indeed corresponding solutions w^k and by Lemma 2 there is a subsequence for which these corresponding solutions converge uniformly with $v_t^k = w_{tt}^k$ converging weakly in $L^2(\mathcal{Q})$. Since u^k, u_t^k are bounded with u^k bounded away from 0 by Lemma 2, the corresponding functions $\nu^k = n(u^k, u_t^k)$ then also converge uniformly to the appropriate limit; we note, in particular, that the limit boundary condition is then necessarily satisfied. With smooth test functions we easily see that the limit of the solutions satisfies (1.3-*ii*) with the limit data, as desired, and such ω are dense. Since this solution is unique by Lemma 1, we have convergence for the full sequence. ∎

PROBLEM 1: Suppose *that* $f \in \mathcal{U}$ and initial data as in (2.1) are already specified. We may then consider the determination of the boundary contact force $\hat{\nu}(\cdot)$ in (1.1), (1.2) as an optimal control problem once we have specified a cost functional $\mathcal{J}$ for optimization. We take $\mathcal{J} = \mathcal{J}_1(\hat{\nu}) + \mathcal{J}_2(w)$ (subject to (1.1), (1.2)) with, for example,

$$\mathcal{J}_1(\hat{\nu}) := \max_{[0,T]}\{|\hat{\nu}(t)|\} + a\int_0^T |\hat{\nu}_t(t)|^{4/3}\, dt \tag{2.4}$$

and, tracking a target trajectory $\bar{w}$,

$$\begin{aligned} \mathcal{J}_2(w) \quad &:= b\sup\nolimits_{\mathcal{Q}}\{|w_{tt}|\} + \chi_*(|w-\bar{w}|/c) \\ &\qquad \text{where } \chi_*(\omega) := \begin{cases} 0 & \text{if } \omega \le 1 \text{ on } \mathcal{Q}, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

Thus, we wish to control the rod 'gently' (so that $\hat{\nu}$ stays small and does not change too abruptly) and to keep the rod from accelerating too violently (so that w_{tt} stays small) while demanding that the rod match the specified target trajectory $\bar{w} = \bar{w}(s,t)$ to within a tolerance c. We do not expect that $\mathcal{J}$ is finite for arbitrary boundary data $\hat{\nu} \in \mathcal{V}$, but assume *a priori* — presumably as a condition on the target $\bar{w}$ under consideration — that the set $\mathcal{V}_{\text{ad}}$ of admissible controls is nonempty: there is at least one $\hat{\nu} \in \mathcal{V}$ for which (1.1), (1.2) gives a solution w with w_{tt} bounded and $|w - \bar{w}| \le c$ everywhere on $\mathcal{Q}$.

THEOREM 1: *With $f, \bar{w}$ and the initial data as above and under the hypotheses on the constitutive function $n(\cdot\cdot)$ of Section 3, there is an optimal control $\bar{\nu}$ for Problem 1, i.e., the cost functional $\mathcal{J}$ given by (2.4) , (2.5) attains its minimum.*

PROOF: Given Lemmas 1, 2 and Corollary 3, the argument has a fairly standard pattern. Let $(\hat{\nu}^k)$ be an infimizing sequence for $\mathcal{J}$ with corresponding solutions (w^k) so

$$\mathcal{J}^k := \mathcal{J}_1(\hat{\nu}^k) + \mathcal{J}_2(w^k) \searrow \mathcal{J}_* := \inf\{\mathcal{J}\}$$

Note that the assumption $\mathcal{V}_{\rm ad} \neq \emptyset$ means that $\mathcal{J}_* < \infty$ and we can assume that $\mathcal{J}^k <$[bound]iy for each k. Since this bounds $\{\hat{\nu}^k\}$ in $\mathcal{V}$, we may assume, without loss of generality, that $(\hat{\nu}^k)$ converges uniformly on $[0,T]$ with weak convergence in $L^{4/3}(0,T)$ of $(\hat{\nu}^k_t)$, i.e., $\hat{\nu}^k \rightharpoonup \bar{\nu}$, $\hat{\nu}^k_t \rightharpoonup \bar{\nu}_t$. Since $\mathcal{J}_1(\cdot)$ is lower semicontinuous in this topology, we have $\mathcal{J}_1(\bar{\nu}) \leq \liminf \mathcal{J}_1(\hat{\nu}^k)$.

Next, using Corollary 3, we have convergence of (w^k) to the solution corresponding to this control $\bar{\nu}$. Since $\chi_* = 0$ for each w^k, the uniform convergence ensures that this also holds in the limit. We only have weak $L^2(\mathcal{Q})$ convergence for the accelerations w^k_{tt}, but we note that $\omega \mapsto \sup_{\mathcal{Q}}\{|\omega|\}$ is lower semicontinuous with respect to the weak L^2 topology (since $\{\omega \in L^2(\mathcal{Q}) : |\omega| \leq \alpha\}$ is convex and strongly closed for each α). Thus, in the limit we have $\mathcal{J}_2 \leq \liminf \mathcal{J}_2(w^k)$ so $\mathcal{J} \leq \mathcal{J}_*$ and the minimum is attained at $\bar{\nu}$. ■

Some characterization of this optimal control $\bar{\nu}$ through (formal) computation of first-order optimality conditions (expressed in terms of a linear adjoint equation) would certainly be possible, if rather messy for the particular cost functional we have treated here, but we do not pursue this.

PROBLEM 2: Now suppose that $\hat{\nu}$ and the initial data are specified and that we seek an optimal distributed control $f \in \mathcal{U}$. We consider two variants of this problem: we may insist on minimizing the $\mathcal{U}$-norm while exactly matching the target state at $t = T$ or we may penalize deviation from the target as measured in some norm. Thus, we either consider

$$\mathcal{J} = \mathcal{J}_1(f) := \|f\|_{\mathcal{U}} \quad \text{subject to: } [w, w_t]\Big|_{t=T} = [\bar{w}, \bar{w}_t] \tag{2.6}$$

or, for example, topologizing deviation in $C^2[0,\ell] \times C^1[0,\ell]$ for the second variant, we may take $\mathcal{J} = \mathcal{J}_1(f) + \mathcal{J}_2(w)$ with $\mathcal{J}_1(\cdot)$ as in (2.6) and

$$\begin{aligned}\mathcal{J}_2(w) := \sup\{ \quad & |w(s,T) - \bar{w}(s)|,\ |w_{ss}(s,T) - \bar{w}''(s)|, \\ & |w_t(s,T) - \bar{w}_t(s)|,\ |w_{ts}(s,T) - [\bar{w}_t]'(s)| : s \in [0,\ell]\}.\end{aligned} \tag{2.7}$$

[Note that (2.6) is equivalent to introducing $\mathcal{J}_2(w) := 0$ if there is such a match, and $\mathcal{J}_2(w) := +\infty$ otherwise.] For neither variant do we expect $\mathcal{J}$ to be finite for arbitrary $f \in \mathcal{U}$ (for (2.7), because our estimates do not bound w_{ss} at all), but we do assume *a priori* that the admissible control set $\mathcal{U}_{\rm ad} := \{f \in \mathcal{U} : \mathcal{J} < \infty\}$ is nonempty for whichever variant is under consideration. This restricts our choice of target states to consider and, in particular, for the first variant it means requiring that the target be exactly reachable.

THEOREM 2: *With $\hat{\nu}, \bar{w}$ and the initial data as above and under the hypotheses on the constitutive function $n(\cdot\cdot)$ of Section 3, there is an optimal control $\bar{\nu}$ for Problem 2, i.e., the cost functional $\mathcal{J}$, given by (2.6) or (2.7) as appropriate, attains its minimum.*

PROOF: The proof is similar enough to that given for Theorem 1 that we only comment on it briefly. We can now begin by finding an infimizing sequence (f^k) for which $f_k \rightharpoonup \bar{f}$ in $L^2(\mathcal{Q})$ and $f_t^k \rightharpoonup \bar{f}_t$ in $L^p\left([0,T] \to L^2(0,\ell)\right)$, the topology for which $\mathcal{J}_1(\cdot)$ is both coercive and lower semicontinuous. As before, we now extract, if necessary, a subsequence for which Lemma 2 gives convergence of the corresponding solutions (w^k). For the first variant we then need only note that Corollary 3 ensures that the terminal condition in (2.6) holds in the limit since it holds for each w^k. For (2.7) we note, much as in the proof of Theorem 1, that the convexity of the set $\{w : \mathcal{J}_2(w) \le \alpha\}$ for each α gives the needed lower semicontinuity of $\mathcal{J}_2(\cdot)$. In any case, we have $\mathcal{J} \le \mathcal{J}_*$ in the limit and the minimum is attained at $\bar{f}$. ■

PROBLEM 3: Finally, we suppose that $f \in \mathcal{U}$ and the initial data (subject to (2.1)) are specified and that we have also specified a target state $[\bar{w}, \bar{w}_t] \in C[0,\ell] \times C[0,\ell]$ and constraints. These are to include (the possibility of) both *control constraints* (for which the boundary control $\hat{\nu}$ lies in a specified subset $\mathcal{V}_{\text{ad}} \subset \mathcal{V}$) and *state constraints* (for which the trajectory $t \mapsto w$ determined by using this $\hat{\nu}$ in (1.2) lies in a specified subset $\mathcal{W} \subset C(\bar{\mathcal{Q}})$). While one could consider more general examples, we assume, for expository simplicity, that $\mathcal{V}_{\text{ad}}$ is convex, closed, and bounded in $\mathcal{V}$ and that W is defined (almost) pointwise so that

$$\mathcal{W} = \{w(\cdot) : [w, w_t]\Big|_t \in \hat{\mathcal{W}}(t) \text{ for } 0 \le t \le \tau = \tau(\hat{\nu})\} \tag{2.8}$$

where each $\hat{\mathcal{W}}(t)$ is closed in $[C^1(0,\ell)]^2$. We also ask, of course, that there be some control $\hat{\nu} \in \mathcal{V}_{\text{ad}}$ for which $w(\cdot)$ is not only in$\mathcal{W}$, but also matches the target, i.e.,

$$[w, w_t]\Big|_{t=\tau(\hat{\nu})} = [\bar{w}, \bar{w}_t] \tag{2.9}$$

Our goal is to minimize the control time τ for this match.

THEOREM 3: *With $f, \bar{w}$, the initial data, and the constraint sets $\mathcal{V}_{\text{ad}}, \mathcal{W}$ as above and under the hypotheses on the constitutive function $n(\cdot\cdot)$ of Section 3, there is an optimal control $\bar{\nu}$ for Problem 3, i.e., there is a control for which the corresponding solution reaches the target state at the minimum control time τ_*.*

PROOF: As before, we consider an infimizing sequence $(\hat{\nu}^k)$ with corresponding solutions w^k for which

$$\tau^k = \tau(\hat{\nu}^k) \searrow \tau_* := \inf\{\tau(\hat{\nu}) : \hat{\nu} \text{ is admissible}\}$$

where 'admissible' means that $\hat{\nu} \in \mathcal{V}_{\text{ad}}$, $w(\cdot) \in \mathcal{W}$, and (2.9). Without loss of generality, since $\mathcal{V}_{\text{ad}}$ is bounded, we may assume that $(\hat{\nu}^k)$ is weakly convergent: $\hat{\nu}^k \rightharpoonup \bar{\nu}$ and our assumptions on $\mathcal{V}_{\text{ad}}$ ensure that we also have $\bar{\nu} \in \mathcal{V}_{\text{ad}}$. The admissibility of each $\hat{\nu}^k$ ensures that $[w^k, w_t^k]\Big|_t \in \hat{\mathcal{W}}(t)$ for $0 \le t \le \tau_*$ since that $\tau^k \ge \tau_*$. Since Lemma 2 ensures (for a subsequence, to which we now restrict our attention) that w^k and $u^k = w_s^k$ converge uniformly on $\bar{Q}$, we have $w^k(\cdot,t)$ converging in $C^1(0,\ell)$ and, similarly we get uniform convergence of $v^k = w_t^k$ and of w_{ts}^k, so that $w_t^k(\cdot,t)$ converges in $C^1(0,\ell)$. Hence, the limit solution w^*, whose existence is given by Corollary 3, also satisfies the defining condition of $\mathcal{W}$ on $[0,\tau_*]$ and so will be in $\mathcal{W}$, as desired, once we show that w^* satisfies (2.9) at $t=\tau_*$, making $\tau(\bar{\nu})=\tau_*$. To verify (2.9), we need only note that

$$[w^k, w_t^k]\Big|_{\tau^k} \longrightarrow [w^*, w_t^*]\Big|_{\tau_*}$$

is ensured by the uniform convergence provided by Lemma 2. ■

3. Hypotheses and first estimates

In this section and the next we provide the estimates giving the compactness of Lemma 2 and underlying the existence proof (which we will not present here) of Lemma 1.

We split the constitutive function $n(y,z)$ for the contact force into an elastic part $n(y,0) =: \varphi'(y)$ and a dissipative part $\sigma(y,z)$ (with $\sigma(y,0)\equiv 0$):

$$n(y,z) = \varphi'(y) + \sigma(y,z) \qquad \text{for } y>0 \text{ and all } z. \tag{3.1}$$

We assume that φ,σ are smooth where defined, e.g., φ',σ are C^1.

We impose two hypotheses on the constitutive function $n(\cdot\cdot)$, i.e., on φ,σ: the first is a rather natural assumption that there be some minimal dissipation while the second, formulated in terms of an auxiliary function ψ, is rather technical. Our first hypothesis is that

$$\sigma_z(y,z) \geq \mu \qquad \text{(for some fixed } \mu>0). \tag{3.2}$$

Note that this gives $\mu|z|^2 \leq z\,\sigma(y,z)$. Next, we introduce $\psi : (0,\infty) \to \mathbb{R}_+$, requiring that $\psi(y)\to\infty$ as $y\to 0$ to enforce our prohibition against total compression, and then impose our second hypothesis:

(a) For each $c>0$ there is a constant $\lambda=\lambda(c)$ such that

$$[z\,n_y(y,z)]^2 \leq \lambda\sigma_z(y,z)\,[1 + z\,\sigma(y,z) + \varphi(y)] \ \text{ when } \psi(y)\leq c \tag{3.3}$$

while

(b) There are $\beta,\bar{c}>0$ such that

$$z\,\psi'(y) \leq n(y,z) + \beta \ \text{ when } \psi(y) > \bar{c}. \tag{3.4}$$

We include in this section some estimation which requires only L^2 bounds for $f,\hat{\nu}$, beginning with the usual energy estimate. Taking $\omega = v$ in (1.3-ii) and noting (2.1)) gives

$$\begin{aligned} &\tfrac{1}{2}\|v\|^2 + \langle\varphi(u)\rangle + \int_0^t \langle v_s\sigma(u,v_s)\rangle \\ &\quad = \tfrac{1}{2}\|\mathring{v}\|^2 + \langle\varphi(\mathring{u})\rangle + \textstyle\int_0^t v(\ell,\cdot)\hat{\nu} + \int_0^t\langle v,f\rangle \end{aligned} \tag{3.5}$$

where $\langle\varphi(u)\rangle = \int_0^\ell \varphi(u(s,t))\,ds$, etc. We have used (3.1) and the fact that $v_s\varphi'(u) = [\varphi(u)]_t$ since $v_s = u_t$. Estimating $|v(\ell,t)|^2 \le \varepsilon\|v_s\|^2 + C_\varepsilon\|v\|^2$ and recalling that $\mu|v_s|^2 \le v_s\sigma(u,v_s)$ so we may choose $\varepsilon > 0$ small and absorb this term $\varepsilon\|v_s\|^2$ on the left. We apply the Gronwall Inequality to obtain

$$\|v\|^2, \langle\varphi(u)\rangle, \int_0^t \langle v_s\sigma\rangle \le C\left(1 + \|\hat{\nu}\|^2 + \|f\|^2\right) \le C \tag{3.6}$$

with C denoting a positive constant depending only on T, μ and bounds on the data, as indicated.

Next we show that our hypotheses, as hoped, ensure the impossibility of total compression: we will bound u away from 0. We first integrate (1.1) over $(0,\bar{s})$ to obtain

$$n(u,u_t)\Big|_{(\bar{s},t)} =: \nu(\bar{s},t) = \int_0^{\bar{s}} [v_t - f]\,ds. \tag{3.7}$$

We may assume in (3.4) that $\bar{c} > \max_s\{\psi(\mathring{u})\}$ so if $\psi(u(\bar{s},\bar{t})) > \bar{c}$, then there is some $\tau > 0$ with $\psi(u(\bar{s},\tau)) = \bar{c}$ and $\psi(u(\bar{s},\cdot)) > \bar{c}$ on $(\tau,\bar{t})$. Thus (3.4) applies to give $u_t\psi'(u) \le \beta + n(u,u_t)$ there, and use (3.7); integrating in t over $(\tau,\bar{t})$ then gives

$$\psi(u(\bar{s},\bar{t})) \le \bar{c} + (\bar{t}-\tau)\beta + \left[\int_0^{\bar{s}} v\right]\Big|_\tau^{\bar{t}} - \int_\tau^{\bar{t}}\int_0^{\bar{s}} f.$$

The terms on the right can all be estimated, by using an $L^2(\mathcal{Q})$ bound on f for the last and (3.6) for the penultimate term; thus, $\psi(u) \le C$ and we will later be able to apply (3.3). Further, since $\psi(y) \to \infty$ as $y \to 0$, this shows that $u = w_t$ is uniformly bounded away from 0.

4. Further estimates and compactness

We continue our estimation, now assuming $\mathcal{U}, \mathcal{V}$ bounds for $f, \hat{\nu}$ and using (3.3) with the appropriate λ.

We begin by setting $\zeta := v_t = w_{tt}$. Note, first, that (2.1) gives $\mathring{u} = \mathring{w}_s$ in $H^1(0,\ell)$, so that $\mathring{u}$ is continuous (and bounded pointwise on $[0,\ell]$) as well as bounded away from 0; similarly, $u_t = [\mathring{w}_t]_s$ is bounded. These bounds ensure that the arguments remain in a compact set for which we have bounds on n_y and $n_z = \sigma_z$. This will enable us to verify the regularity of $\mathring{\zeta} := \zeta(\cdot,0)$. Using (1.1) at $t = 0$, we have

$$\mathring{\zeta} = [n(\mathring{u},\mathring{v}_s)]_s + f(\cdot,0) = n_y\,\mathring{w}_{ss} + \sigma_z[\mathring{w}_t]_{ss} + f$$

which, with n_y, σ_z bounded, is in $L^2(0,\ell)$.

Differentiating (1.3-ii) in t one obtains

$$\langle\omega,\zeta_t\rangle + \langle\omega_s,\sigma_z\zeta_s\rangle = \omega(\ell)\hat{\nu}_t + \langle\omega,f_t\rangle - \langle\omega_s,n_y u_t\rangle.$$

Taking $\omega = \zeta$ and integrating over $(0,t)$ then gives

$$\begin{aligned}\tfrac{1}{2}\|\zeta(t)\|^2 + \textstyle\int_0^t \langle \zeta_s, \sigma_z \zeta_s\rangle \\ \le \tfrac{1}{2}\| \mathring{\zeta} \|^2 + C\left[\textstyle\int_0^t |\zeta(\ell,\cdot)|^4\right]^{1/4} + C\bar{z}(t) \\ + \textstyle\int_0^t \left[\varepsilon\langle\zeta_s,\sigma_z\zeta_s\rangle + \tfrac{\lambda}{4}\varepsilon\left(1 + \langle v_s\sigma(u,v_s)\rangle + \langle\varphi(u)\rangle\right)\right]\end{aligned} \tag{4.1}$$

where we have set $\bar{z}(t) := \max\{\|\zeta(\bar{t})\| : 0 \le \bar{t} \le t\}$, have applied (3.3) after noting that

$$|\zeta_s n_y u_t| \le \varepsilon\sigma_z(\zeta_s)^2 + (1/4\varepsilon\sigma_z)|n_y u_t|^2,$$

have used the relevant bounds on $f_t, \hat{\nu}_t$, and have noted that the final set of terms were already bounded by (3.6). Since $|\zeta(\ell,t)|^2 \le \left[(1/\ell)\|\zeta\|^2 + 2\bar{z}\|\zeta_s\|\right]$ we have

$$\left[\int_0^t |\zeta(\ell,\cdot)|^4\right]^{1/4} \le \varepsilon\|\zeta(t)\|^2 + \varepsilon\|\zeta_s\|^2 + C_\varepsilon\left[1 + \bar{z}(t)\right].$$

Using this in (4.1) with ε chosen to absorb those terms on the left then gives

$$\|z(t)\|^2 \le C[1 + \bar{z}(t)] \le C[1 + \bar{z}(\bar{t})]$$

with $\bar{t}$ arbitrarily fixed so $t \le \bar{t}$. Taking the $\max_t$ over $[0, tb]$ then gives $\bar{z}^2 \le C[1 + \bar{z}]$ (uniformly for $\bar{t} \in [0,T]$) and we have bounded[2] $\zeta = v_t = w_{tt}$ in $L^\infty((0,T) \to L^2(0,\ell))$.

In view of (3.7), this gives us also a uniform pointwise bound for ν. Returning to (4.1) we see that we have also bounded the integral on the left and so have bounded $\zeta_s = w_{stt} = u_{tt}$ in $L^2(\mathcal{Q})$; integrating this over $(0,t)$, using (2.1), bounds $u_t = v_s = w_{st}$ in $L^\infty((0,T) \to L^2(0,\ell))$. Summarizing, we have by now shown

$$\|w_{tt}(\cdot,t)\| \le C, \qquad |\nu(s,t)| \le C, \qquad \|w_{stt}\|_{\mathcal{Q}} \le C, \qquad \|w_{st}(\cdot,t)\| \le C \tag{4.2}$$

as well as (3.6) and the bound away from 0 obtained for $u = w_s$.

Now consider $\varphi'(u) + \sigma(u,u_t) =: n(u,u_t) = \nu(\bar{s},\cdot)$ with $s = \bar{s}$ arbitrarily fixed in $(0,\ell)$. Multiplying this by u_t and integrating the product over $[0,T]$ gives, using the bound on ν,

$$\mu\|u_t\|^2_{[0,T]} \le \varphi(u) + \int_0^T u_t\sigma(u,u_t)\,dt = \int_0^T u_t\nu\,dt \le C\|u_t\|_{[0,T]}$$

which bounds $u_t = v_s = w_{st}$ in $L^2(0,T)$ on each $s = \bar{s}$. Integrating over $(0,t)$ and noting (2.1) then uniformly bounds u pointwise on $\mathcal{Q}$ and so, with our previous bound away from 0, restricts the argument of $\varphi(\cdot)$ to a compact subinterval of its domain, whence $\varphi(u), \varphi'(u)$ are necessarily also bounded. Now we apply $|z| \le |\sigma(y,z)|/\mu$ to bound u_t pointwise, noting that $\sigma(u,u_t) = \nu - \varphi'(u)$ has been bounded.

[2]This and a density argument show that ζ is actually in $C([0,T] \to L^2(0,\ell))$.

This, of course, only bounds $u = w_s$ and $u_t = v_s = w_{st}$ in $L^\infty(\mathcal{Q})$ and, although a density argument would suffice to show that these remain within the closed subspace $C(\bar{\mathcal{Q}})$, we wish to show that they lie in a compact subset of that. [Our bound on f_t is in $L^p\left([0,T] \to L^2(0,\ell)\right)$, by (2.2), but we note that it would have been sufficient, up to this point, only to have required a bound on f_t in the dual space of $C([0,T] \to L^2(0,\ell))$.]

We prepare for the next step by recalling a compactness lemma appearing in a footnote in [4] (note also [5]):

LEMMA 4: *Let $\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ with the embedding $\mathcal{X} \hookrightarrow \mathcal{Y}$ compact. Suppose that $\mathcal{F}$ is a set of functions on $[0,T]$ which is bounded in $L^\infty([0,T] \to \mathcal{X})$ and for which $\{\psi_t : \psi \in \mathcal{F}\}$ is bounded in $L^p([0,T] \to \mathcal{Z})$ for some $p > 1$. Then $\mathcal{F}$ is a precompact subset of $C([0,T] \to \mathcal{Y})$.*

We now apply this to $\mathcal{F} = \{\nu\}$ with $\mathcal{X} = H^1(0,\ell)$, $\mathcal{Y} = C[0,\ell]$, $\mathcal{Z} = L^2(0,\ell)$, which satisfy the embedding requirements. From (3.7) we already have $\nu_s = \zeta - f$ bounded in $L^2(0,\ell)$ so ν is bounded in $H^1(0,\ell)$. On the other hand,

$$\nu_t = [n(u,u_t)]_t = n_y u_t + \sigma_z \zeta_s \tag{4.3}$$

on noting that $u_{tt} = v_{st} = \zeta_s$. At this point, since the arguments of $n(u,u_t)$ have been restricted to a fixed compact subset of the domain $(0,\infty) \times \mathbb{R}$ we have uniform bounds on the derivatives n_y, $n_z = \sigma_z$. As we already have bounds on u_t in $L^\infty(\mathcal{Q})$ and on ζ_s in $L^2(\mathcal{Q})$, the right hand side of (4.3) is bounded in $L^2([0,T] \to \mathcal{Z})$ as desired and we have shown that ν lies in a fixed compact subset of $C([0,T] \to \mathcal{Y}) = C(\bar{\mathcal{Q}})$.

Next we notice that the ordinary differential equation: $n(u,u_t) = \nu$ (with specified initial data $\mathring{u} \in C[0,\ell]$) defines a continuous solution map[3]

$$\mathbf{S} : \nu \mapsto u_t : C(\bar{\mathcal{Q}}) \to C(\bar{\mathcal{Q}})$$

and the set $\mathcal{K}$ of (2.3) is the image under $\mathbf{S}$ of the compact subset of $C(\bar{\mathcal{Q}})$ obtained to contain ν. The compactness of $\mathcal{K}$ follows immediately, of course, from the continuity of $\mathbf{S}$.

This effectively completes the proof of Lemma 2. The estimates here provide the core of the existence proof but we omit the actual proof, referring the reader to [1] for further detail. ■

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[3]Since $n(y,z)$ is not globally smooth and is not defined on all of $\mathbb{R}^2$ (which has forced us to use the restriction of the arguments which we have obtained), one must redefine $\mathbf{S}$. E.g., suppose the rectangle $\mathcal{D}$ is the compact subset of $\mathbb{R}^2$ to which we have restricted values of $[u,u_t]$. We redefine $n(y,z)$ when $|z|$ is too large for $\mathcal{D}$ so as to be smooth (still), uniformly Lipschitzian in z, and still satisfying $n_z \geq \mu$; then further redefine n (now for y outside the relevant range) to coincide with the values for the nearest admissible y-value. There is then no obstruction to the global (re)definition of $\mathbf{S}$ and we observe that the results must coincide with the original results for all relevant inputs so the redefinition is nugatory.

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