QUASILINEAR HYPERBOLIC–PARABOLIC EQUATIONS OF ONE–DIMENSIONAL VISCOELASTICITY

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This paper is dedicated to Jack Hale on the occasion of his sixty-fifth birthday

ABSTRACT. We study the global existence of solutions of initial-boundary-value problems for a quasilinear hyperbolic-parabolic equation describing the longitudinal motion of a one-dimensional viscoelastic rod. We treat a variety of nonhomogeneous boundary conditions, requiring separate analyses, because they lead to distinctive physical effects. We employ a constitutive equation giving the stress as a general nonlinear function of the strain and the strain rate. All global analyses of this and related problems, except that of Dafermos [**13**], have employed a stress that is merely affine in the strain rate. Dafermos's assumptions are far more appropriate for shearing motions than for longitudinal motions. Our constitutive equation satisfies the physically natural requirement that an infinite amount of compressive stress is needed to produce a total compression at any point of the rod. This requirement is the source of a severe singularity in the governing partial differential equations, which is particularly acute when time-dependent Dirichlet data are prescribed. The further novel, yet physically reasonable, restrictions we impose on the constitutive function yield estimates that preclude a total compression anywhere at any finite time. The resulting estimates are crucial for the global existence theory we obtain.

1. INTRODUCTION

In this paper we study the global existence of solutions of initial-boundary-value problems for the following quasilinear hyperbolic-parabolic equation describing one-dimensional longitudinal motions of a class of viscoelastic bodies:

$$\rho(s)w_{tt} = n(s, w_s, w_{st})_s + f(s,t). \tag{1.1}$$

The unknown $w(s,t)$ denotes the position at time t of the material point whose coordinate in a natural reference state is $s \in [0,1]$. Thus $w_s(s,t)$ is the *stretch* at (s,t), i.e., the local ratio of deformed to reference length at (s,t). $\rho(s)$ denotes the prescribed mass density per unit reference length at s, $n(s, w_s(s,t), w_{st}(s,t))$

denotes the stress resultant (contact force) experienced by the material point s at time t, and $f(s,t)$ denotes the prescribed body force per unit of reference length at (s,t).

If the forces applied to the rod are well behaved, we would expect the stretch to be everywhere positive, except possibly on a set of measure zero. We incorporate this intuitive restriction into our analysis by requiring that $n(x,y,z) \to -\infty$ as $y \to 0$, which says that an infinite amount of compressive force is needed to effect a total compression at any point of the body. This requirement causes (1.1) to be singular. To handle this singularity without compromising the generality inherent in the nonlinear constitutive function n, we introduce a number of new restrictions on n, which enable us to obtain estimates ensuring that total compression cannot occur at any finite time. These restrictions lie at the heart of our analysis: They must be both physically reasonable and mathematically useful. It is perhaps surprising that when (time-dependent) Dirichlet data are prescribed at each end of the rod, we require stronger constitutive restrictions and a more delicate analysis to obtain the requisite bounds. The estimates we obtain support the global existence and regularity theory given in Sections 8 and 9. We may regard parts of our work as exploring the kinds of dissipative mechanisms that are strong enough to smooth out certain kinds of shocks associated with total compression.

Under restrictions on the constitutive function n, different from those we shall impose and under different interpretations of w, equation (1.1) can also model the one-dimensional motion of a compressible viscous gas and the shearing motion of a viscoelastic solid. (See Fig. 1.2.) Versions of this equation and relatives of it have been intensively studied over the past twenty years. To appreciate the mathematical status of our work, we describe a few of the most pertinent analyses of special cases of (1.1).

The analysis of (1.1) was initiated by Greenberg, MacCamy, and Mizel [**21**], who studied functions n affine in their last arguments:

$$n(x,y,z) = f(y) + mz \tag{1.3}$$

where m is a positive constant. Major advances in the theory for this equation were made by Andrews [**2**] and by Andrews and Ball [**3**].

Kanel' [**24**] and MacCamy [**31**] independently developed the theory for problems in which there is a positive number m such that

$$n(x,y,z) = f(y) + g(y)z \quad \text{with} \quad g \geq m. \tag{1.4}$$

Andrews [**1**] (cf. [**3**]) developed a very simple device for proving that w_s is everywhere positive for solutions of (1.1) when (1.4) holds.

The material (Lagrangian) formulation of the one-dimensional Navier-Stokes equations for compressible fluids has the form (1.4), with $g(y) = \mu(y)/y$ where μ is the viscosity, allowed to depend on the specific volume y, which is the reciprocal of the actual density. Now the three-dimensional Navier-Stokes equations for fluids can be generalized by adopting a constitutive equation for non-Newtonian fluids in place of those that are affine in the velocity gradient. It happens that the naive replacement of the affine dependence on the velocity gradient with a nonlinear dependence yields a theory with properties that do not reflect those of a single

Figure 1.2. Typical graphs of the elastic response $y \mapsto n(x, y, 0)$ for (a) gas dynamics, in which $n(x, y, 0)$ is the negative of the pressure corresponding to the specific volume y, (b) shearing of a viscoelastic layer, in which $n(x, y, 0)$ is the shear stress corresponding to the shear strain y, (c) longitudinal motion of a viscoelastic rod. We limit our attention to (c). A detailed interpretation of the variables for this case is given in Section 2. Note that in (a) and (c) the function n has the property that $n(x, y, 0) \searrow -\infty$ as $y \searrow 0$. Our analysis of problems for viscoelasticity differs from that for gas dynamics in that the dependence of n on its last argument is affine for gas dynamics and is far more general here.

known real fluid. Consequently generalizations of (1.4) in which n is allowed to be nonlinear in z, which are the object of our study, are apparently irrelevant for the dynamics of viscous gases. On the other hand, the form of (1.4) is not ideally suited for solids. E.g., on physical grounds one might wish to postulate that n_y be positive everywhere, or at least that n_y be positive in certain large regions. If, however, g is not constant, then there must regions of (y, z)-space where $n_y(x, y, z)$ is not positive. In particular, if $g(y) = 1/y$ or, more generally, if g is a strictly decreasing function of y, then for any $y > 0$, there is a positive z such that $n_y(x, y, z) < 0$, a result that is counter to our intuition.

Dafermos [**13**] studied the fully nonlinear problem (1.1) for functions $(y, z) \mapsto n(y, z)$ having the property that there are positive numbers m and M such that

$$n_z \geq m, \quad (n_y)^2 \leq M n_z \tag{1.5a,b}$$

for all z and for all $y > 0$. The assumption (1.5b) is perfectly suitable for the description of shearing motions of solids, but, as is demonstrated in [**6**], it is incompatible with physically natural restrictions on the longitudinal motions of solids that require that an infinite compressive force accompany a total compression, a mild version of which is

$$n(x, y, 0) \searrow -\infty \quad \text{as} \quad y \searrow 0. \tag{1.6}$$

(Refinements of such conditions form much of the content of Section 3.)

While equations modelled on (1.4) are reasonable for problems describing one-dimensional motions of compressible Navier-Stokes fluids, there seems to be no physical reason to restrict the description of solids to (1.4). Thus it is desirable to generalize (1.4), but in doing so we cannot use (1.5) for longitudinal motions. Since the analyses of **[1,2,3]**, **[13]**, **[21]**, **[24]**, **[31]**, depend crucially on the specific forms of (1.3)–(1.5), we must construct effective alternatives to the procedures of these papers. (In particular the clever change of variables introduced in **[2]** is not available to us.)

The objective of our paper is thus to replace (1.4) and (1.5) with a general class of constitutive laws appropriate for the longitudinal response of viscoelastic rods, to show that in this setting we can obtain a suitable a priori lower bound for w_s, and to use this bound to obtain a global existence and regularity theory. (This bound has a central role in constructing replacements for the estimates that are lost by suspending (1.4) and (1.5).) The class of materials we use was suggested by the study **[4]** of a discrete problem, but our analysis here is of course technically much more challenging than that. Our most primitive hypothesis allows us to exploit the fundamental idea of **[1]** in far more general circumstances. (We replace Andrews' equality (1.4), which he requires to hold everywhere, with an inequality (3.7) that holds only on part of the domain of n.) Our more refined hypotheses, needed to account for Dirichlet data at each end of the rod, bear no resemblance to Andrews'.

There has been an extensive body of work on equations with n of the form (1.3) or (1.4), among which are **[8]**, **[14]**, **[15]**, **[18]**, **[20]**, **[22]**, **[25]**, **[26]**, **[27]**, **[36]**, **[42]**, **[43]**, and **[47]**. For studies of analogous equations in more independent spatial dimensions see **[12]**, **[16]**, **[17]**, **[32]**, **[33]**, **[38]**, and **[48]**. Several of these papers are devoted to the compressible Navier-Stokes equations. Our listing of references is meant merely to be representative of an extensive literature. See **[45]** for a treatment of special n's nonlinear in z and see **[11]** for a study of the motions of a more general system near equilibrium states. Finally, there is a rich literature on n's depending on the past history of w_s, for which **[37]** should be consulted. We are not concerned here with problems in which such memory effects occur.

We formulate our initial-boundary-value problem in Section 2. In Section 3 we give a comprehensive discussion of the various constitutive assumptions that we employ in the rest of the paper. In Section 4 we study energy estimates useful for the subsequent analysis. Here we first encounter the difficulties that are caused by time-dependent Dirichlet conditions: Conditions like (1.6) make it very difficult to get standard estimates of boundary integrals (corresponding to the power of the boundary forces), which in virtually all previous work is zero because of zero boundary data. To compensate for these difficulties, we invoke a more refined set of constitutive restrictions, which are needed to support the more intricate analysis of Section 6. In Section 4 we incidentally give conditions on the elastic response that ensure that a total compression can occur only on a set of spatial measure 0 for each fixed time.

The heart of our paper is in Sections 5–7, where we prove that our new conditions on the viscoelastic response give a positive pointwise lower bound for w_s. This bound supports the existence theory of Sections 8 and 9. In Section 5 we treat the simplest problems (in which there is at least one Neumann boundary condition) under the strongest constitutive assumption, Hypothesis 3.7. The analysis of this section and all that precedes it is completely elementary. The easiest way to get the gist of this paper is thus to read Section 3 only through the discussion of

Hypothesis 3.7, and then to proceed on to Sections 4 and 5. At this stage, the reader can continue with the remaining sections, referring back to Section 3 for discussions of the constitutive hypotheses used there.

In Section 6 an analysis of a system of differential inequalities yields a lower bound for the strain for the difficult problem with Dirichlet conditions at each end. This analysis relies on further constitutive hypotheses. In Section 7, we use a sort of phase-plane analysis to show how to get the requisite lower bound when we sacrifice the analytic virtues of Hypothesis 3.7 for the weaker and physically more natural Hypothesis 3.9. The existence and regularity theory of Sections 8 and 9, relying on these bounds, is based on the Galerkin method. In Section 10 we show that conditions comparable to those used in Section 3 ensure that total compression cannot occur for the hyperbolic equations of nonlinear elasticity.

In the course of our study we give a detailed treatment of distinct boundary conditions, not only because they are physically important, but also because they lead to significant differences in the behavior of solutions. On the other hand, we content ourselves with a reasonable set of regularity assumptions on the data, sufficient to produce solutions with regularity adequate to our needs; we make no effort to produce a scale of the sharpest results.

Notation. We let m and M denote typical positive constants that are supplied as data or that can be estimated in terms of data. Their meanings typically change with each appearance (even in the same equation or inequality. M may be regarded as increasing and m as decreasing with each appearance). Similarly $c(T)$ and $C(T)$ denote typical positive constants depending on the time parameter T. Tacit in an inequality of the form $\|u\| \leq M$ is an assertion that there exists a positive number M such that this estimate holds.

We only use real function spaces. For each nonnegative integer k, $C^k[0,T]$ denotes the space of functions that are k times continuously differentiable on the interval $[0,T]$, $C^{k,\alpha}[0,T]$ denotes the space of functions whose kth derivatives are Hölder continuous with exponent $\alpha \in (0,1]$ on the interval $[0,T]$, $H^k(0,1)$ denotes the Sobolev space of functions defined on the interval $(0,1)$ whose distributional derivatives of order k are square integrable, $H^0(0,1) = L^2(0,1)$, and $L^\infty(0,T)$ denotes the space of essentially bounded functions on $[0,T]$.

2. Formulation of the Initial-Boundary-Value Problem

Let a viscoelastic rod have a straight unstressed reference configuration, with its axis lying along the segment (0,1) of the real line. A typical material point of this axis (or, equivalently, a typical material cross-section of the rod) is labelled by its distance s from the origin in this reference configuration. We study longitudinal motions of this rod, in which the axis remains straight. Let $w(s,t)$ denote the distance from the origin of the material point s at time t. We shall seek solutions of the governing equations that never suffer a total compression, i.e., solutions for which the stretch is everywhere positive:

$$w_s(s,t) > 0 \quad \forall\ s \in [0,1],\ t \geq 0. \tag{2.1}$$

(The same restriction holds in gas dynamics with w_s playing the role of specific volume.) Under suitable constitutive restrictions, we shall show that if $w_s(s,0) > 0$ for all $s \in [0,1]$, then (2.1) holds.

Let $\rho(s)$ be the given mass density per unit reference length of the rod at s in the reference configuration. We assume that it is bounded above and that it has a positive lower bound on $[0,1]$. Let $\hat{n}(s,t)$ be the contact force at $s \in (0,1)$ at time t exerted by the material of $(s,1]$ on the material of $[0,s]$. Let $f(s,t)$ be the body force per unit reference length. Then the classical form of the equation of motion is

$$\rho w_{tt} = \hat{n}_s + f. \tag{2.2}$$

We assume that there is a continuously differentiable constitutive function

$$[0,1] \times (0,\infty) \times \mathbb{R} \ni (x,y,z) \mapsto n(x,y,z) := \phi_y(x,y) + \sigma(x,y,z) \in \mathbb{R} \tag{2.3a}$$

with

$$\phi_y(x,y) := n(x,y,0), \quad \sigma(x,y,0) = 0, \quad n(x,1,0) = \phi_y(x,1) = 0 \tag{2.3b,c,d}$$

such that

$$\hat{n}(s,t) = n(s, w_s(s,t), w_{st}(s,t)). \tag{2.3e}$$

Condition (2.3d) merely says that there is no contact force in the reference configuration, in which the rod is unstretched and at rest, i.e., so that the reference configuration is natural. ϕ is the stored-energy function for the elastic response.

The constitutive equation (2.3) defines a class of nonlinearly viscoelastic materials without memory. The substitution of (2.3) into (2.2) produces our partial differential equation (1.1).

It is convenient to recast (1.1) as a system. We set

$$u = w_s, \quad v = w_t \tag{2.4}$$

so that (1.1) is equivalent to

$$u_t = v_s, \tag{2.5}$$

$$\rho v_t = n(s,u,v_s)_s + f. \tag{2.6}$$

We assume that each end of the rod either has a prescribed motion or else is subjected to a prescribed force, so that there are functions v_0, v_1, n_0, n_1 for which

$$\text{either} \quad v(0,t) = v_0(t) \quad \text{or} \quad n(0,u(0,t),v_s(0,t)) = n_0(t), \tag{2.7a,b}$$

$$\text{either} \quad v(1,t) = v_1(t) \quad \text{or} \quad n(1,u(1,t),v_s(1,t)) = n_1(t). \tag{2.7c,d}$$

Thus at $s=0$ and at $s=1$ one factor of the power $\hat{n}v$ is prescribed, and the other is unknown. Virtually all previous work on related equations takes the prescribed factors to be zero. We shall devote considerable effort to estimating terms containing this nonzero power, because it is a source of interesting physical consequences.

For the energy estimates of Section 4, we tacitly use the assumption that the two functions prescribed in (2.7) satisfy the appropriate conditions from

$$v_0, v_1 \in H^1_{\text{loc}}(0,\infty), \quad n_0, n_1 \in L^2_{\text{loc}}(0,\infty). \tag{2.8a}$$

In Section 8 we impose the stronger restrictions

$$v_0, v_1 \in H^2_{\text{loc}}(0,\infty), \quad n_0, n_1 \in H^1_{\text{loc}}(0,\infty). \tag{2.8b}$$

Our initial conditions are

$$u(s,0) = \bar{u}(s), \quad v(s,0) = \bar{v}(s) \quad \forall s \in [0,1] \tag{2.9a,b}$$

where $\bar{u} \in H^1(0,1)$ and $\bar{v} \in H^2(0,1)$ are prescribed functions. We assume that the following zeroth–order compatibility conditions hold:

$$\begin{aligned} v_0(0) &= \bar{v}(0) \quad \text{if (2.7a) is prescribed,} \\ v_1(0) &= \bar{v}(1) \quad \text{if (2.7c) is prescribed,} \end{aligned} \tag{2.10}$$

These limiting values are assumed in the sense of trace.

The classical form of our initial-boundary value problem is (2.5), (2.6), (2.7), (2.9). We refer to this problem as Problem DD when (2.7a,c) hold, as Problem ND when (2.7a,d) or (2.7b,c) hold, and as Problem NN when (2.7b,d) hold. D designates *Dirichlet* and N, *Neumann*.

3. Constitutive Assumptions

In this section we assemble all the constitutive restrictions we use. The novel features of our analysis depend crucially upon Hypotheses 3.7, 3.9, 3.17, 3.18. The remaining hypotheses are either standard or technical. The consequences of Hypothesis 3.7, presented in Section 5, are particularly simple; they illustrate the fundamental ideas of our approach.

General Restrictions. We assume that an increase in the contact force must accompany an increase in the rate of stretch $v_s = w_{st}$ by requiring that there be a positive number m such that

$$n_z \equiv \sigma_z \geq m. \tag{3.1}$$

This condition, which is the same as (1.5a) and which generalizes (1.3) and (1.4), ensures that the mechanical process described by (1.1) is dissipative in a uniform way and that for a given function u, equation (2.6) is parabolic. (The form of (2.5), (2.6) thus indicates why it is called a hyperbolic-parabolic system.)

We require that in statics an infinite compressive force accompany a total compression:

$$\phi_y(x,y) := n(x,y,0) \to -\infty \quad \text{as} \quad y \to 0. \tag{3.2}$$

We complement (3.2) with the requirement that in statics an infinite tensile force accompany an infinite stretch

$$\phi_y(x,y) := n(x,y,0) \to \infty \quad \text{as} \quad y \to \infty. \tag{3.3}$$

It would also not be unreasonable to require that an increase in the stretch $u = w_s$ produce an increase in n, so that $n_y > 0$. This restriction, however, is incompatible with models for phase changes studied by [**3**], [**13**], and [**36**]. Rather than enforce this restriction, we adopt the policy that the totality of constitutive restrictions we actually employ should not prohibit it. For simplicity of exposition we do adopt a much weaker restriction: There are numbers y_* and y^* with $0 < y_*,\ y^* < \infty$ such that

$$\phi_{yy}(x,y) := n_y(x,y,0) > 0 \quad \text{for} \quad y \in (0,y_*) \cup (y^*,\infty). \tag{3.4}$$

The restrictions (3.1)–(3.4) we have imposed on n are tacitly assumed to hold throughout our analysis. We now describe further constitutive restrictions introduced as formal hypotheses. These are not assumed to hold universally, but are the objects of our study.

Constitutive hypotheses relating to total compression. We could strengthen (3.2) by requiring

3.5. Hypothesis.

$$n(x,y,z) \to -\infty \quad as \quad y \to 0 \quad \forall\, z. \tag{3.5}$$

Although we do not need this condition for much of our analysis, we shall exclude constitutive restrictions that are incompatible with it.

Another kind of strengthening of (3.2) is

3.6. Hypothesis.

$$\phi(x,y) \to \infty \quad as \quad y \to 0. \tag{3.6}$$

We investigate the consequences of (3.6) in Section 4.

We impose a further restriction, inspired partly by (1.4) and by [**1**], that says that viscous effects become more pronounced as the stretch w_s becomes small:

3.7. Hypothesis. *There are numbers $y_* \in (0,1)$, $M \geq 0$, and $A \geq 0$, and there is a continuously differentiable function ψ on $(0,y_*)$ with*

$$\psi(y) \to \infty \quad \text{as} \quad y \to 0, \quad \psi \geq 0 \tag{3.7a}$$

such that

$$n(x,y,z) \leq -\psi'(y)z + M\psi(y) + A \tag{3.7b}$$

when

$$y \leq y_*, \quad z \in \mathbb{R}. \tag{3.7c}$$

This hypothesis generalizes that of [**4**] by allowing the presence of the term with M on the right-hand side of (3.7b); it is more restrictive than that of [**4**] in that here (3.7b) must hold for all z. Consequently this hypothesis prohibits a superlinear dependence of n on z for large z. Since $-\psi'$ is positive, (3.7b) is mild for z positive and bounded. Note that there is no loss of generality in taking $A = 0$ when $M > 0$, because we can always take ψ to have a positive lower bound. It is nevertheless convenient to retain A. Replacing $M\psi(y)$ in (3.7b) with $M\psi(y)^p$ where $p > 1$ does not prove useful. Since $\phi_y(y) < 0$ for $y < y_*$, condition (3.7b) is much milder than that obtained by replacing n with σ. (Kanel' [**24**], MacCamy [**31**], and Andrews [**2**] each take $\sigma(y, z) = -\psi'(y)z$ for all y.)

The hypotheses presented so far suffice for the analysis of Sections 4 and 5.

We can relax the restrictiveness inherent in (3.7c) by replacing Hypothesis 3.7 with Hypothesis 3.9. To appreciate the geometrical significance of our assumptions, see Figure 3.8.

Figure 3.8. Geometry of Hypothesis 3.9. Here x is fixed. The curves $n(x, y, z) =$ const are asymptotic to the z-axis as $y \searrow 0$ if (3.5) holds. The shaded region is that defined by (3.11c).

3.9. Hypothesis. *There is a positive number y_* smaller than $\inf_x\{y : n(x, y, 1) = 0\}$, there is a number $n_* > \max_x n(x, y_*, 1)$, there is a positive number A_+, and there is a continuously differentiable, strictly decreasing, convex, positive function ψ_+ on $(0, y_*)$ with*

$$\text{(3.10a)} \qquad \psi_+(y) \to \infty \quad \textit{as} \quad y \to 0,$$

such that

$$\text{(3.10b)} \quad n(x, y, z) \le \phi_y(x, y) - \psi_+'(y)z - A_+, \quad \textit{i.e.,} \quad \sigma(x, y, z) \le -\psi_+'(y)z - A_+,$$

for

$$\text{(3.10c)} \qquad y \le y_*, \quad 1 \le z, \quad n(x, y, z) \le n_*.$$

Inequality (3.7b) *holds when* $y \le y_*$ *and* $z \le 1$.

Moreover, there is a number $n_{**} < n_*$ with the property that $z \geq 1$ if $n(x,y,z) = n_{**}$ and $y \leq y_*$, there is a number $A_- \geq A_+$, and there is a continuously differentiable, strictly decreasing, convex, positive function ψ_- on $(0, y_*)$ with

$$\psi_-(y) \to \infty \quad \text{as} \quad y \to 0 \tag{3.11a}$$

such that

$$n(x,y,z) \geq \phi_y(x,y) - \psi'_-(y)z - A_- \tag{3.11b}$$

for

$$y \leq y_*, \quad 1 \leq z, \quad n_{**} \leq n(x,y,z) \leq n_*. \tag{3.11c}$$

Note that if this hypothesis holds, then it continues to hold if we replace y_* by any smaller positive number. Hence there is no loss of generality in assuming that n_{**} has the indicated property. Note that (3.10b) ensures that (3.7b) with $M = 0$ holds for (3.10c). We need the sharper form of (3.10b) to ensure that the contact forces oppose total compression, because $-\psi'(y)z$ is positive for $z > 0$. This fact makes the requirements that $A_\pm > 0$ be reasonable. Our mathematical analysis would be quite easy if we were to assume in place of (3.11) that $n(x,y,z) \geq -\psi'_-(y)z - A_-$, but this assumption would prevent (3.5) from holding. Condition (3.11) allows both this property and infinite viscosity provided that ϕ_y dominates ψ'_- for small y.

Condition (3.1) and the fact that $\phi_y < 0$ for $y < y_*$ imply that $n(x,y,z) \geq \phi_y(x,y)z + mz$ for $y \leq y_*$, $z \geq 1$. Although $\phi_y(x,y)z$ has the same form as $-\psi'(y)z$, it has the opposite sign, so that this inequality, which does not account for infinite viscosity, is not sharp enough for our needs.

If we assume that (3.5) holds, then the curves $n(x,y,z) = \mathit{const}$ are asymptotic to the z-axis as $y \searrow 0$ because of (3.1). Thus the replacement of (3.7c) with (3.10c) allows n to be superlinear in z for large z. We can of course replace the restriction in (3.11c) that $z \geq 1$ with the restriction that z have any positive lower bound. We can, however, retain the threshold $z = 1$ without loss of generality by simply rescaling variables.

Refined hypotheses for Problem DD. In our treatment of Problem DD in Section 6, we find it necessary to impose some further technical conditions. In particular, we shall sometimes be able to estimate integrals of the stress-power $v_s\sigma(\cdot, u, v_s)$ and shall need to estimate integrals of $|\sigma(\cdot, u, v_s)|$. For this purpose it is tempting to assume that there are numbers $M > 0$ and $\alpha \in (0,1)$ such that

$$|\sigma(x,y,z)| \leq M\{[z\sigma(x,y,z)]^\alpha + 1\}, \tag{3.12}$$

If the effects associated with the behavior of σ for y small were ignored, then this condition would be tantamount to assuming that σ have a polynomial rate of growth in z. Unfortunately the singular dependence of σ on y implies that (3.12) is incompatible with Hypothesis 3.7, as we show at the end of this section. We accordingly require a suitable replacement for (3.12).

Now clearly (3.1) and (2.3c) imply that

$$|\sigma(x,y,z)| \leq \varepsilon z \sigma(x,y,z) \quad \text{for} \quad |z| \geq 1/\varepsilon, \tag{3.13}$$

$$\sigma(x,y,-\tfrac{1}{\varepsilon}) \leq \sigma(x,y,z) \leq \sigma(x,y,\tfrac{1}{\varepsilon}) \quad \text{for} \quad z \in \left(-\tfrac{1}{\varepsilon},\tfrac{1}{\varepsilon}\right), \tag{3.14}$$

so we need a useful bound only for $|\sigma(x,y,\pm 1/\varepsilon)|$. Since $\sigma(x,y,\pm 1/\varepsilon)$ is bounded when y lies in a compact subset of $(0,\infty)$ we need only bound this function for y very small and very large.

We now give an intuitive motivation for the bound we adopt for y small. We may assume that $\psi'(y)$ typifies the viscosity of $\sigma_z(x,y,\pm 1/\varepsilon)$ for y small. Thus we might assume that $|\sigma(x,y,z)| \sim |\psi'(y)||z|$ for $|z| < 1/\varepsilon$ as $y \to 0$. Here "$\sim$" denotes asymptotic equivalence as $y \to 0$. Thus $|\sigma| \sim \sqrt{|\psi'|}\sqrt{z\sigma}$ for $|z| < 1/\varepsilon$ as $y \to 0$. Now if there is a positive number μ such that $\psi(y) \sim My^{-\mu}$, then $\psi'(y) \sim -M\psi(y)^{(\mu+1)/\mu}$, and

$$|\sigma| \sim M|\psi|^{(\mu+1)/2\mu}\sqrt{z\sigma} \quad \text{for} \quad |z| < 1/\varepsilon \quad \text{as} \quad y \to 0. \tag{3.15}$$

Note that $\alpha \equiv \frac{\mu+1}{2\mu} > \frac{1}{2}$. We shall want $\alpha < 1$, which would correspond to $\mu > 1$. This says that the effect of compression on the dissipation should be reasonably strong. We can carry out an analogous development for y large, in which we assume that $\psi(y) \sim My^{\nu}$ for $\nu > 0$; we obtain

$$|\sigma| \sim M|\psi|^{(\nu-1)/2\nu}\sqrt{z\sigma} \quad \text{for} \quad |z| < 1/\varepsilon \quad \text{as} \quad y \to \infty. \tag{3.16}$$

We now generalize these considerations with

3.17. Hypothesis. *For each $\varepsilon > 0$ there are numbers $M > 0$, $\alpha \in (0,1)$, and $\delta \in (0,1)$ such that*

$$|\sigma(x,y,z)| \leq M\left\{|\psi(y)|^{\alpha}[z\sigma(x,y,z)]^{\delta} + 1\right\} \quad \textit{for} \quad |z| < 1/\varepsilon, \quad y \in (0,y_*) \cup (y^*,\infty). \tag{3.17}$$

The next condition says that the dissipative behavior, as embodied in the properties of ψ, must compensate to some extent for the fact that ϕ_y grows at a rate faster than ϕ for y small.

3.18. Hypothesis. *There are constants $M > 0$, $\beta \in (0,1]$, $\gamma \in (0,1]$ such that*

$$|\phi_y(x,y)| \leq M\left[1+\phi(x,y)^{\beta}\right]\left[1+\chi(y)^{\gamma}\right] \tag{3.18}$$

where $\chi = \psi$ if Hypothesis 3.7 holds and where $\chi = \max\{\psi,\psi_+\}$ if Hypothesis 3.9 holds.

Constitutive hypotheses relating to large extension. The analog of Hypothesis 3.7 when y is large is:

3.19. Hypothesis. *There are numbers $y^* > 1$, $M \geq 0$, and $A \geq 0$, and there is a continuously differentiable function ψ on (y^*,∞) with*

$$\psi(y) \to \infty \quad \textit{as} \quad y \to \infty \tag{3.19a}$$

such that

$$n(x,y,z) \geq \psi'(y)z - M\psi(y) - A \quad \text{for} \quad y \geq y^* \quad \text{for all} \quad z. \tag{3.19b}$$

Note that $\psi'(y)$ can be a decreasing function, e.g., a multiple of $1/y$, so that the viscosity can weaken in large tension.

We extend ψ as a continuously differentiable function to all $(0,\infty)$. We assume that it is nonnegative. We can likewise readily construct the analog of Hypothesis 3.9, but do not pause to do so.

An alternative to Hypothesis 3.19 is a restricted version of Dafermos's assumption (1.5b), which lacks the disadvantages of (1.5b) described above:

3.20. Hypothesis. *There are numbers $y^* > 1$ and $M > 0$ such that*

$$n_y(x,y,z)^2 \leq M n_z(x,y,z) \quad \text{when} \quad y > y^*. \tag{3.20}$$

Constitutive hypotheses relating to moderate extensions. Hypothesis 3.20 can sometimes be used in place of Hypothesis 3.19 to get an upper bound on u. When we can get such an upper bound directly without recourse to Hypothesis 3.20 (and when we have a lower bound on u as a consequence of hypotheses like 3.7), we can take advantage of Dafermos's hypothesis (1.5b) in our estimates of Section 8 under circumstances that cannot be disqualified as unphysical, simply by using the following very mild version of Hypothesis 3.20:

3.21. Hypothesis. *There is an increasing continuous function λ from $[1,\infty)$ to $\mathbb{R}$ such that*

$$n_y(x,y,z)^2 \leq \lambda(M) n_z(x,y,z) \quad \text{when} \quad M^{-1} \leq y \leq M. \tag{3.21}$$

For the a priori estimates of Section 8, which support our existence theory, we need a technical constitutive assumption, reminiscent of those imposed by [**28**, Eq. (V.6.9)] and [**23**, Eq. (1.6.4)]:

3.22. Hypothesis. *There are increasing continuous functions μ from $[1,\infty)$ to $\mathbb{R}$ and p from $[1,\infty)$ to $[0,2)$ such that such that*

$$|n_x(x,y,z)| \leq \mu(M)\left[1 + |z|^{p(M)}\right] \quad \text{when} \quad M^{-1} \leq y \leq M. \tag{3.22}$$

This condition is trivially satisfied in the important special case in which the rod is uniform, so that the constitutive function n is independent of x.

Consistency of the hypotheses. An example. It is an easy matter to demonstrate that all our constitutive restrictions are compatible by producing a class of examples that satisfies them. We accordingly consider constitutive functions of the form

$$\begin{aligned}
\phi(y) &= A_1 y^{-a_1} - A_2 y^{1/(1+a_2)} + A_3 + B y^{b+1}, \\
\psi(y) = \psi_+(y) &= C y^{-c} + D y^{d+1}, \\
\psi_-(y) &= H y^{-h} + J y^j, \\
\sigma(y,z) &= (E + F y^{-f} + G y^g) \begin{cases} z^k & \text{for} \quad z \geq 1, \\ p(z) & \text{for} \quad |z| \leq 1, \\ -|z|^l & \text{for} \quad z \leq -1. \end{cases}
\end{aligned} \tag{3.23}$$

Here the lower-case symbols $a_1, \dots, l$ represent positive real numbers and the upper-case symbols $A_1, \dots, J$ represent nonnegative real numbers. The function p is a polynomial satisfying $p(0) = 0$, $p(\pm 1) = \pm 1$, $p'(1) = k$, $p'(-1) = l$, $p'(z) > 0$ for $|z| \le 1$. We readily note that (3.1) is satisfied if $k \ge 1$, $l \ge 1$ and if either $E > 0$ or $F > 0$ and $G > 0$, that (3.2) is satisfied if $A_1 + A_2 > 0$, that (3.3) is satisfied if $B > 0$, that (3.4) is satisfied everywhere if $A_1 + A_2 + B > 0$, that (3.6) is satisfied if $A_1 > 0$, and that (3.7a) is satisfied if $C > 0$. Conditions (3.7b,c) are satisfied if

$$A_1 + A_2 > 0, \quad k \le 1, \quad l \ge 1, \quad F = cC, \quad f = c + 1 \tag{3.24}$$

provided y_* is taken sufficiently small. Note that the conditions prescribed in (3.23) are sufficient, not necessary. The restrictiveness of the condition that $k \le 1$ is the basic motivation for introducing Hypothesis 3.9. The virtue of Hypothesis 3.9 is that the third inequality of (3.10c) reduces the proof of (3.10b) when (3.23) holds to showing that

$$\phi_y + n_* \le [cCy^{-c-1} + (d+1)Dy^d]z - A_+ \tag{3.25}$$

for $z \ge 1$. Inequality (3.24) is satisfied if $A_1 > 0$ and $a_1 < c$ or if $A_1 = 0$. The remaining hypotheses are easily treated. E.g., (3.17) holds for y small if $f < c\alpha + f\delta$, and (3.18) holds for y small if $a_1 + 1 < a_1\beta + c\gamma$ and $A_1 > 0$ or if $1 - (1 + a_2)^{-1} < c\gamma$, $A_1 = 0$, $A_2 > 0$. A consistent set of inequalities is easily chosen from the full list we have just developed.

Discussion of (3.12). We now show that the imposition of (3.12) would be incompatible with (3.7). It follows from (2.3b) and (3.1) that $\sigma(y, z)z > 0$ for $z \ne 0$. Then (3.12) and Young's inequality imply that

$$\begin{aligned} |\sigma| &\le M(\epsilon^\alpha |\sigma|^\alpha |z|^\alpha \epsilon^{-\alpha} + 1) \\ &\le M\left[\frac{(\epsilon^\alpha|\sigma|^\alpha)^{1/\alpha}}{1/\alpha} + \frac{(|z|^\alpha \epsilon^{-\alpha})^{1/(1-\alpha)}}{1/(1-\alpha)} + 1\right] \\ &= M(\epsilon\alpha|\sigma| + (1-\alpha)\epsilon^{-\frac{\alpha}{1-\alpha}}|z|^{\frac{\alpha}{1-\alpha}} + 1) \end{aligned} \tag{3.25}$$

for any $\epsilon > 0$. By choosing $\epsilon = 1/2M\alpha$ we obtain

$$|\sigma| \le M(|z|^\mu + 1) \quad \text{with} \quad \mu = \frac{\alpha}{1-\alpha} \tag{3.26}$$

for all z and y as above, with a suitable M that does not depend on y or z. Now (3.7b) with $z < 0$ implies that

$$\psi'(y)z \le M\psi - \sigma + a = M\psi + |\sigma| + a \le M\psi + M(|z|^\mu + 1), \tag{3.27}$$

and thus

$$\psi'(y) \ge -M\frac{\psi}{z} - M\left(|z|^{\mu-1} + \frac{1}{|z|}\right) \ge -M\left(|z|^{\mu-1} + \frac{1}{|z|}\right) \tag{3.28}$$

for all negative z and all small y. Since the right-hand side of (3.28) does not depend on y, it follows that that $\psi(0)$ must be finite, contradicting (3.7a). This argument is due to Hans Engler.

4. Energy Estimates

In this section we obtain energy estimates (i) that are essential for the subsequent analysis, (ii) that ensure that for Problems ND and NN, $u(\cdot, t)$ can vanish only on a set of measure zero when (3.6) holds, and (iii) illustrate the source of essential difficulties for Problem DD.

Let $\mathfrak{Q}(t) \equiv [0,1] \times [0,t]$. Throughout our development we use the inequality $2|ab| \le \varepsilon^{-1}a^2 + \varepsilon b^2$ for $\varepsilon > 0$ and the Cauchy-Bunyakovskiĭ-Schwarz inequality without comment. We first get some preliminary identities.

We define the *kinetic energy* K, the *strain energy* Φ, the *total energy* E, and the *dissipative stress work* W by

(4.1)
$$K(t) := \tfrac{1}{2}\int_0^1 \rho(s)v(s,t)^2\,ds, \quad \Phi(t) := \int_0^1 \phi\big(s,u(s,t)\big)\,ds, \quad E(t) := K(t)+\Phi(t),$$
$$W(t) := \int_0^t\int_0^1 \sigma\big(s,u(s,\tau),v_s(s,\tau)\big)v_s(s,\tau)\,ds\,d\tau.$$

Let us multiply (2.6) by v, integrate the resulting equation by parts over $\Omega(t)$, and use (2.3) to obtain the *energy equation*

$$E(t)+W(t) = E(0)+\int_0^t\int_0^1 f(s,\tau)v(s,\tau)\,ds\,d\tau + \int_0^t \hat{n}(s,\tau)v(s,\tau)\,d\tau\bigg|_{s=0}^{s=1}. \tag{4.2}$$

The focus of our attention is the last term in (4.2), which is the work of the forces acting on the ends of the rod. (Most of the other authors who have treated similar problems adopted boundary conditions that cause this work to be zero.) We always assume that the total energy $E(0)$ at time 0 is finite. Of course, this assumption alone does not ensure that the total energy remains finite for all time.

We now want a useful representation for the boundary values of $\hat{n}$, appearing on the right-hand side of (4.2). Let us replace s in (2.6) with ξ. We multiply (2.6) by ξ and integrate the resulting equation with respect to ξ over $[0,s]$. Next, we multiply (2.6) by $\xi-1$ and integrate the resulting equation with respect to ξ over $[s,1]$. We add the resulting equations to obtain

$$\hat{n}(s,t) = \int_0^1 \Gamma(s,\xi)\rho(\xi)v_t(\xi,t)\,d\xi + \int_0^1 \hat{n}(\xi,t)\,d\xi - \int_0^1 \Gamma(s,\xi)f(\xi,t)\,d\xi \tag{4.3}$$

where

$$\Gamma(s,\xi) = \begin{cases} \xi & \text{for} \quad \xi < s, \\ \xi - 1 & \text{for} \quad s < \xi. \end{cases}$$

From (4.3) we readily obtain the equation of motion for the rod as a whole:

$$\hat{n}(1,t)-\hat{n}(0,t) = \int_0^1 \rho(s)v_t(s,t)\,ds - \int_0^1 f(s,t)\,ds, \tag{4.4}$$

which we could have obtained more directly by integrating (2.6) with respect to s from 0 to 1.

Since the initial energy is finite and since

$$\left|\int_{\Omega(t)} fv\,ds\,d\tau\right| \le \frac{1}{2\varepsilon\min\rho}\int_{\Omega(t)} f^2\,ds\,d\tau + \frac{\varepsilon}{2}\int_{\Omega(t)} \rho v^2\,ds\,d\tau, \tag{4.5}$$

we obtain from (4.2) that

$$E(t)+W(t) \le C(t) + \varepsilon\int_0^t K(\tau)\,d\tau + \int_0^t \hat{n}(s,\tau)v(s,\tau)\,d\tau\bigg|_{s=0}^{s=1} \tag{4.6}$$

(where $C(t)$ is the sum of $E(0)$ and the first term on the right-hand side of (4.5)).

We now use (4.3) and the boundary conditions to estimate (4.6).

4.7. Proposition. *For Problem NN, for Problem ND, and for Problem DD with $v_1 = v_0$, the left-hand side of (4.2) is dominated by $C(t)$. Consequently, in each of these cases, if ϕ satisfies (3.6), then $u(\cdot, t) > 0$ a.e.*

Proof for Problem NN. Let μ be the total mass of the rod and let $V(t)$ be the velocity of the mass center of the rod:

$$\mu := \int_0^1 \rho(s)\,ds, \quad \mu V(t) := \int_0^1 \rho(s) v(s,t)\,ds. \tag{4.8a,b}$$

Then (4.4) and (4.8a) imply that

$$|V(t)| \le C(t). \tag{4.9}$$

We need estimates of $v(0,t)$ and $v(1,t)$. Since for each t, $v(s,t) - V(t)$ must vanish at some point $s^*(t) \in [0,1]$, it follows that

$$|v(s,t) - V(t)| \le \left| \int_{s^*(t)}^{s} v_s(\xi, t)\,d\xi \right| \le \int_0^1 |v_s(s,t)|\,ds. \tag{4.10}$$

From (3.1) we obtain

$$\sigma(x,y,z)z \ge mz^2, \tag{4.11}$$

so that (4.10) yields

(4.12)

$$\begin{aligned} \left| \int_0^t n_1(\tau) v(1,\tau)\,d\tau \right| &\le \int_0^t \left\{ |n_1(\tau)V(\tau)| + \frac{1}{2\varepsilon}|n_1(\tau)|^2 + \frac{\varepsilon}{2}\left[\int_0^1 |v_s(s,\tau)|\,ds\right]^2 \right\} d\tau \\ &\le C(t) + \frac{\varepsilon}{2}\int_0^t \int_0^1 v_s(s,\tau)^2\,ds\,d\tau \le C(t) + \frac{\varepsilon}{2m} W(t) \end{aligned}$$

for any $\varepsilon \in (0,1)$. We use (4.12) and the analogous inequality for $s = 0$ to bound the last term of (4.6). By choosing $\varepsilon < m$ we obtain from (4.2) that

$$K(t) \le C(t) + M \int_0^t K(\tau)\,d\tau. \tag{4.13}$$

The Gronwall inequality then implies that $K(t) \le C(t)$, so that (4.2) is likewise bounded. $\square$

Proof for Problem ND. To be specific, suppose that (2.7a,d) hold. We must estimate $v(1,t)$ and $\hat{n}(0,t)$. As in (4.10) we find that

$$|v(1,t)| \le |v_0(t)| + \int_0^1 |v_s(s,t)|\,ds. \tag{4.14}$$

Equation (4.4) yields

$$
\begin{aligned}
(4.15)\quad &\int_0^t \hat{n}(0,\tau)v_0(\tau)\,d\tau \\
&= \int_0^t \left\{ v_{0t}(\tau)\int_0^1 \rho(s)v(s,\tau)\,ds + v_0(\tau)n_1(\tau) + v_0(\tau)\int_0^1 f(s,\tau)\,ds \right\} d\tau \\
&\qquad - v_0(t)\int_0^1 \rho(s)v(s,t)\,ds + v_0(0)\int_0^1 \rho(s)\bar{v}(s)\,ds.
\end{aligned}
$$

Now we use the estimates

$$
\begin{aligned}
(4.16)\quad &\left| \int_0^t v_{0t}(\tau)\int_0^1 \rho(s)v(s,\tau)\,ds\,d\tau \right| \\
&\le \int_0^t \left\{ \frac{\mu}{2}v_{0t}(\tau)^2 + \frac{1}{2\mu}\left[\int_0^1 \rho(s)v(s,\tau)\,ds \right]^2 \right\} d\tau \le C(t) + \int_0^t K(\tau)\,d\tau,
\end{aligned}
$$

$$
(4.17)\qquad \left| v_0(t)\int_0^1 \rho(s)v(s,t)\,ds \right| \le \frac{\mu}{2\varepsilon}v_0(t)^2 + \varepsilon K(t)
$$

and make a suitable choice of ε to deduce from (4.6), (4.14)–(4.17) that (4.13) again holds, so that $K(t) \le C(t)$. We again conclude that the left-hand side of (4.2) is dominated by $C(t)$. □

Proof for Problem DD when $v_0 = v_1$. We use (4.4) to obtain estimates just like those for Problem ND. □

Now we study problem DD when the length of the rod is allowed to vary. Let D be a function satisfying

$$
(4.18)\qquad D(\tau) \ge |v_1(\tau) - v_0(\tau)|.
$$

We introduce D with an inequality because we shall later impose restrictions on D, e.g., that it be decreasing, that would be unduly restrictive were they imposed on $|v_1 - v_0|$ itself.

We can now use (4.3) and (2.3) to dominate the last term of (4.6) with

$$
\begin{aligned}
(4.19)\quad &\left| \int_0^t [v_1(\tau) - v_0(\tau)] \int_0^1 n\big(s, u(s,\tau), v_s(s,\tau)\big)\,ds\,d\tau \right| \\
&\qquad \le \int_0^t D(\tau)\int_0^1 \{ |\phi_y(s, u(s,\tau))| + |\sigma\big(s, u(s,\tau), v_s(s,\tau)\big)| \}\,ds\,d\tau
\end{aligned}
$$

together with other terms that are readily controlled by the methods used above. Since $\phi_y(x,y)$ is much more badly behaved than $\phi(x,y)$ itself for y small, and since $\sigma(x,y,z)$ is not automatically dominated by a simple term involving $z\sigma(x,y,z)$, we only get useful energy estimates when ϕ_y and σ meet further conditions, such as those embodied in (3.18) and (3.17). This is not surprising: If the material strongly resists having its length changed, then changing its length against a large resistance might cause the generation of so much work that the right-hand side of (4.6) cannot be controlled. We show how (3.17) and (3.18) affect (4.19) in Section 6.

The following simple example is illuminating. Suppose that $f = 0$ and that $n_x = 0$. Then

$$u(s,t) = 1 - t, \quad v(s,t) = -s \tag{4.20}$$

defines a solution of (2.5), (2.6) satisfying the initial conditions $u(s,0) = 1$, $v(s,0) = -s$, and boundary conditions $v(0,t) = 0$, $v(1,t) = -1$. This solution corresponds to a total compression at time $t = 1$. Note that, as a function of t, the second boundary condition is ostensibly innocuous. Thus we cannot expect to extend Proposition 4.7 without further restrictions. Note that for this example we could replace the second boundary condition with the "equivalent" condition that $\hat{n}(1,t) = n(1, 1-t, -1)$, which becomes infinite as $t \nearrow 1$ provided that (3.5) holds. Clearly this boundary condition is not innocuous; it violates the boundedness imposed on n_1 in (2.7).

5. Pointwise Lower Bounds on the Strains for Problems NN and ND

We now establish strengthened versions of (2.1) for Problems ND and NN, and accordingly assume that the boundary condition (2.7b) holds. As was suggested in the comments at the end of Section 4, Problem DD requires special treatment, which we postpone to the next section. In contrast to the development of Section 4, we give a primary role to an equation related to (4.4), and a subsidiary role to the energy estimate.

The existence theory of Section 8 shows that if the data are sufficiently regular, then so is the solution as long as it exists. We accordingly assume that u is continuous. We suppose that n satisfies (3.4) and Hypothesis 3.7 and that $\min_s \bar{u}(s) > 0$. Then without loss of generality we may choose the number y_* introduced in (3.4) and in Hypothesis 3.7 so that $y_* \le 1$ and

$$y_* \le \min_s \bar{u}(s). \tag{5.1}$$

We shall show that $u(s,t)$ is positive for all s, t, and in certain cases that it has a positive lower bound. Of course it suffices for us to prove this result only for those (s,t) for which $u(s,t) < y_*$. Thus suppose that there is a point (ξ, ω) such that $u(\xi,\omega) < y_*$. If $\xi = 0$, then the continuity of u enables us to show that there is a $\xi > 0$ with the same property. Since u is continuous, there is a $\vartheta \in (0,\omega)$ such that $u(\xi,\vartheta) = y_*$, $u(\xi,t) < y_*$ for $\vartheta < t \le \omega$. It then follows from (3.4) that $\phi_y\big(\xi, u(\xi,t)\big) < 0$ for $\vartheta < t < \omega$. We integrate (2.6) over $[0,\xi] \times [\vartheta, t]$ and use (2.5) and (3.7b) to obtain

$$\begin{aligned}
&\int_0^\xi \rho(s) v(s,\tau)\, ds \Big|_{\tau=\vartheta}^{\tau=t} = \int_\vartheta^t n\big(s, u(s,\tau), v_s(s,\tau)\big)\, d\tau \Big|_{s=0}^{s=\xi} + \int_\vartheta^t \int_0^\xi f(s,\tau)\, ds\, d\tau \\
&\le -\int_\vartheta^t \psi'(u(\xi,\tau)) u_t(\xi,\tau)\, d\tau + M \int_\vartheta^t \psi(u(\xi,\tau))\, d\tau + A(t-\vartheta) - \int_\vartheta^t n_0(\tau)\, d\tau + C(t) \\
&\le \psi\big(u(\xi,\vartheta)\big) - \psi\big(u(\xi,t)\big) + M \int_\vartheta^t \psi(u(\xi,\tau))\, d\tau + C(t)
\end{aligned} \tag{5.2}$$

for all $t \in [\vartheta, \omega]$. Proposition 4.7 implies that

$$\left| \int_0^\xi \rho v\, ds \right| \le \sqrt{\int_0^\xi \rho\, ds} \sqrt{\int_0^\xi \rho v^2\, ds} \le C(\omega) \tag{5.3}$$

for all $t \in [\vartheta, \omega]$. The $C(\omega)$ in (5.3) can be replaced with a constant M if $f = 0$ and if the applicable right-hand sides of (2.7) are independent of t. Inequalities (5.2) and (5.3) yield

$$\psi(u(\xi,t)) \le \psi(y_*) + M \int_\vartheta^t \psi\big(u(\xi,\tau)\big)\, d\tau + C(t) \tag{5.4}$$

for all $t \in [\vartheta, \omega]$. It follows from the Gronwall inequality that

$$\psi\big(u(\xi,\omega)\big) \le C(\omega), \tag{5.5a}$$

so that (3.7a) gives

$$u(\xi,\omega) \ge c(\omega). \tag{5.5b}$$

(The continuity of u ensures that (5.4) holds for $\xi = 0$.) Thus we obtain the pointwise bound

$$u(s,t) \ge c(t) \quad \forall\ (s,t) \in \mathfrak{Q}(\infty). \tag{5.5c}$$

An analogous argument clearly yields (5.5) if (2.7d) holds. In summary, we have

5.6. Theorem. *Let u be continuous, let n satisfy Hypothesis 3.7, and let $\bar{u}$ be everywhere positive. Then (5.5) holds for Problems ND and NN. If $a = 0$, $K = 0$, $f = 0$ and if the applicable right-hand sides of (2.7) are independent of t, then the lower bound for u given in (5.5) is independent of t.*

Note that this result is valid whether or not (3.6) holds. It is interesting to observe that for *static* one-dimensional problems, the preclusion of total compression can be achieved by using (3.6) alone. In Section 10 we show that a condition like (3.6) precludes total compression for the dynamical hyperbolic problem in which there is no dissipation.

We likewise obtain

5.7. Theorem. *Let u be continuous, let n satisfy Hypothesis 3.19, and let $\bar{u}$ be everywhere positive. Then*

$$u(s,t) \le C(t) \quad \forall\ (s,t) \in \mathfrak{Q}(\infty) \tag{5.8}$$

for Problems ND and NN. If Hypothesis 3.19 holds with $a = 0$, K=0, if $f = 0$, and if the applicable right-hand sides of (2.7) are independent of t, then the upper bound for u given in (6.4) is independent of t.

6. Pointwise Lower Bounds on the Strains for Problem DD

We now handle the difficult Problem DD by supplementing the basic assumptions used in Section 5 with Hypotheses 3.17 and 3.18. We again assume that u is continuous. Here we show that extreme compression is precluded for finite time by obtaining an upper bound for

$$\bar{\psi}(t) := \max\{\psi\big(u(s,\tau)\big) : 0 \le s \le 1,\ 0 \le \tau \le t\} \tag{6.1}$$

that depends only on the data. Our chief aim in this section is to demonstrate the efficacy of Hypotheses 3.17 and 3.18, which exhibit a subtle interaction between elasticity and viscosity at extreme compression and extension.

Let y_* be chosen to satisfy (5.1). We need only consider the case in which

$$\bar{\psi}(\omega) = \psi\big(u(\xi,\omega)\big) \quad \text{with} \quad 0 < u(\xi,\omega) < y_*. \tag{6.2}$$

(Were there no such (ξ,ω), we would have $u \geq y_*$.) We define ϑ just as in the remarks following (5.1). Then by (3.7) and (4.3) we obtain

(6.3)
$$\begin{aligned}
\bar{\psi}(t) &= \psi(y_*) + \int_\vartheta^t \psi'\big(u(\xi,\tau)\big)u_t(\xi,\tau)\,d\tau \\
&\leq \psi(y_*) + M\int_\vartheta^t \psi\big(u(\xi,\tau)\big)\,d\tau + A(t-\theta) - \int_\vartheta^t n\big(\xi,u(\xi,\tau),v_s(\xi,\tau)\big)\,d\tau \\
&= \psi(y_*) + M\int_\vartheta^t \psi\big(u(\xi,\tau)\big)\,d\tau + A(t-\theta) - \int_\vartheta^t\int_0^1 \varGamma(\xi,s)\rho(s)v_t(s,\tau)\,ds\,d\tau \\
&\qquad - \int_\vartheta^t\int_0^1 n\big(s,u(s,\tau),v_s(s,\tau)\big)\,ds\,d\tau + \int_\vartheta^t\int_0^1 \varGamma(\xi,s)f(s,\tau)\,ds\,d\tau \\
&\leq C(t) + M\int_0^t \bar{\psi}(\tau)\,d\tau + M\max_{\tau\in[0,t]} K(\tau) \\
&\qquad + \int_0^t\int_0^1 \big|\sigma\big(s,u(s,\tau),v_s(s,\tau)\big)\big|\,ds\,d\tau + \int_0^t\int_0^1 \big|\phi_y\big(s,u(s,\tau)\big)\big|\,ds\,d\tau
\end{aligned}$$

for all $t \in (\vartheta,\omega)$, and a fortiori, by the definition of $\bar{\psi}$, for all $t \in [0,\omega]$.

Let us set

(6.4)
$$\begin{aligned}
\mathfrak{Q}_1(\omega) &:= \{(s,t) \in \mathfrak{Q}(\omega) : |v_s(s,t)| \geq 1/\varepsilon\} \equiv \{(s,t) \in \mathfrak{Q}(t) : |\sigma| \leq \varepsilon\sigma v_s(s,t)\}, \\
\mathfrak{Q}_2(\omega) &:= \{(s,t) \in \mathfrak{Q}(\omega) : |v_s(s,t)| < 1/\varepsilon,\ u(s,t) \in [y_*, y^*]\} \\
&\qquad \subset \{(s,t) \in \mathfrak{Q}(t) : |\sigma| \leq M(\varepsilon,\omega)\}, \\
\mathfrak{Q}_3(\omega) &:= \{(s,t) \in \mathfrak{Q}(\omega) : |v_s(s,t)| < 1/\varepsilon,\ u(s,t) \in (0,y_*) \cup (y^*,\infty)\} \\
&\qquad \subset \{(s,t) \in \mathfrak{Q}(\omega) : (3.17) \text{ holds with } (x,y,z) = (s,u(s,t),v_s(s,t))\}.
\end{aligned}$$

Then (3.13), (3.17), and (3.18) imply that

$$\begin{aligned}
&\int_0^t\int_0^1 |\sigma(s,u,v_s)|\,ds\,d\tau \\
&\qquad \leq \varepsilon\int_{\mathfrak{Q}_1(t)} \sigma(s,u,v_s)v_s\,ds\,d\tau + M\int_{\mathfrak{Q}_2(t)} ds\,d\tau \\
&\qquad\quad + M\int_{\mathfrak{Q}_3(t)} \big\{\psi(u)^\alpha[\sigma(s,u,v_s)v_s]^\delta + 1\big\}\,ds\,d\tau \\
&\qquad \leq \varepsilon W(t) + Mt + C(t)\bar{\psi}(t)^\alpha W(t)^\delta,
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\int_0^t\int_0^1 |\phi_y(s,u)|\,ds\,d\tau &\leq M\int_0^t [1+\bar{\psi}(\tau)^\gamma]\int_0^1 [1+\phi(s,u)^\beta]\,ds\,d\tau \\
&\leq M\int_0^t [1+\bar{\psi}^\gamma]\,\big[1+\varPhi^\beta\big]\,d\tau.
\end{aligned} \tag{6.6}$$

Let us first examine the case in which $D = 0$. We take $\beta = 1$, $\gamma = 1$ in (3.18). Then using (6.5), (6.6), and Proposition 4.7, we deduce from (6.3) that

$$\bar{\psi}(t) \leq C(t) + C(t) \int_0^t \bar{\psi}\, d\tau + C(t)\bar{\psi}(t)^\alpha. \tag{6.7}$$

Let $\mathfrak{E}$ be the set of $t \in [0, \omega]$ for which $C(t) + C(t) \int_0^t \bar{\psi}(\tau)\, d\tau \leq C(t)\bar{\psi}(t)^\alpha$. On $\mathfrak{E}$, inequality (6.7) reduces to $\bar{\psi}(t) \leq 2C(t)\bar{\psi}(t)^\alpha$, so that $\bar{\psi}(t) \leq C(t)$ on $\mathfrak{E}$. On $[0, \omega] \setminus \mathfrak{E}$,

$$\bar{\psi}(t) \leq 2C(t) + 2C(t) \int_0^t \bar{\psi}\, d\tau. \tag{6.8}$$

But the bounds we have already obtained on $\mathfrak{E}$ enable us to deduce from (6.7) that (6.8) holds on $[0, \omega]$. We apply the Gronwall inequality to (6.8) to conclude that $\bar{\psi}(\omega) \leq C(\omega)$. Thus

6.9. Theorem. *Let u be continuous, let n satisfy Hypotheses 3.19 and 3.18 with $\beta = 1 = \gamma$, and let $\bar{u}$ be everywhere positive. For Problem DD with $v_1 = v_0$, inequality* (5.5) *holds.*

Now let us study the problem with boundary conditions (2.7a,c) with $v_1 \neq v_0$. We assume that $t \mapsto D(t)$ is decreasing and that $\int_0^t D(\tau)\, d\tau$ is bounded. We shall obtain further restrictions on D that ensure that (5.5c) holds. For the sake of simplicity of exposition we express these restrictions in terms of the catch-all constant M and the catch-all function C. A meticulous and boring analysis would enable us to express the crucial bounds on D in terms of specific properties of the data. Our purpose is not to exhibit sharp bounds, but rather to indicate the kinds of constitutive assumptions that suffice to deliver reasonable bounds preventing total compression.

Throughout our analysis we use without comment the standard inequalities

$$\begin{aligned} &(x + y)^\lambda \leq x^\lambda + y^\lambda \quad \text{for} \quad x > 0,\ y > 0,\ 0 < \lambda < 1, \\ &x^p \leq 1 + x^{p+q} \quad \text{for} \quad x > 0,\ p > 0,\ q > 0, \\ &x^\lambda y^{1-\lambda} \leq \lambda x + (1 - \lambda) y \quad \text{for} \quad 0 < \lambda < 1. \end{aligned} \tag{6.10}$$

For notational simplicity we let μ and ν denote positive exponents, whose meaning, like that of M, can change in each appearance. In every instance, however, they can be expressed in terms of constitutive parameters.

We set

$$\begin{aligned} &\bar{\Phi}(t) := \max\{\Phi(\tau) : 0 \leq \tau \leq t\}, \quad \Psi(t) := \int_0^t \bar{\psi}(\tau)\, d\tau, \\ &Q(t) := \int_0^t D(\tau)\bar{\psi}(\tau)\, d\tau, \quad \Delta(t) := \int_0^t D(\tau)\, d\tau. \end{aligned} \tag{6.11}$$

The relevant version of the energy inequality, coming from (4.6), (4.19), and from (3.17), (3.18) via estimates like (6.5), (6.6) has the form

$$
\begin{aligned}
&K(t)+\varPhi(t)+W(t)\\
&\le C(t)+\int_0^t D(\tau)\int_0^1\{|\phi_y(s,u)|+|\sigma(s,u,v_s)|\}\,ds\,d\tau+M\int_0^t K\,d\tau\\
&\le C(t)+MB(t)+\varepsilon\int_0^t\int_0^1 D\sigma(s,u,v_s)v_s\,ds\,d\tau\\
&\quad+M\int_0^t\int_0^1 D\bar\psi^\alpha[\sigma(s,u,v_s)v_s]^\delta\,ds\,d\tau+M\int_0^t K\,d\tau\\
&\le C(t)+MB(t)+\varepsilon D(0)W(t)+MH(t)^{1-\delta}W(t)^\delta+M\int_0^t K\,d\tau
\end{aligned}
\tag{6.12a}
$$

for all $t\in[0,\omega]$, where

$$
B(t):=\int_0^t D[\bar\psi^\gamma+\varPhi^\beta+\bar\psi^\gamma\varPhi^\beta]\,d\tau,\quad H(t):=\int_0^t [D\bar\psi^\alpha]^{1/(1-\delta)}\,ds\,d\tau. \tag{6.12b,c}
$$

We now choose the ε introduced in Hypothesis 3.17 so that $\varepsilon D(0)<1$. Then (6.12a) holds without the term $\varepsilon D(0)W(t)$ appearing on the extreme right-hand side. We now treat W in (6.12a) just as we treated $\bar\psi$ in (6.7), (6.8), obtaining

$$
K(t)+\varPhi(t)+W(t)\le C(t)+MB(t)+MH(t)+M\int_0^t K\,d\tau \tag{6.13}
$$

for all $t\in[0,\omega]$. Estimating B by the Hölder inequality we obtain from (6.13) that

$$
\begin{aligned}
\varPhi(t)\le C(t)+M\varDelta(t)^{1-\gamma}Q(t)^\gamma+M\bar\varPhi(t)^\beta[\varDelta(t)+\varDelta(t)^{1-\gamma}Q(t)^\gamma]&\\
+MH(t)+M\int_0^t K\,d\tau.&
\end{aligned}
\tag{6.14}
$$

We replace t on the right-hand side of (6.14) by any number r exceeding t. It follows that (6.14) holds for all t and r with $t\le r\le\omega$. We can now take the maximum of the left-hand side 0f (6.14) over $t\in[0,r]$. Replacing r with t we recover (6.14) with its left-hand side replaced by $\bar\varPhi(t)$. We now treat $\bar\varPhi$ in this version of (6.14) just as we treated $\bar\psi$ in (6.7), (6.8), obtaining

$$
\begin{aligned}
\varPhi(t)\le C(t)+M\varDelta(t)^{1-\gamma}Q(t)^\gamma+M[\varDelta(t)+\varDelta(t)^{1-\gamma}Q(t)^\gamma]^{1/(1-\beta)}&\\
+MH(t)+M\int_0^t K\,d\tau,&
\end{aligned}
\tag{6.15}
$$

which we substitute into (6.13) to obtain

$$
\begin{aligned}
K(t)+\varPhi(t)+W(t)\le C(t)+M\varDelta(t)^{1-\gamma}Q(t)^\gamma+M[\varDelta(t)+\varDelta(t)^{1-\gamma}Q(t)^\gamma]^{1/(1-\beta)}&\\
+MH(t)+M\int_0^t K\,d\tau.&
\end{aligned}
\tag{6.16}
$$

Thus

$$K(t) \le C(t) + M\Delta(t)^\mu Q(t)^\nu + MH(t) + M\int_0^t K\,d\tau \tag{6.17}$$

for all $t \in [0, \omega]$. We use the Gronwall inequality to obtain an estimate for K in terms of the other variables in this inequality, and substitute this estimate into (6.16) to get

$$K(t) + \Phi(t) + W(t) \le C(t)[1 + \Delta(t)^\mu Q(t)^\nu + H(t)]. \tag{6.18}$$

Let λ be any number satisfying $0 < \lambda < \min\{1, \frac{\alpha}{1-\delta}\}$, and set $\frac{1}{1-\delta} - \lambda = \mu + 1 - \lambda$. Then the integrand of H equals $D^\mu D^{1-\lambda}(D\bar{\psi})^\lambda \bar{\psi}^\eta$ where $\eta = \frac{\alpha}{1-\delta} - \lambda$. Thus

$$H \le D(0)^\mu \bar{\psi}(t)^\eta \Delta(t)^{1-\lambda} Q(t)^\lambda. \tag{6.19}$$

We substitute (6.19) into (6.18), substitute the resulting (6.18) into (6.5) and (6.6), and finally substitute the resulting (6.5) and (6.6) into the rightmost term of (6.3) to obtain

$$\bar{\psi} \le C\left[1 + \Psi + \Delta^\mu Q^\nu + (1 + \Delta^\mu Q^\nu)\bar{\psi}^\zeta\right] \tag{6.20}$$

for $0 \le t \le \omega$ where $\zeta := \max\{\eta, \alpha, \alpha + \eta\delta, \gamma + \eta\beta\}$. We now require that $\zeta < 1$, which is ensured by

$$\alpha < 1 - \delta^2, \quad \gamma + \frac{\alpha\beta}{1-\delta} < 1 + \beta. \tag{6.21}$$

Note that these inequalities are compatible with the discussion in the paragraph preceding Hypothesis 3.17. Since $0 < \zeta < 1$ we can once again use the method leading to (6.8) to obtain

$$\bar{\psi} \le C\left[1 + \Psi + \Delta^\mu Q^\nu\right]. \tag{6.22}$$

Thus

$$Q' \le CD\left[1 + \Psi + \Delta^\mu Q^\nu\right] \le C\left[1 + Q + D\Delta^\mu Q^\nu\right] \tag{6.23}$$

since D is decreasing. We substitute

$$1 + Q(t) = R(t)\exp\left(\int_0^t C(\tau)\,d\tau\right) \tag{6.24}$$

into (6.23) to obtain

$$R(t)^{-\nu}R'(t) \le C(t)D(t)\Delta(t)^\mu \exp\left((\nu-1)\int_0^t C(\tau)\,d\tau\right) := C_1(t)D(t)\Delta(t)^\mu. \tag{6.25}$$

We integrate (6.25) from 0 to t. Since $R \mapsto J(R) := \int_1^R r^{-\nu}\,dr$ is an increasing function on $[1, \infty)$, it follows from the integral of (6.25) that $R(t) \le C(t)$ if

$$\int_0^\infty C_1(t)D(t)\Delta(t)^\mu\,dt < J(\infty). \tag{6.26}$$

It then follows that Q is bounded. We substitute this bound into (6.22) and apply the Gronwall inequality to deduce that Ψ and $\bar{\psi}$ are bounded. Thus we have proved

Figure 7.1. A possible graph of $u(\xi,\cdot)$.

6.27. Theorem. *Let u be continuous, let n satisfy Hypotheses 3.7, 3.17, and 3.18 with $\beta,\gamma<1$ and with* (6.21) *holding, and let $\bar{u}$ be everywhere positive. Let the boundary conditions* (2.7a,c) *hold. Let $t\mapsto D(t)$ be decreasing. If* (6.26) *holds, then* (5.5c) *holds, and $K(t)+\Phi(t)+W(t)\le C(t)$.*

Note that the choice of the origin of time is arbitrary. Thus, if u is known to have a positive lower bound for $0\le t\le t_0$, then Theorem 6.27 holds with the lower limit of integration 0 in (6.26) replaced with t_0.

7. Pointwise Lower Bounds on the Strains under Weaker Hypotheses

We now study the question of total compression when Hypothesis 3.9 replaces Hypothesis 3.7. In Figure 7.1 we sketch a possible graph of $u(\xi,\cdot)$ on the interval $[\vartheta,\omega]$, defined at the beginning of Section 5. To fix ideas, consider the Problem NN or ND with $M=0$. Suppose that (3.7b) holds only for $y\le y_*$ and $z\le 0$. What prevents us from carrying out the procedures of Section 5?

Let ξ,ϑ,ω be defined as in the beginning of Section 5. On the intervals (ϑ,ω_1), (ϑ_2,ω_2), etc., on which $u(\xi,\cdot)$ is decreasing, we obtain, just as in Section 5, that $\psi(u(\xi,\omega_1))\le\psi(y_*)+C(\omega_1)$, $\psi(u(\xi,\omega_2))\le\psi(u(\xi,\vartheta_2))+C(\omega_2)$, etc. Since $u(\xi,\vartheta_2)>u(\xi,\omega_1)$, we have $\psi(u(\xi,\vartheta_2))\le\psi(u(\xi,\omega_1)$, and therefore, $\psi(u(\xi,\omega_2))\le\psi(y_*)+C(\omega_1)+C(\omega_2)$, etc. Thus we would find that $\psi(u(\xi,\omega))\le C(\omega)$ and that (5.5c) would hold *provided that $u(\xi,\cdot)$ does not have an infinite number of oscillations on $[\vartheta,\omega]$, or, more generally, that the series $C(\omega_1)+C(\omega_2)+\cdots$ converges.*

We could alternatively demonstrate (5.5c) if each local maximum $u(\xi,\vartheta_k)$ of $u(\xi,\cdot)$ were to exceed the threshold y_*. Indeed, we could then identify each ϑ_k with the ϑ of Sections 5 and 6 and thereby construct the requisite bounds. We shall actually use Hypothesis 3.9 to construct a new threshold y_*, denoted y_{**}, for which a variant of this idea can be exploited. Moreover, our approach also delivers a demonstration of a version of the desideratum discussed in the last paragraph, namely, that the number of "bad" oscillations is locally finite.

Figure 7.2. A dangerous trajectory of $u(\xi, \cdot)$ in the (y, z)-phase plane.

Our strategy for overcoming the obstacles associated with Figure 7.1 is to study the trajectory of $t \mapsto (u(\xi, t), u_t(\xi, t))$ in the (y, z)-phase plane shown in Figure 7.2. (On the basis of the existence theory of Section 9, we may suppose that u and u_t are continuous.) We use Hypothesis 3.9 and the properties of the phase plane to control where this trajectory can go.

Condition (5.1) ensures that this trajectory starts in the half plane $y > y_*$. Since trajectories move the the right for $z > 0$ and to the left for $z < 0$, we might as well assume that there is a (ξ, ϑ) such that $u(\xi, \vartheta) = y_*$, and that $u(\xi, t) < y_*$ and $u_t(\xi, t) \leq 0$ for t in some interval of the form (ϑ, ω). (Otherwise there would be nothing to prove.) Let ω_1 be the supremum of such ω's. We set $U(t) := u(\xi, t)$.

Either $\omega_1 = \infty$, in which case the developments of Sections 5 and 6 show that we are done, or else these same developments show that $U(\omega_1) \geq c(\omega_1) > 0$, $U_t(\omega_1) = 0$, and the trajectory penetrates the region

$$y \leq y_*, \quad z > 0, \quad n(x, y, z) \leq n_*, \tag{7.3}$$

Since trajectories move to the right in this region, and since Hypothesis 3.9 ensures that (3.7a) holds, we deduce that $U(t) \geq c(\omega_1) > 0$ as long as the trajectory lies in the region (7.3). The trajectory can leave the region (7.3) only by piercing one of the boundaries $y = y_*$, $z = 0$, $n(x, y, z) = n_*$. If it leaves through $y = y_*$, then we can repeat the analysis just given. If it leaves through $z = 0$, then the developments of Sections 5 and 6 and the geometry of the phase plane show that the bound $U(t) \geq c(t) > 0$ holds as long as the trajectory does not pierce the line $n = n_*$, because Hypothesis (3.9) requires that (3.7b) hold for $z \leq 1$ and that (3.7b) with $M = 0$ hold on (3.10c). The only difficulty is in the third alternative, in which the trajectory leaves the region (7.3) through $n(x, y, z) = n_*$, as shown in Figure 7.2. We term such a trajectory *dangerous*. The momentum a dangerous trajectory builds up above the region (7.3) could conceivably carry it across the segment $z = 0$, $y < y^*$ and to the boundary $y = 0$. We now show how Hypothesis 3.9 can prevent this total compression.

Now for a trajectory that enters the region (7.3) across the line $z = 0$ to leave this region through $n(x,y,z) = n_*$, it must cross the strip (3.11c):

$$y \le y_*, \quad 1 \le z, \quad n_{**} \le n(x,y,z) \le n_*, \tag{7.4}$$

which is shaded in Fig. 3.8. In particular, there must be an interval $[t_1, t_2]$ such that $n(\xi, U(t_1), U_t(t_1)) = n_{**}$, $n_{**} \le n(\xi, U(t), U_t(t)) \le n_*$ for $t_1 \le t \le t_2$, $n(\xi, U(t_2), U_t(t_2)) = n_*$. Since trajectories move to the right in region (7.4), it follows that $u_1 \equiv U(t_1) < u_2 \equiv U(t_2)$.

Let us set

$$\begin{aligned} \Omega(t) := -\int_{t_1}^{t} \phi_y(\xi, U(\tau))\, d\tau, \quad \alpha_+ := A_+ + n_{**}, \quad \alpha_- := A_- + n_*, \\ \Omega_+(t) := \Omega(t) + \alpha_+(t - t_1), \quad \Omega_-(t) := \Omega(t) + \alpha_-(t - t_1). \end{aligned} \tag{7.5}$$

We assume that $0 < \alpha_+ < \alpha_-$. We use (3.10) and (3.11) to deduce that

$$\psi_-(U(t)) \ge \psi_-(u_1) - \Omega_-(t), \tag{7.6}$$

$$\psi_+(U(t)) \le \psi_+(u_1) - \Omega_+(t). \tag{7.7}$$

for $t_1 \le t \le t_2$. The properties of ψ_+ and ϕ ensure that

$$U(t) \ge \psi_+^{-1}\left(\psi_+(u_1) - \Omega_+(t)\right), \tag{7.8}$$

$$\Omega_+'(t) = -\phi_y(U(t)) + \alpha_+ \le -\phi_y\left(\psi_+^{-1}\left(\psi_+(u_1) - \Omega_+(t)\right)\right) + \alpha_+. \tag{7.9}$$

We define the function F to be the integral of

$$F'(\chi) := \frac{1}{-\phi_y\left(\psi_+^{-1}(\chi)\right) + \alpha_+} > 0. \tag{7.10}$$

Setting

$$\zeta := \psi_+(u_1) - \Omega_+(t_2), \tag{7.11}$$

we integrate (7.9) over (t_1, t_2) to obtain

$$F\left(\psi_+(u_1)\right) \le F(\zeta) + t_2 - t_1 = F\left(\zeta + F^{-1}\left(F(\zeta) + t_2 - t_1\right) - F^{-1}\left(F(\zeta)\right)\right). \tag{7.12}$$

Using the fact that F is increasing, the Mean Value Theorem, and the fact that $-\phi_y\left(\psi_+^{-1}(\cdot)\right)$ is increasing, we obtain

$$\begin{aligned} \psi_+(u_1) &\le \zeta + F^{-1}\left(F(\zeta) + t_2 - t_1\right) - F^{-1}\left(F(\zeta)\right) \\ &= \zeta + (t_2 - t_1)\int_0^1 \left[-\phi_y\left(\psi_+^{-1}\left(F(\zeta) + \eta(t_2 - t_1)\right)\right) + \alpha_+\right] d\eta \\ &\le \zeta + (t_2 - t_1)\left[-\phi_y\left(\psi_+^{-1}\left(F(\zeta) + (t_2 - t_1)\right)\right) + \alpha_+\right]. \end{aligned} \tag{7.13}$$

Using (7.11), (7.5), and (7.6), we deduce from (7.13) that

$$\psi_-(u_1) \leq \psi_-(u_2) + (t_2 - t_1)\left[-\phi_y\left(\psi_+^{-1}\left(F(\zeta) + (t_2 - t_1)\right)\right) + \alpha_-\right]. \tag{7.14}$$

Since $z > 1$ when (7.4) holds, it follows that $U_t \geq 1$ on $[t_1, t_2]$, so that $t_2 - t_1 \leq u_2 - u_1 \leq y_*$. Thus (7.11) implies that

$$F(\zeta) + t_2 - t_1 \leq F(\psi_+(y_*)) + y_*. \tag{7.15}$$

We then obtain from (7.14) that

$$\psi_-(u_1) \leq \psi_-(u_2) + \mu(u_2 - u_1), \quad \mu := \left[-\phi_y\left(\psi_+^{-1}\left(F(\psi_+(y_*)) + y_*\right)\right) + \alpha_-\right]. \tag{7.16}$$

Since ψ_- is convex, we obtain from (7.16) that

$$\psi_-'(u_2) \geq \frac{\psi_-(u_2) - \psi_-(u_1)}{u_2 - u_1} \geq \psi_-'(u_1) \geq -\mu. \tag{7.17}$$

Thus there is a number $y_{**} > 0$ such that $u_2 \geq y_{**}$. Note that y_{**} depends only on the constitutive restrictions and on the initial value of U through the choice of y_*.

Let us now repeat our discussion of the trajectory of $(U(\cdot), U_t(\cdot))$, but with y_{**} replacing y_*, *except* in (7.15) and (7.16). We again find that either $U(t)$ is positive for all finite t, or else, after time t_2 this trajectory passes through the region $z < 0$ and at a later time t_3 satisfies $n(\xi, U(t_3), U_t(t_3)) = n_*$, $U(t_3) \geq y_{**}$. Moreover, in the latter case, there must be a time interval (τ_1, τ_2) such that $U_t(\tau_1) = 0$, $U_t(t) > 0$ and $n(\xi, U(t), U_t(t)) < n_*$ for $t \in (\tau_1, \tau_2)$, $U(\tau_2) = y_{**}$. If $U(\tau_1) \leq \frac{1}{2}y_{**}$, then $U_t(t) \leq \sup_x\{z : n(x, y, z) = n_*, \frac{1}{2}y_{**} \leq y \leq y_{**}\}$ for $t \in (\tau_1, \tau_2)$, so that $\tau_2 - \tau_1$ has a positive lower bound depending only on y_{**}. From our discussion of Figure 7.1 then follows

7.18. Theorem. *Let u and u_t be continuous, let n satisfy Hypothesis 3.9 with $0 < \alpha_+ < \alpha_-$, and for Problem DD let the hypotheses of Theorem 6.27 also hold. Then for each problem the pointwise lower bound* (5.5c) *holds.*

8. A Priori Estimates

In this section we use the bounds for u obtained in Sections 5–7 to derive a priori estimates for the L^2-norms of $v_t(\cdot, t)$, $v_{ss}(\cdot, t)$, and v_{st}. These will be used in Section 9 to prove the actual existence of a solution to our problem. At certain stages we use Hypothesis 3.21 in imitation of Dafermos's [**13**] use of (1.5b) to obtain estimates for Problem NN, which was the only problem he treated. Our existence theory, however, is quite different from his. (As we mentioned in Section 3, Hypothesis 3.21 suffers from none of the objections to (1.5b).) Alternatively, at least for Problem NN, we could follow the approach of Dafermos exactly. In particular, if we employ Hypothesis 3.20, which is more restrictive than (3.21), in place of Hypothesis 3.19, then his methods also deliver an upper bound for u. We do not pursue his approach, however, because it requires major modifications to handle other boundary conditions and because our methods seem more readily applicable to more complicated problems.

Notation. Throughout this and the next section, T is a fixed, arbitrary positive number. We denote the norm on a Banach space $\mathfrak{X}$ by $\|\cdot\|_{\mathfrak{X}}$, but omit the subscript for the norm on $L^2(0,1)$. If $\mathfrak{X}$ is a Banach space of functions on the interval $(0,1)$ and if $\mathfrak{Y}$ is a Banach space of real-valued functions on the interval $[0,T]$, then we denote by $\mathfrak{Y}(\to \mathfrak{X})$ the Banach space of measurable mappings $[0,T] \ni t \mapsto w(t) \in \mathfrak{X}$ with norm $\|t \mapsto \|w(t)\|_{\mathfrak{X}}\|_{\mathfrak{Y}}$. In particular, a norm of w in $L^2(\to H^1(0,1))$ is

$$\|w\|_{L^2(\to H^1(0,1))} = \sqrt{\int_0^T \int_0^1 [w(s,t)^2 + w_s(s,t)^2]\, ds\, dt}.$$

Let $\mathfrak{W}$ denote the subspace of $H^1(0,1)$ consisting of functions that vanish in the sense of trace at 0 if (2.7a) holds, and that vanish in the sense of trace at 1 if (2.7b) holds. Let $(\cdot,\cdot)$ be the inner product on $L^2(0,1)$. Let $\mathfrak{W}^*$ be the space dual to $\mathfrak{W}$ with the duality pairing $\langle\cdot,\cdot\rangle$ between $\mathfrak{W}^*$ and $\mathfrak{W}$ pivoted on $L^2(0,1)$. Thus $\langle\varphi,\omega\rangle = (\varphi,\omega)$ when $\varphi \in L^2(0,1)$ and $\omega \in \mathfrak{W}$.

Let

$$\delta_0 := \begin{cases} 0 & \text{if (2.7a) holds,} \\ 1 & \text{if (2.7b) holds,} \end{cases} \qquad \delta_1 := \begin{cases} 0 & \text{if (2.7c) holds,} \\ 1 & \text{if (2.7d) holds.} \end{cases} \tag{8.1}$$

$$a(s,t) := [(1-s)(1-\delta_0)v_0(t) + s(1-\delta_1)v_1(t)]. \tag{8.2}$$

Thus (2.7) implies that if $v(\cdot,t) \in H^1(0,1)$, then

$$v^\sharp(\cdot,t) := v(\cdot,t) - a(\cdot,t) \in \mathfrak{W}. \tag{8.3}$$

We now suppose throughout this and the following section that upper and lower pointwise bounds on u, of the sort derived in Sections 5–7, are available. Thus

$$0 < C(t)^{-1} \le u(s,t) < C(t) \tag{8.4}$$

for all $s \in [0,1]$ and all $t > 0$. Under Hypothesis 3.21 (which is weaker than Hypothesis 3.20) we then get a priori estimates for $\|v_t(\cdot,t)\|$, $\|v_{ss}(\cdot,t)\|$, and $\|v_{st}\|_{L^2(\Omega(T))}$, which must be satisfied by a sufficiently regular solution of our problem.

We assume that

$$f_t \in L^2(\to \mathfrak{W}^*), \tag{8.5}$$

$$v_{0tt},\ v_{1tt},\ n_{0t},\ n_{1t} \in L^2(0,T). \tag{8.6}$$

Thus $a_{tt}(s,\cdot) \in L^2(0,T)$ for each s.

Formally differentiating (2.6) with respect to t and using (2.5) and (8.3), we obtain

$$\rho v^\sharp_{tt} = [\sigma_z v_{ts} + n_y v_s]_s + f_t - \rho a_{tt}. \tag{8.7}$$

Multiplying (8.7) by $2v_t^\sharp$ and integrating the result by parts with respect to s we obtain:
(8.8)

$$\left(\|\sqrt{\rho}v_t^\sharp\|^2\right)_t = -2\int_0^1 v_{ts}^\sharp(\sigma_z v_{ts} + n_y v_s)\,ds + 2\langle f_t, v_t^\sharp\rangle - 2(\rho a_{tt}, v_t^\sharp) + 2[b_1(t) - b_0(t)]$$

where

$$(8.9)\quad b_0(t) := v_t^\sharp(0,t)n\big(0, u(0,t), v_s(0,t)\big)_t, \quad b_1(t) := v_t^\sharp(1,t)n\big(1, u(1,t), v_s(1,t)\big)_t.$$

We use the brackets for the action of f_t rather than parentheses because we do not restrict f_t to $L^2(\mathfrak{Q}(T))$. (We justify these formal procedures at the end of Section 9.)

Now for any $\theta > 0$ we have

$$(8.10)\quad \begin{gathered} -2(\rho a_{tt}, v_t^\sharp) \le \|\sqrt{\rho}a_{tt}\|^2 + \|\sqrt{\rho}v_t^\sharp\|^2, \\ 2\langle f_t, v_t^\sharp\rangle \le \|f_t\|_{\mathfrak{W}^*}\|v_t^\sharp\|_{\mathfrak{W}} \le \frac{1}{\theta}\|f_t\|^2_{\mathfrak{W}^*} + \theta\|v_t^\sharp\|^2_{\mathfrak{W}}, \\ \|v_t^\sharp\|^2_{\mathfrak{W}} = \|v_t^\sharp\|^2 + \|v_{ts}^\sharp\|^2 \le \frac{\|\sqrt{\rho}v_t^\sharp\|^2}{\min\rho} + \frac{1}{m}\int_0^1 \sigma_z {v_{ts}^\sharp}^2\,ds. \end{gathered}$$

We take $\theta \in (0, m)$. From (8.8) and (3.21) it then follows that

$$(8.11)\quad \begin{aligned} &\left(\|\sqrt{\rho}v_t^\sharp\|^2\right)_t + (2 - \tfrac{\theta}{m})\int_0^1 \sigma_z {v_{ts}}^2\,ds \\ &\quad \le \int_0^1 \sigma_z {v_{ts}}^2\,ds + \lambda(C(T))\|v_s\|^2 + \left(1 + \frac{\theta}{\min\rho}\right)\|\sqrt{\rho}v_t\|^2 \\ &\quad + \|\sqrt{\rho}a_{tt}\|^2 + \frac{1}{\theta}\|f_t\|^2_{\mathfrak{W}^*} + 2a_{ts}\int_0^1 (\sigma_z v_{ts} + n_y v_s)\,ds + 2[b_1(t) - b_0(t)], \end{aligned}$$

so that

$$(8.12)\quad \begin{aligned} &G(t) := \|\sqrt{\rho}v_t(\cdot,t)\|^2 + (1 - \tfrac{\theta}{m})\int_0^t\int_0^1 \sigma_z {v_{ts}}^2\,ds\,d\tau \\ &\quad \le \|\sqrt{\rho}v_t(\cdot,0)\|^2 + \lambda(C(T))\int_0^T \|v_s\|^2\,d\tau + M\int_0^t \|\sqrt{\rho}v_t\|^2\,d\tau + C(T) \\ &\quad + 2\int_0^t [(1-\delta_1)v_{1t} - (1-\delta_0)v_{0t}]\int_0^1 (\sigma_z v_{ts} + n_y v_s)\,ds\,d\tau + 2\int_0^t (b_1 - b_0)\,d\tau \end{aligned}$$

for $t \le T$. We assume that

$$(8.13)\quad \|\sqrt{\rho}v_t(\cdot,0)\|^2 = \left\|\frac{n_s(\cdot,\bar{u},\bar{v}_s) + f(\cdot,0)}{\sqrt{\rho}}\right\|^2 \le M,$$

(cf. (2.6)). It follows from (3.1) and from either Proposition 4.7 or Theorem 6.27 that $\lambda(C(T))\int_0^T \|v_s\|^2\,d\tau \le C(T)$. Thus the first three terms on the right-hand side of (8.12) are bounded. If we can show that there are $\varepsilon, \eta \in [0,1)$ such that

$$\begin{aligned}(8.14)\quad \int_0^t [(1-\delta_1)v_{1t} - (1-\delta_0)v_{0t}] \int_0^1 (\sigma_z v_{ts} + n_y v_s)\,ds\,d\tau + \int_0^t (b_1 - b_0)\,d\tau \\ \le \varepsilon G + C(T)\left[1 + G^\eta + \int_0^t G(\tau)\,d\tau\right],\end{aligned}$$

then we can use a modification of the Gronwall inequality to bound G on $[0,T]$. We would thus have bounds for v_t in $L^\infty(\to L^2(0,1))$ and in $L^2(\to H^1(0,1))$, which are critical for our subsequent analysis. We now establish (8.14) by different methods for each of the three classes of boundary conditions we are considering. Our treatment of the left-hand side of (8.14) parallels that of the boundary power in Section 4. (We could also carry out the treatment of Section 4 by using (8.3).)

Problem NN ($\delta_0 = 1 = \delta_1$). Integrating the inequality $|v_t(s,t)| \le |v_t(\xi,t)| + \left|\int_0^1 |v_{ts}(\eta,t)|\,d\eta\right|$ with respect to ξ over $(0,1)$, using the Cauchy-Bunyakovskiĭ-Schwarz inequality, and finally using (3.1) we obtain

$$(8.15)\quad |v_t^\sharp(s,t)| = |v_t(s,t)| \le \|v_t\| + \|v_{ts}\| \le \frac{\|\sqrt{\rho}v_t\|}{\min\{\sqrt{\rho}\}} + \frac{1}{\sqrt{m}}\left[\int_0^1 \sigma_z {v_{ts}}^2\,ds\right]^{1/2}.$$

Since $a = 0$, conditions (2.7b,d) and (8.9) imply that (8.14) holds with $\varepsilon = 0$, $\eta = 0$.

Problem ND. We study (2.7b,c), so that $\delta_0 = 1, \delta_1 = 0$ The left-hand side of (8.14) reduces to $\int_0^t v_{1t}\int_0^1(\sigma_z v_{ts} + n_y v_s)\,ds\,d\tau - \int_0^t v_t(0,t)n_{0t}\,d\tau$, which is dominated by $C(T)[1 + \int_{\Omega(t)}(\sigma_z v_{ts} + n_y v_s)\,ds\,d\tau]$ by virtue of by (8.15). Now

$$\begin{aligned}&\int_{\Omega(t)} (\sqrt{\sigma_z}v_s + \sigma_z v_{st})\,ds\,d\tau \\ (8.16)\quad &\le \sqrt{\int_{\Omega(t)} \sigma_z {v_s}^2\,ds\,d\tau} + \sqrt{\int_{\Omega(t)} \sigma_z\,ds\,d\tau}\sqrt{\int_{\Omega(t)} \sigma_z {v_{st}}^2\,ds\,d\tau} \\ &\le \sqrt{W(t)} + M\sqrt{\int_{\Omega(t)} \sigma_z\,ds\,d\tau}\sqrt{G(t)}.\end{aligned}$$

By Proposition 4.7, $W \le C(T)$. Wherever $|v_s| \le 1$, our bounds on u ensure that $\sigma_z \le C(T)$. Elsewhere, $\sigma_z \le \sigma_z {v_s}^2$. Thus the right-hand side of (8.16) is dominated by $C(T)[1 + \sqrt{G(t)}]$, and (8.14) holds with $\varepsilon = 0$ and $\eta = \frac{1}{2}$.

Problem DD ($\delta_0 = 0 = \delta_1$). The left-hand side of (8.14) here reduces to $\int_0^t (v_{1t} - v_{0t})\int_0^1(\sigma_z v_{ts} + n_y v_s)\,ds\,d\tau$, which is treated exactly as in Problem ND, except that Proposition 6.27 is used in place of Proposition 4.7. Note that if $v_{0t} = v_{1t}$, then the left-hand side of (8.14) is 0.

We summarize these results:

8.17. Proposition. *Let T be any positive number and let (u, v) be a solution of (2.5)–(2.9) on $\mathfrak{Q}(T)$ that is smooth enough for the time integral of (8.8) to make sense, for u to be continuous, and for (8.4) to hold. Let (3.1), (8.5), (8.6), (8.13) hold. Let Hypothesis 3.20 hold or let $u < C(T)$ and Hypothesis 3.21 hold. Then $G < C(T)$. This $C(T)$ depends only on T, the constitutive functions, and the bounds for the data.*

To proceed further, we invoke Hypothesis 3.22, which we shall use in conjunction with the following estimate, obtained just as in (8.15):

$$|v_s(s,t)| \le \|v_s\| + \frac{1}{\sqrt{3m}} \left[\int_0^1 \sigma_z {v_{ss}}^2 \, ds \right]^{1/2}. \tag{8.18}$$

8.19. Corollary. *Let Hypothesis 3.22 and the hypotheses of Proposition 8.17 hold. Let $\bar{u}_s \in L^2(0,1)$. If $n_x \neq 0$, let $u < C(T)$. Then*

$$m\|v_{ss}\|^2 \le \int_0^1 \sigma_z {v_{ss}}^2 \, ds < C(T) \tag{8.19}$$

for $t \le T$.

Proof. Since $\sigma_z v_{ss} = \rho v_t - f - n_x - n_y u_s$ and $u_s(s,t) = \bar{u}_s(s) + \int_0^t v_{ss}(s,\tau)\, d\tau$, it follows from (3.1), (3.20) or (3.21), (3.22), (8.18) that

(8.20)
$$\begin{aligned} \sigma_z {v_{ss}}^2 &= (\sqrt{\sigma_z} v_{ss}) \frac{1}{\sqrt{\sigma_z}} (\rho v_t - f - n_x - n_y u_s) \\ &\le \tfrac{1}{2} \sigma_z {v_{ss}}^2 + \tfrac{1}{2} \left[\frac{1}{\sqrt{m}} (\rho |v_t| + |f| + |n_x|) + \sqrt{\lambda} \left(|\bar{u}_s| + \int_0^t |v_{ss}| \, d\tau \right) \right]^2. \end{aligned}$$

Thus (3.22), (8.18) yield

$$\begin{aligned} \sigma_z {v_{ss}}^2 \le & \frac{6}{m} \left[\rho {v_t}^2 + f^2 + C(T) + C(T) \left(\int_0^1 \sigma_z {v_{ss}}^2 \, ds \right)^{p/2} \right] \\ & + 4\lambda \left[{\bar{u}_s}^2 + T \int_0^t {v_{ss}}^2 \, d\tau \right]. \end{aligned} \tag{8.21}$$

Integrating (8.21) with respect to s over $(0,1)$ and using the bound on $\|\rho v_t\|^2$ given by Proposition 8.17, we obtain

$$\int_0^1 \sigma_z {v_{ss}}^2 \, ds \le C(T) \left[1 + \left(\int_0^1 \sigma_z {v_{ss}}^2 \, ds \right)^{p/2} + \int_0^t \int_0^1 \sigma_z {v_{ss}}^2 \, ds \, d\tau \right]. \tag{8.22}$$

We obtain the desired bound by using a version of the Gronwall inequality. □

In our next step we make use of the following result proved independently in [**39**] and [**40**]:

8.23. Lemma. *Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be Banach spaces of functions with $\mathfrak{X}$ compactly embedded in $\mathfrak{Y}$ and with $\mathfrak{Y}$ embedded in $\mathfrak{Z}$. Let $\mathfrak{C}$ be a set of functions w for which w_t lies in a bounded subset of $L^p(\to \mathfrak{Z})$ with $p > 1$ and for which w lies in a bounded subset of $L^\infty(\to \mathfrak{X})$. Then $\mathfrak{C}$ lies in a compact subset of $C^0(\to \mathfrak{Y})$.*

We can now prove

8.24. Corollary of Proposition 8.17. *Let the hypotheses of Proposition 8.17 and Corollary 8.19 hold. Then u and $u_t = v_s$ lie in compact subsets of $C^0(\mathfrak{Q}(T))$, which depend only on T, the constitutive functions, and the bounds for the data.*

Proof. From (8.18), Corollary 8.19, and the Arzelà-Ascoli Theorem it follows that $v_s(\cdot, t)$ lies in a fixed compact subset of $C^0[0,1]$. Proposition 8.17 yields an L^2 bound on v_{st}. That v_s lies in compact subsets of $C^0(\mathfrak{Q}(T))$ then follows from Lemma 8.23. (We take $p = 2$ and identify $\mathfrak{X} = H^1(0,1)$, $\mathfrak{Y} = C^0[0,1]$, $\mathfrak{Z} = L^2(0,1)$. Actually, embedding theory allows us to take $\mathfrak{Y} = C^{0,\alpha}[0,1]$ with $\alpha \in (0, \frac{1}{2})$, and thereby strengthen this lemma by replacing $C^0(\mathfrak{Q}(T))$ with $C^0(\to C^{0,\alpha}[0,1])$.) The corresponding result for u comes from integrating (2.5) with respect to t. (Note that this last step gives an upper bound for u in those cases in which such a bound had not been derived previously as a consequence of Hypothesis 3.19 and in which the existence of such a bound is not hypothesized in Proposition 8.17 or Corollary 8.19. Thus this bound is valid in certain cases in which Hypothesis 3.20 holds, but Hypothesis 3.19 does not.) □

9. Global Existence Theory

Since we have just shown that there is a $C(T)$ such that $C(T)^{-1} \le u \le C(T)$, $|v_s| \le C(T)$, only the restriction of $n(s,\cdot,\cdot)$ to the corresponding values of the arguments y and z enters our initial-boundary-value problem for $t \le T$. We exploit this fact in constructing our existence theory. For this fixed $C(T)$ we first define the cut-off functions $[\cdot]$ and $[[\cdot]]$ by

$$
(9.1) \qquad [y] := \begin{cases} \gamma^{-1} & \text{if} \quad y \le \gamma^{-1}, \\ \gamma & \text{if} \quad y \ge \gamma, \\ y & \text{otherwise}, \end{cases} \quad , \quad [[z]] := \begin{cases} -\gamma & \text{if} \quad z \le -\gamma, \\ \gamma & \text{if} \quad z \ge \gamma, \\ z & \text{otherwise} \end{cases}
$$

where $\gamma = 2C(T)$. For $x \in [0,1]$, $y, z \in \mathbb{R}$, we define

$$
(9.2) \qquad N(x,y,z) := \phi_y(x,[y]) + \int_0^z \sigma_z(x,[y],[[\zeta]])\,d\zeta.
$$

Note that $N(x,\cdot,\cdot)$ is uniformly Lipschitz continuous, that $N_z \ge m$, and that $N(\cdot, U(\cdot,\cdot), Z(\cdot,\cdot)) \in L^2(\mathfrak{Q}(T))$ if $U(\cdot,\cdot)$, $Z(\cdot,\cdot) \in L^2(\mathfrak{Q}(T))$. We now show that our initial-boundary-value problem in which N replaces n has a sufficiently regular solution. Since $N(x,\cdot,\cdot)$ agrees with $n(x,\cdot,\cdot)$ for $C(T)^{-1} \le y \le C(T)$, $|z| \le C(T)$, we can identify this solution with that of our original problem.

Recall that if $V(\cdot,t) \in H^1(0,1)$, then

$$
(9.3) \qquad V(\cdot,t) - a(\cdot,t) \in \mathfrak{W}.
$$

Let us define the operator $\mathcal{N}$ by

$$\langle \mathcal{N}(U(\cdot,t),V(\cdot,t)),\eta\rangle := \int_0^1 N(s,U(s,t),V_s(s,t))\eta_s(s)\,ds \tag{9.4}$$

for all $\eta \in \mathfrak{W}$. By multiplying (2.6) by $\eta \in \mathfrak{W}$ and using the boundary conditions to integrate the resulting expression by parts over $(0,1)$, we formally obtain the (spatially) weak form of the momentum equation (2.6) for N:

$$\langle \rho V_t(\cdot,t),\eta\rangle + \langle \mathcal{N}(U(\cdot,t),V(\cdot,t)),\eta\rangle = \langle f,\eta\rangle - \delta_0 n_0(t)\eta(0) - \delta_1 n_1(t)\eta(1) \tag{9.5}$$

for all $\eta \in \mathfrak{W}$. We supplement this with the integral form of (2.5):

$$U(s,t) = \bar{u}(s) + \int_0^t V_s(s,\tau)\,d\tau, \tag{9.6}$$

which is more convenient than the weak form of (2.5).

We now prove

9.7. Theorem. *Let T be a fixed positive number. Let $\bar{u},\bar{v} \in H^1(0,1)$ and $f_t \in L^2(\to \mathfrak{W}^*)$ be prescribed. Let the hypotheses on the initial and boundary data given in the compatibility condition (2.10), in Proposition 8.17, and in Corollary 8.19 hold. Then there is a unique pair U,V with*

$$V - a \in C^0(\to C^1[0,1]) \quad \text{with} \quad (V-a)_t \in L^2(\to \mathfrak{W}) \tag{9.8}$$

satisfying (9.6) and satisfying (9.5) in the sense that

$$\int_{s_1}^{s_2} \rho V_t(s,t)\,ds = N(s,U(s,t),V_s(s,t))\Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} f(s,t)\,ds \tag{9.9}$$

for almost every $s_1,s_2 \in (0,1)$ and for almost every $t \in [0,T]$. Moreover, the initial conditions and boundary conditions are satisfied pointwise.

Proof. Step 1. Galerkin's method. Let $\{\lambda_i\}$, with $0 \le \lambda_1 < \lambda_2 < \cdots$, be the eigenvalues, and let $\{\eta_i\}$ be the corresponding eigenfunctions for the eigenvalue problem $\eta'' + \lambda\rho\eta = 0$ on $(0,1)$ with homogeneous Dirichlet or Neumann conditions corresponding to (2.7). We take $(\rho\eta_i,\eta_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, in which case $(\eta_{is},\eta_{js}) = \lambda_i\delta_{ij}$. Then $\{\eta_i\}$ is a basis both for $\mathfrak{W}$ and for $L^2(0,1)$.

We seek approximate solutions of (9.5), (9.6) in the form

$$\begin{gathered}\tilde{U}^k(s,t) := \bar{u}(s) + \int_0^t a_s(s,\tau)\,d\tau + U^k(s,t), \quad U^k(s,t) := \sum_{i=1}^k U_i(t)\eta_{is}(s), \\ \tilde{V}^k(s,t) := a(s,t) + V^k(s,t) := a(s,t) + \sum_{i=1}^k V_i(t)\eta_i(s).\end{gathered} \tag{9.10}$$

We replace U, V in (9.5), (9.6) with $\tilde{U}^k, \tilde{V}^k$ of (9.10), and replace η with η_j to obtain

$$(9.11a) \qquad U_{jt} = V_j,$$

(9.11b)

$$V_{jt} \equiv \langle \rho V_t^k, \eta_j \rangle = -\int_0^1 N(s, \tilde{U}^k(s,t), \tilde{V}_s^k(s,t))\eta_{js}(s)\,ds + \langle f, \eta_j \rangle - \int_0^1 \rho a_t \eta_j\,ds - \delta_0 n_0(t)\eta_j(0) - \delta_1 n_1(t)\eta_j(1)$$

for $j = 1, \dots, k$. Motivated by (9.10) we take initial conditions

$$(9.12a,b) \qquad \lambda_i U_i(0) = 0, \quad V_i(0) = (\bar{v} - a(\cdot, 0), \eta_i).$$

(For Problem NN, $\lambda_1 = 0$ and η_1 is constant. Thus (9.12a) for $i = 1$ is identically satisfied, which causes no difficulty because U_1 does not enter into (9.10).)

The properties of N and the hypotheses on the data ensure that the right-hand sides of (9.11) satisfy the Carathéodory conditions. Standard theory of ordinary differential equations then implies that there is a $T_k > 0$ such that the initial-value problem (9.11), (9.12) has an absolutely continuous solution $(U_1, \dots, U_k, V_1, \dots, V_k)$ on $[0, T_k]$. We now get an energy estimate that will allow us to take $T_k = T$ and to show that there are suitable weakly convergent subsequences of the solutions of (9.11).

Step 2. Estimates. We multiply (9.11b) by V_j and sum the resulting expression from $j = 1$ to $j = k$ to obtain

$$(9.13) \quad \frac{1}{2}\frac{d}{dt}\int_0^1 \rho (V^k)^2\,ds + \int_0^1 N(s, \tilde{U}^k(s,t), \tilde{V}_s^k(s,t))\tilde{V}_s^k(s)\,ds$$
$$= \langle f(\cdot, t), V^k \rangle - \int_0^1 \rho(s) a_t(s,t) V^k(s,t)\,ds + \int_0^1 N(s, \tilde{U}^k(s,t), \tilde{V}_s^k(s,t)) a_s(s,t)\,ds - \delta_0 n_0(t) V^k(0,t) - \delta_1 n_1(t) V^k(1,t).$$

We now use the techniques of Section 4, without any of the difficulties associated with total compression (which are absent because of the properties of N), to get an energy estimate that implies that

$$(9.14) \qquad \max_{t \in [0,T]} \|V^k\| \le C(T), \qquad \int_0^T \int_0^1 (V_s^k)^2\,ds\,dt \le C(T)$$

where $C(T)$ is independent of k. The consequent boundedness of $U_1, \dots, U_k$ and of $V_1, \dots, V_k$ enables us to apply the continuation theory for ordinary differential equations to show that solutions exist on the entire interval $[0, T]$.

We now get further a priori estimates. Fix $\xi \in \mathfrak{W}$ with $\|\xi\|_{\mathfrak{W}} \le 1$. Let ξ, as an element of the space of square-integrable functions with weight ρ, have the orthogonal projection η onto $\operatorname{span}\{\eta_1, \dots, \eta_k\}$, so that $(\rho\xi, \eta_j) = (\rho\eta, \eta_j)$ for $j =$

$1, \dots, k$, and $(\rho\eta, \eta_j) = 0$ for $j > k$. Since the form of V^k ensures that $\langle \rho V_t^k, \xi \rangle = (\rho V_t^k, \xi) = (\rho V_t^k, \eta)$, we obtain from (9.11b) that

$$\begin{aligned} \langle \rho V_t^k, \xi \rangle = &- \int_0^1 N(s, \tilde{U}^k(s,t), \tilde{V}_s^k(s,t))\eta_s(s)\, ds + \langle f(\cdot, t), \eta \rangle \\ &- \int_0^1 \rho(s) a_t(s,t)\eta(s)\, ds - \delta_0 n_0(t)\eta(0) - \delta_1 n_1(t)\eta(1). \end{aligned} \tag{9.15}$$

By the properties of N, by the Sobolev Embedding Theorem (to control $\eta(0)$ and $\eta(1)$), and by (9.14) we obtain

$$\begin{aligned} |\langle \rho V_t^k, \xi \rangle| \le &\int_0^1 |N(s,1,0)||\eta_s(s)|\, ds + \int_0^1 \left[|N_y||\tilde{U}^k(s,t) - 1| + N_z|\tilde{V}_s^k(s,t)| \right] |\eta_s(s)|\, ds \\ &+ \|f(\cdot,t)\|_{\mathfrak{W}^*}\|\eta\|_{\mathfrak{W}} + C(t)\|\eta\| + C(t)|\eta(0)| + C(t)|\eta(1)| \\ &\le C(T)\left[1 + \|V^k(\cdot,t)\|_{\mathfrak{W}^*}\right]\|\eta\|_{\mathfrak{W}} \le C(T)\left[1 + \|V^k(\cdot,t)\|_{\mathfrak{W}^*}\right]\|\xi\|_{\mathfrak{W}}. \end{aligned} \tag{9.16}$$

In the second integral of (9.16) the partial derivatives of N are evaluated at intermediate values. It then follows from (9.14) that

$$\|V_t^k\|_{L^2(\to\mathfrak{W}^*)} \equiv \sup_{\xi \ne 0} \frac{|\langle V_t^k, \xi \rangle|}{\|\xi\|_{\mathfrak{W}}} \le C(T). \tag{9.17}$$

In the process of deducing (9.16) we demonstrated that

$$\|\mathcal{N}(\tilde{U}^k(\cdot,t), \tilde{V}^k(\cdot,t))\|_{\mathfrak{W}^*} \le C(T)\left[1 + \|V^k(\cdot,t)\|_{\mathfrak{W}}\right]. \tag{9.18a}$$

Thus (9.14) implies that

$$\|\mathcal{N}(\tilde{U}^k, \tilde{V}^k)\|_{L^2(\to\mathfrak{W}^*)} \le C(T). \tag{9.18b}$$

The bounds we have obtained so far would suffice to give us the existence and uniqueness for the quasilinear parabolic problem (9.5) in which N is independent of y. Our next estimates are used to handle the dependence of N on y. We first observe that the properties of N, the hypotheses on the data, and the absolute continuity of the solution to (9.11), (9.12) ensure that the right-hand sides of (9.11) are absolutely continuous with respect to t. Thus the U_j and the V_j have absolutely continuous first derivatives. We can accordingly differentiate (9.11) with respect to t obtaining

$$\begin{aligned} V_{jtt} \equiv \langle \rho V_{tt}^k, \eta_j \rangle = &- \int_0^1 \left[N_y(s, \tilde{U}^k, \tilde{V}_s^k)\tilde{V}_s^k + N_z(s, \tilde{U}^k, \tilde{V}_s^k)\tilde{V}_s^k \right] \eta_{js}\, ds \\ &+ \langle f_t, \eta_j \rangle - \int_0^1 \rho a_{tt}\eta_j\, ds - \delta_0 n_{0t}(t)\eta_j(0) - \delta_1 n_{1t}(t)\eta_j(1). \end{aligned} \tag{9.19}$$

We multiply (9.19) by V_{jt} and sum the resulting expression over j to obtain an equation which contains the expression $\langle \rho V_{tt}^k, V_t^k \rangle \equiv \frac{d}{dt}\int_0^1 \rho (V_t^k)^2\, ds$. This equation

is completely analogous to (8.8). By following the techniques leading to Proposition 8.17 and Corollary 8.19 we find that
(9.20a,b,c)

$$\max_{t\in[0,T]} \|V_t^k\| \le C(T), \quad \int_0^T \int_0^1 (V_{ts}^k)^2\, ds\, dt \le C(T), \quad \max_{t\in[0,T]} \|V_{ss}^k\| \le C(T).$$

Step 3. Convergence. The bounds of (9.14) say that V^k is a bounded sequence in $C^0(\to L^2(0,1)) \cap L^2(\to \mathfrak{W})$. The bound (9.17) implies that V^k is a bounded sequence in $H^1(0,T:\mathfrak{W}^*)$. The bound (9.20a) says that V_t^k is a bounded sequence in $C^0(\to L^2(0,1))$, so that V^k is a bounded sequence in $C^{0,1}(\to L^2(0,1)) \subset H^1(\to L^2(0,1))$. The bound (9.20b) says that V_{st}^k is a bounded sequence in $L^2(\to L^2(0,1))$, so that V^k is a bounded sequence in $H^1(\to H^1(0,1))$. The bound (9.20c) says that V_{ss}^k is a bounded sequence in $C^0(\to L^2(0,1))$, so that V^k is a bounded sequence in $L^\infty(\to H^2(0,1)) \subset L^2(\to H^2(0,1))$. Thus

$$V^k \text{ is a bounded sequence in } H^1(\to \mathfrak{W}) \cap C^0(\to H^2(0,1)). \tag{9.21a}$$

From (9.21a) and (9.18b), from the weak compactness of reflexive Banach spaces, from Alaoglu's Theorem, and from the distributional definition of derivatives in the function spaces we are using follows the existence of a V with

$$V - a \in H^1(\to \mathfrak{W}) \cap C^0(\to H^2(0,1)), \tag{9.21b}$$

of a $\chi \in L^2(\to \mathfrak{W}^*)$, and of a subsequence of $\{V^k\}$ (denoted the same way) such that

$$V^k \rightharpoonup V - a \quad \text{in} \quad H^1(\to \mathfrak{W}), \tag{9.22a}$$

$$V^k \overset{*}{\rightharpoonup} V - a \quad \text{in} \quad C^0(\to H^2(0,1)), \tag{9.22b}$$

$$\mathcal{N}(\tilde{U}^k, \tilde{V}^k) \rightharpoonup \chi \qquad \text{in} \quad L^2(\to \mathfrak{W}^*). \tag{9.22c}$$

These results induce via (9.10) and (9.11) a corresponding convergence of $\tilde{U}^k$ to U. In particular, (9.22a) implies that

$$U^k \rightharpoonup U - \bar{u} - \int_0^t a_s\, d\tau \quad \text{in} \quad H^1(\to H^1(0,1)), \tag{9.23}$$

Since $H^1(\to H^1(0,1))$ is compactly embedded in $L^2(\mathfrak{Q}(T))$, we obtain from (9.23) the strong convergence

$$U^k \to U - \bar{u} - \int_0^t a_s\, d\tau \quad \text{i.e.,} \quad \tilde{U}^k \to U \quad \text{in} \quad L^2(\mathfrak{Q}(T)). \tag{9.24}$$

To exploit these convergences in spaces of time-dependent functions, we need to use time-dependent test functions. Let us fix an integer l, let $\gamma_1 \dots \gamma_l \in C^1(0,T)$, and let $\gamma(s,t) = \sum_{i=1}^{l} \gamma_i(t)\eta_i(s)$. Such functions γ are dense in $L^2(\to \mathfrak{W})$. From (9.11b) we obtain

$$\int_0^T \langle \rho(V_t^k + a_t), \gamma\rangle \, dt + \int_0^T \int_0^1 N(s, \tilde{U}^k, \tilde{V}_s^k)\gamma \, ds \, dt \\ = \int_0^T \langle f, \gamma\rangle \, dt - \int_0^T [\delta_0 n_0(t)\gamma(0,t) + \delta_1 n_1(t)\gamma(1,t)] \, dt \tag{9.25}$$

for all γ in $L^2(\to \mathfrak{W})$. Taking the subsequential limit in (9.25), we obtain from (9.22) that

$$\int_0^T \langle \rho V_t, \gamma\rangle \, dt + \int_0^T \langle \chi, \gamma\rangle \, dt \\ = \int_0^T \langle f, \gamma\rangle \, dt - \int_0^T [\delta_0 n_0(t)\gamma(0,t) + \delta_1 n_1(t)\gamma(1,t)] \, dt \tag{9.26}$$

for all γ in $L^2(\to \mathfrak{W})$. In particular, if we chose $\gamma = \gamma_i\eta_i$, then each integrand in (9.26) has the arbitrary function γ_i as a factor. We can thus use the Fundamental Lemma of the Calculus of Variations to obtain

$$\langle \rho V_t, \eta\rangle = -\langle \chi, \eta\rangle + \langle f, \eta\rangle - \delta_0 n_0(t)\eta(0) - \delta_1 n_1(t)\eta(1) \tag{9.27}$$

for $\eta = \eta_i$ and therefore for any $\eta \in \mathfrak{W}$.

To show that (U,V) satisfies its partial differential equations we must show that $\chi = \mathcal{N}(U,V)$. Recalling (9.4) we set

$$X^k := \int_0^T \langle \mathcal{N}(\tilde{U}^k, \tilde{V}^k) - \mathcal{N}(\tilde{U}^k, \gamma + a), V^k - \gamma\rangle \, dt \tag{9.28}$$

for γ in $L^2(\to \mathfrak{W})$. Since $N_z > m$, it follows that $X^k \geq 0$. In (9.25) we choose $\gamma = V^k$ to obtain an expression for $\int_0^T \langle \mathcal{N}(\tilde{U}^k, \tilde{V}^k), V^k\rangle \, dt$, which we substitute into (9.28) to get

$$\begin{aligned} 0 \leq X^k =& \tfrac{1}{2}\|\sqrt{\rho}V^k(\cdot,0)\|^2 - \tfrac{1}{2}\|\sqrt{\rho}V^k(\cdot,T)\|^2 \\ &+ \int_0^T \langle f, V^k\rangle \, dt - \int_0^T \int_0^1 \rho a_t V^k \, ds \, dt \\ &- \int_0^T [\delta_0 n_0(t)V^k(0,t) + \delta_1 n_1(t)V^k(1,t)] \, dt \\ &- \int_0^T \langle \mathcal{N}(\tilde{U}^k, \tilde{V}^k), \gamma\rangle \, dt - \int_0^T \langle \mathcal{N}(U, \gamma + a), V^k - \gamma\rangle \, dt \\ &- \int_0^T \langle \mathcal{N}(\tilde{U}^k, \gamma + a) - \mathcal{N}(U, \gamma + a), V^k - \gamma\rangle \, dt. \end{aligned} \tag{9.29}$$

We use (9.12b) to replace $V^k(\cdot,0)$ with $\sum_{i=1}^{k}(\bar{v} - a_s(\cdot,0), \eta_i)\eta_i$, which converges strongly to $\bar{v} - a_s(\cdot,0)$ in $L^2(0,1)$. Now we take the subsequential limit of (9.29) using (9.22), using the fact that norms are weakly lower semicontinuous, so that $\|\sqrt{\rho}V(\cdot,T)\| \le \liminf \|\sqrt{\rho}V^k(\cdot,T)\|$, and using (9.24) to show that the last term of (9.29) approaches 0, to obtain

$$\begin{aligned} 0 \le &\tfrac{1}{2}\|\sqrt{\rho}[\bar{v} - a_s(\cdot,0)]\|^2 - \tfrac{1}{2}\|\sqrt{\rho}[V(\cdot,T) - a(\cdot,T)]\|^2 \\ &+ \int_0^T \langle f, V-a\rangle\,dt - \int_0^T\int_0^1 \rho a_t(V-a)\,ds\,dt \\ &- \int_0^T \{\delta_0 n_0(t)[V(0,t)-a(0,t)] + \delta_1 n_1(t)[V(1,t)-a(1,t)]\}\,dt \\ &- \int_0^T \langle \chi,\gamma\rangle\,dt - \int_0^T \langle \mathcal{N}(U,\gamma+a), V-a-\gamma\rangle\,dt. \end{aligned} \tag{9.30}$$

Let us now take $\gamma = V - a$ in (9.26). We substitute the resulting equation into (9.30) to obtain

$$0 \le \int_0^T \langle \chi - \mathcal{N}(U,\gamma+a), V-a-\gamma\rangle\,dt. \tag{9.31}$$

Choosing $\gamma = V - a - \lambda\zeta$ where $\lambda > 0$ and ζ is an arbitrary element of $L^2(\to \mathfrak{W}^*)$, we obtain from (9.31) that

$$0 \le \int_0^T \langle \chi - \mathcal{N}(U, V-\lambda\zeta),\zeta\rangle\,dt. \tag{9.32}$$

Let $\lambda \to 0$. The properties of N allow us to use the Lebesgue Dominated-Convergence Theorem to obtain $0 \le \int_0^T \langle \chi - \mathcal{N}(U,V),\zeta\rangle\,dt$ for all ζ, and this implies that $\chi = \mathcal{N}(U,V)$, as desired.

To prove (9.9) we replace γ in (9.26) with

$$\eta = \begin{cases} 0 & \text{for} 0 \le s \le s_1, \\ (s-s_1)/\varepsilon & \text{for} s_1 \le s \le s_1+\varepsilon, \\ 1 & \text{for} s_1+\varepsilon \le s \le s_2-\varepsilon, \\ (s_2-s)/\varepsilon & \text{for} s_2-\varepsilon \le s \le s_2, \\ 0 & \text{for} s_2 \le s \le 1 \end{cases}$$

where ε is sufficiently small. By using the Lebesgue Dominated Convergence Theorem to take the limit in the resulting form of (9.26) as $\varepsilon \to 0$ we obtain (9.9).

Step 4. Regularity. Note that (9.21b) gives much of the regularity needed to justify the analysis given at the beginning of this section. Let us identify $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ of Lemma 8.23 with $H^2(0,1)$, $C^1[0,1] \cap \mathfrak{W}$ (or $C^{1,\alpha}[0,1] \cap \mathfrak{W}$ with $\alpha \in (0,\frac{1}{2})$), $\mathfrak{W}$. Then Lemma 8.23 and (9.21b) imply that $V - a \in C^0(\to C^1[0,1] \cap \mathfrak{W})$. This fact ensures that V satisfies whatever Dirichlet conditions are prescribed. Since $v_0, v_1 \in H^1(0,T)$, it follows that $a \in C^0(\to C^1[0,1])$, so that $V \in C^0(\to C^1[0,1])$. Thus

$$U_t = V_s \in C^0(\mathfrak{Q}(T)). \tag{9.33}$$

This and the continuity of $\bar{u}$ ensure the continuity of U corresponding to that of u, which was basic for the development in Sections 5 and 6, and ensure the continuity of U_t corresponding to that of u_t, which was basic for the development in Section 7.

Step 5. Initial and Neumann Conditions. We integrate the first terms of (9.25) and (9.26) by parts with respect to t, and then subtract the resulting equation. We get an equation of the form

$$\big(\rho[V^k(\cdot,0)+a(\cdot,0)-\rho V(\cdot,0)],\gamma\big)=R^k. \tag{9.34}$$

Since (9.12b) says that $V^k(\cdot,0)+a(\cdot,0)$ converges in $L^2(0,1)$ to $\bar{v}$, we can use the results obtained in Step 4 to take the subsequential limit of (9.34) to show that $R^k \to 0$ and thus that $V(\cdot,0)=\bar{v}$. That U satisfies its initial condition is accounted for by (9.11a) and (9.12a).

To handle the Neumann boundary conditions, we integrate the second term of (9.5) by parts with respect to s and substitute (9.9) into the resulting equation to cancel most of its terms. The allowable arbitrariness of η at 0 and 1 gives whatever Neumann conditions are prescribed.

Step 6. Uniqueness. We let $(U_{(1)},V_{(1)})$ and $(U_{(2)},V_{(2)})$ be two solutions of our problem. From (9.26) we then find that

$$\int_0^T \langle \rho(V_{(1)}-V_{(2)})_t, V_{(1)}-V_{(2)}\rangle\,dt+\int_0^T\langle \mathcal{N}(U_{(1)},V_{(1)})-\mathcal{N}(U_{(2)},V_{(2)}),V_{(1)}-V_{(2)}\rangle\,dt=0. \tag{9.35}$$

We follow the techniques of Sections 4 and 8 to obtain an inequality that can be handled by the Gronwall inequality to yield that $(U_{(1)},V_{(1)})=(U_{(2)},V_{(2)})$. □

In view of the remarks at the end of the paragraph containing (9.2) we have proved that our original initial-boundary-value problem has a unique solution with the indicated regularity.

The proof of Theorem 9.7 is roughly modelled on the development of Lions [**30**, Chap. 2]. The chief difference is our need to account for elastic response, which causes our equations to form a hyperbolic-parabolic system instead of a parabolic equation. To handle the concomitant difficulties, we had to obtain the bounds (9.20), leading to (9.24), which would have been otherwise unnecessary for the existence theory. (We needed these further bounds, however, for our regularity theory.) The culmination of Lions' treatment is his abstract Theorem 2.1.2, a slightly more general version of which is Theorem III.4.2 of [**9**], which is proved by semigroup methods. We can apply these abstract theorems to our problem for U,V in which U is held fixed, to get a solution V of (9.5) depending on U. We can then use the Contraction Mapping Principle with carefully chosen norms to prove the existence of a unique solution of (9.5), (9.6) under slightly weaker hypotheses than those used above. We would then have to use our stronger hypotheses to demonstrate the requisite regularity. The approach we used above incorporates the regularity theory more naturally and efficiently into the existence theory.

10. Pointwise Bounds on the Strains for Elastic Bodies

We now show that for the hyperbolic equations of nonlinear elasticity, for which $\sigma=0$, we get pointwise bounds on u under a constitutive restriction like (3.6). To ensure hyperbolicity, we assume that $n_y(\cdot,\cdot,0):=\phi_{yy}>0$. Then (2.5), (2.6) reduces to

$$\begin{pmatrix} u \\ \rho(s)v\end{pmatrix}_t-\begin{pmatrix}0 & 1\\ \phi_{yy}(s,u) & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_s=\begin{pmatrix}0\\ \phi_{xy}(s,u)+f(s,t)\end{pmatrix}. \tag{10.1}$$

The Riemann invariants for the left-hand side of (10.1) are

$$r^{\pm}(s,t) = v(s,t) \mp \int_1^{u(s,t)} \sqrt{\frac{\phi_{yy}(s,y)}{\rho(s)}}\, dy. \tag{10.2}$$

They satisfy the partial differential equations

$$r_t^{\pm} \pm \sqrt{\frac{\phi_{yy}}{\rho(s)}}\, r_s^{\pm} = \frac{\phi_{xy}+f}{\rho} \pm \frac{\sqrt{\phi_{yy}}}{2\rho^2} \int_1^u \frac{\rho(s)\phi_{yyx}(s,y) - \rho_s \phi_{yy}(s,y)}{\sqrt{\phi_{yy}(s,y)}}\, dy. \tag{10.3}$$

Note that our assumptions ensure that we can solve (10.2) uniquely for u and v in terms of $r^{\pm}$. It is these representations that we substitute into the right-hand side of (10.3).

Now we could use results like those obtained by **[19]**, **[34]**, **[35]** and others who show how Riemann invariants are controlled in a shock. It is more convenient, however, to exploit an idea of **[46]**. Suppose, for simplicity of exposition, that there is no time-dependent forcing, i.e., we take the right-hand sides of (2.7) and we take f to be independent of t. We extend our boundary-value problem to all space by periodicity, thus obtaining an initial-value problem with periodic initial data. Now we add $\varepsilon r_{ss}^{\pm}$ where $\varepsilon > 0$ to the right-hand side of (10.3) to make each of its equations parabolic. We apply the Maximum Principle (for weakly coupled parabolic systems) to the resulting equations to get bounds independent of ε on the Riemann invariants in terms of the initial data. In particular,

$$\int_1^{u(s,t)} \sqrt{\frac{\phi_{yy}(s,y)}{\rho(s)}}\, dy = \frac{r^-(s,t) - r^+(s,t)}{2} \geq \frac{r^-(s,0) - r^+(s,0)}{2} > -\infty \tag{10.4}$$

provided that $\bar{u} \geq m > 0$. If $\phi_{yy}(s,\cdot)$ is not integrable on $(0,1)$, then $u(s,t) \geq m$ for all (s,t). This conclusion is almost the same as those coming from the energy estimates of Section 4.

It is tempting to try to extend this application of the Maximum Principle for a problem with an artificial viscosity, directly to the problems formulated in Section 2, which have natural dissipative mechanisms. To do so, we could treat the governing equation (2.6) as parabolic with u regarded as given. Unfortunately, the nonlinear coupling of elastic with viscous effects would cause the application of the Maximum Principle to depend on an a priori bound for u_s, which we do not have.

11. Conclusion

We have given simple conditions ensuring that there is no total compression. We do not maintain that such conditions are thereby more physically natural than other conditions, or that blow-up in the equations is always physically unnatural.

The analytic difficulties we encountered with the treatment of time-dependent Dirichlet conditions to some extent reflect the physical difficulties illustrated in example (4.20). We do not have the physical capacity to *impose* arbitrary end motions because some of these can only be maintained by infinite forces at the ends. To deal with this question in a fully satisfactory way, we must use more sophisticated modelling of realistic loading devices (cf. [**10**]). On the other hand, it is physically easy to *observe* the motion of the ends. If it is known that the loading device can exert only finite force, then we get a sharp energy estimate, like that of Proposition 4.7. Indeed, as long as there is a bound on the end forces, we do not require the technical analysis of Section 6. A variant of this principle makes it easy to handle the problem in which one end of the rod is fixed and the other is attached to a tip mass (cf. [**5**]).

In our treatment of problem DD, we imposed conditions on the function D that are clearly crude: They do not distinguish between large compressional and extensional motions. They give yet another illustration, however, of a linkage between elasticity and viscosity that suffices to preclude total compression.

As in many of the references cited in Section 1, we had no need for the convexity of the potential energy. We did, however, use its coercivity.

It is clear that our critical Hypothesis 3.7 generalizes the constitutive assumption (1.4) by replacing it with an inequality, which is not required to hold everywhere. The techniques we have used to exploit Hypothesis 3.7 are not, however, those developed by Kanel' [**24**] and MacCamy [**31**]; they actually generalize those introduced by Andrews [**2**] for (1.3). An inequality like (3.7) might admit natural extensions to systems, generalizing (1.1), that can describe flexural motions of rods, whereas it is not evident that (1.4) can be realistically generalized to such systems.

Rather than treating our equations as evolution equations in Sobolev spaces, we could have followed an approach like that of Dafermos [**13**], who used the Leray-Schauder theory in the setting of Hölder spaces. Many of his estimates were based on the maximum principle for parabolic equations, which are effectively restricted to scalar equations. Our techniques, not employing these estimates, might be effective for systems.

The heart of our model is the nonlinear interaction between elastic effects associated with the stretch w_s and viscous effects associated with the rate of stretch w_{st}. Our constitutive assumptions prevented these derivatives from being uncoupled. Consequently the existence and regularity proofs were quite delicate. Similar phenomena can occur for ordinary differential equations in which the effects of spring forces and dissipative forces are coupled. See [**7**], [**29**], [**41**], [**44**].

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