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[1,2] A nonlocal parabolic equation arising in a turbulence model

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ABSTRACT: We consider a class of parabolic partial differential equations with nonlocal diffusion coefficient. The study is motivated by a model of turbulent viscosity for a developed flow in a duct between parallel walls.

Under rather more general conditions, we use a rescaling in t to prove existence and well-posedness for weak solutions. Sharp comparison estimates enable us to bound from below the spatial derivative at the wall. The steady state problem is also nonstandard and we show local stability by a spectral continuation argument.

Finally, numerical results for the steady state solution are compared with available experimental data.

Key Words: *parabolic, nonlinear partial differential equation, nonlocal, turbulence.*

1. Introduction

We will discuss a class of nonlinear parabolic partial differential equations with nonlocal diffusion coefficient. A particular example arises in considering a 'turbulent viscosity' model of flow in a plane duct. As a matter of mathematical interest, we have embedded this model in a somewhat more general form

$$(1.1) \qquad \begin{cases} u_t = [Du_x]_x + f & \text{on } \mathcal{Q} := (0,T) \times (0,1) \\ u = 0 & \text{at } x = 0,1 \\ u = \overset{\circ}{u} & \text{at } t = 0 \end{cases}$$

in which the diffusion coefficient D is given nonlocally by

$$(1.2) \quad D = G(x; h(t)) := g_0(x) + g_1(x)h(t) \quad \text{where} \begin{cases} h(\cdot) := \varphi(\zeta(\cdot)) \\ \zeta(\cdot) := u_x\big|_{x=0} \end{cases} .$$

For the motivating turbulence model, averaging out the local turbulent fluctuations from the basic Navier-Stokes equation leads to a one-dimensional equation for the mean velocity field u. Following [6], [9], [3], one obtains, as indicated in Section 2, a nonlinear parabolic partial differential equation as in (1.1), (1.2) with the particular form

$$(1.3) \qquad D = D[u] = \nu_0 + \sqrt{u_x(t,0)}\, g(x).$$

In this model f is a modified pressure gradient, $\nu_0 > 0$ is the kinematic viscosity of the fluid, and the positive function g empirically adjusts the turbulence level.

In Sections 3, 4 we obtain well-posedness results for (1.1)–(1.2) under suitable hypotheses on g_0, g_1, φ. Our set of 'minimal hypotheses', sufficient for existence of a solution to (1.1)–(1.2), will be

$$(1.4) \qquad \begin{array}{ll} i) & 0 < \underline{\nu} \le g_j(\cdot) \le \bar{\nu}, \quad \|g_j'\| \le M = M_g \text{ for } j = 0,1; \\ ii) & \varphi : \mathbb{R} \to \mathbb{R}_+ \text{ is continuous;} \\ iii) & \overset{\circ}{u} \in H_0^1(0,1) := \{v \in H^1(0,1) : v(0) = v(1) = 0\}; \\ iv) & \|f\|_{\mathcal{Q}} \le M = M_f. \end{array}$$

We assume these throughout and strengthen them as needed. Note that, while this is irrelevant for the specific function: $\varphi(r) = \sqrt{r}$ of (1.3), we expend some effort to obtain the various results without imposing any growth condition on $\varphi(\cdot)$. On the other hand, since the square root function of (1.3) is 'nice' only for positive arguments, a significant part of Section 5 is a novel comparison theorem needed to bound $\zeta = u_x(\cdot, 0)$ away from 0. Finally, in Sections 6, 7 we treat local stability of the steady state for (1.1)–(1.3) and then compare the steady state results to experimental data.

We do note that (1.1)–(1.2) is a special form of what J. Cannon and H.-M. Yin [4] have called 'nonstandard' parabolic equations. A principal motivation of our treatment is to obtain a theory permitting consideration of (1.1)–(1.3) with weaker assumptions on the forcing term $f \geq 0$ and on the initial data than are given in [3] or than can be utilized with the Hölder space methods of [4]. Our arguments are quite different from those of [4] since we will be working in a Sobolev space setting.

Notation: We will consistently denote u_x by z, $u_x|_{x=0}$ by ζ, $\varphi(\zeta(t))$ by $h(t)$ and $[0,T] \times [0,1]$ by $\mathcal{Q}$. Later we will rescale time, with a new variable $s \leftrightarrow t$ (with $S \leftrightarrow T$ and with $[0,S^*] \times [0,1] =: \hat{\mathcal{Q}} \supset [0,S] \times [0,1] \leftrightarrow \mathcal{Q}$ if we select $S^* \geq S$); for functions written in terms of s we will use the corresponding letters with a caret so, e.g., $\hat{z}(s,x) = z(t,x) = \hat{u}_x(s,x)$ and $\hat{h}(s) = \varphi(\hat{\zeta}(s)) = h(t)$.

We will use K as a generic positive 'absolute constant' (not necessarily the same in each appearance, even within the same formula) as, e.g., $\sqrt{5}$ or 'the norm of the embedding: $H^1(0,1) \hookrightarrow C[0,1]$' or …; we will use C generically as a positive constant which may be dependent on the parameters $T, \underline{\nu}, \bar{\nu}, \ldots$ of (1.4) (and of later sets of hypotheses, as imposed) and on previous instances of C — but which will never explicitly be dependent on S or on the particular data, etc., until this may be reduced to 'allowable dependencies' as above and will not depend on the particular function $\varphi(\cdot)$ except for a bound (or, later, a Lipschitz constant) for the restriction of φ to $[-r,r]$ with r depending on a prior instance of C.

We use $\|\cdot\|_p$ to denote the $L^p(0,1)$-norm, omitting the subscript for $p=2$, and $\|\cdot\|_{\mathcal{Q}}$ for the $L^2(\mathcal{Q})$-norm, etc. We use $\langle\cdot,\cdot\rangle$ both for the $L^2(0,1)$ inner product and for the scalar product between $H^1(0,1)$ and its dual. The norm for $H^1(0,1)$ will be denoted by $\|w\|_{(1)} := [\|w\|^2 + \|w_x\|^2]^{1/2}$ (but we note that $\|u\|_{(1)} \leq K\|z\|^2$ as $u\Big|_{x=0,1} = 0$); in particular, we have $\|z\|_\infty \leq K\|z\|_{(1)}$ and $|\langle z, f_x\rangle| \leq K\|z\|_{(1)}\|f\|$ and, similarly, we have $\|w_x\|_{(-1)} \leq K\|w\|$, etc. We write, e.g., $L^2([0,T] \to H^1_0(0,1))$ for the space of $H^1_0(0,1)$)-valued functions on $[0,T]$ for which $\int_0^T \|u\|_{(1)}^2\,dt < \infty$, suppressing the domain where this may be understood — whence $L^2(\to L^2(0,1))$ is $L^2(\mathcal{Q})$ or $L^2(\hat{\mathcal{Q}})$ according as we are considering $t \in [0,T]$ or $s \in [0,S^*]$.

2. The physical problem

We begin with the Navier-Stokes equations for the fluid velocity $\mathbf{V}$ under the assumption of incompressibility (so the density ρ is constant) and with constant viscosity $\nu_0 > 0$; componentwise, we have

$$\dot{V}_i + \sum_j \frac{\partial (V_i V_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu_0 \Delta V_i. \tag{2.1}$$

Let us now assume a scale separation $\mathbf{V} = \bar{\mathbf{V}} + \mathbf{\Phi}$ where $\bar{\mathbf{V}}$ is the locally averaged flow (ensemble average) and $\mathbf{\Phi}$, essentially the turbulence, is regarded quasi-statistically as an additive local fluctuation with small spatial scale; we similarly assume an averaged pressure $\bar{p}$. Averaging (2.1) gives the equation for the smoothed velocity $\bar{\mathbf{V}}$:

$$\dot{\bar{V}}_i + \sum_j \frac{\partial}{\partial x_j}\left(\bar{V}_i \bar{V}_j + \overline{\Phi_i\,\Phi_j}\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu_0 \Delta \bar{V}_i, \tag{2.2}$$

noting that $\overline{\bar{V}_i} = \bar{V}_i$ and $\overline{\bar{V}_i \Phi_j} = \bar{V}_i \overline{\Phi_j} = 0$, etc.

In order to make this fully determined, closure assumptions must be imposed for the Reynolds stress tensor $\boldsymbol{\tau} = -\overline{\Phi_i\,\Phi_j}$ — predicting, as a constitutive relation, the dependence of this on the averaged flow field. Numerous 'effective-viscosity' hypotheses have been introduced in the literature stemming from Boussinesq's 1877 paper [2] and we adopt here the modeling hypothesis that there is a 'turbulent viscosity' ν_T such that $\boldsymbol{\tau}$ can be well-approximated in terms of local stress for the mean flow, given by velocity gradients, so

$$-\overline{\Phi_i\,\Phi_j} = \frac{1}{3}q^2\delta_{ij} + \nu_T\left(\frac{\partial \bar{V}_i}{\partial x_j} + \frac{\partial \bar{V}_j}{\partial x_i}\right) \tag{2.3}$$

with $q^2 := \overline{\sum_j \Phi_j{}^2}$. Using this ansatz in (2.2) then gives

$$\begin{aligned} &\dot{\bar{V}}_i - \nabla \cdot \left[(\nu_0 + \nu_T)\, \nabla \bar{V}_i \right] \\ &\qquad = -\sum_j \frac{\partial}{\partial x_j}\left(\bar{V}_i \bar{V}_j\right) - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\bar{V}_j\right) + f_i \\ &\qquad\qquad \text{with } \; f_i := \frac{\partial}{\partial x_i}\left[-\frac{1}{\rho}\bar{p} + \frac{1}{3}q^2\right]. \end{aligned} \tag{2.4}$$

Our application is to a description of the motion of a turbulent fluid between two parallel planes. We take these planes a distance 1 apart, with a suitable choice of units, and choose coordinates (x_1, x_2, x_3) with $x = x_1 \in [0, 1]$ transverse to the duct. We now assume that the averaged flow $\bar{\mathbf{V}}$, neglecting the turbulent fluctuations, will be parallel to the walls of the duct (so $\bar{V}_1 \equiv 0$) and remains parallel to a fixed flow vector $\vec{c}$ — so $\bar{V}_3 \equiv 0$ and everything is independent of x_3 if we take the coordinate $y = x_2$ in the direction of $\vec{c}$. The system (2.4) then reduces to a scalar equation for the component $\bar{V}_2$.

Our final modelling assumption, here, is that this is an 'established flow', which is invariant along the flow direction: $\partial\bar{p}/\partial y$ and $\bar{V}_2 = u(t, x)$ are independent of the variable y. With these assumptions, the system (2.4) becomes

$$\dot{u} - \left[(\nu_0 + \nu_T)\, u_x\right]_x = f \tag{2.5}$$

with $f = f_2 = [-\bar{p}/\rho + q^2/3]_y$, taken to be a 'known' function $f = f(t, x)$. As is standard, we adjoin to (2.5) the boundary condition that the fluid be stationary in contact with the duct walls, so

$$u = 0 \quad \text{at } x = 0, 1 \tag{2.6}$$

as in (1.1). Note that ν_T appears in (2.5) as an addition to the intrinsic viscosity ν_0, explaining our use of the phrase 'turbulent viscosity' for this model. The hypothesis (2.3) and that we are considering an established flow with $f \geq 0$ in (2.5) will not be valid assumptions in all settings but appear to give good results for our current concerns; cf., Section 7.

[It would be of interest to show mathematically what one seems to observe in computational practice: that, for a more general flow, the transverse components and the y-dependence damp out even more rapidly than the approach to steady state treated in Section 6. Also, one might try to formulate the specification $f = f_2 = [-\bar{p}/\rho + q^2/3]_y$, at least approximately as an additional 'closure' of the system, in terms of some suitable nonlinearity. We do not at all consider this extension here.]

We have yet to model any constitutive relation for this effective viscosity ν_T due to turbulence. We now assume that the observed turbulence is 'generated' at the walls due to the shear stress imposed by the boundary condition (2.6) and this effect then spreads across the duct. Since this shear stress is just [u_x *at the wall*], this gives a model of the form

$$\nu_T(t,x) = \varphi\left(u_x(t,0)\right) g(x) \tag{2.7}$$

in which $\varphi(\cdot)$ is a 'generation rate' for turbulence and $g(\cdot)$ is a function 'adjusting' the transported effect of the turbulence. The evaluation of $u_x(\cdot,0)$ at the same time t is based on an assumption that turbulence propagates into the mean flow far more rapidly than the time variation of the established flow we are considering here. [The form (2.7) implicitly assumes symmetry for the solution to permit evaluation only at $x = 0$, rather than, say, using $[\varphi\left(u_x(t,0)\right) g(x) + \varphi\left(u_x(t,1)\right) g(1-x)]$. This assumption of symmetry will be justified by the Corollary to Theorem 1.]

The influence function $g(x)$ is designed, in particular, in order to describe the turbulent motion immediately adjacent to the walls, where the molecular viscosity influences directly the process of transport of turbulence; there have been several articles (e.g., [12], [10], [5]) dealing with closure models of this kind. There is no difficulty here with the hypothesis (1.4-i) or, later, with (5.6-i).

Finally, we observe that $U_f = \sqrt{\nu_0 u_x\big|_{wall}}$ is dimensionally a velocity and is, indeed, a characteristic velocity scale for the flow directly adjacent to the wall (in a sublayer within the usual boundary layer) in which the turbulence generation is concentrated. This generation is the form of energy dissipation associated with wall friction. The dissipation (per unit length in y) is given by the frictional force — here proportional to U_f for sliding friction. Thus we get [*generation*] $\sim U_f$, whence $\varphi(r) = \sqrt{r}$ in (2.7), completing our model.

3. Mathematical reformulation and an estimate

We now return to consideration of the general problem (1.1), (1.2), subject only to (1.4). Our approach to 'existence' will be to take the function

$\zeta = u_x|_{x=0}$ as the 'primary unknown', defining a map

$$\boldsymbol{\mathcal{F}}_0 : \zeta_0 \mapsto \zeta : \mathcal{Z} = L^2(0,T) \to \mathcal{Z} \tag{3.1}$$

whose fixpoint would give a solution, and then to obtain estimates giving applicability of the Schauder Theorem. The difficulty with this approach does not lie in obtaining the estimates but in showing continuity of $\boldsymbol{\mathcal{F}}_0$ in the absence of any growth conditions imposed on $\varphi(\cdot)$. We must be somewhat more roundabout and, to use the Dominated Convergence Theorem, will want a uniform bound for D, independent of the specific $h(\cdot)$. Our 'trick' will be to rescale time in an h-dependent way, introducing $t \leftrightarrow s = s(t) = s(t;h)$ with $ds/dt = 1 + h$. Thus, $s(\cdot)$ and its inverse function $\hat{t} = s^{-1}$ are to be given by

$$s(t) := \int_0^t [1 + h(\tau)]\, d\tau, \qquad \hat{t}(s) := \int_0^s \frac{d\sigma}{1 + \hat{h}(\sigma)}; \tag{3.2}$$

this construction must, of course, be justified eventually by showing that h is, indeed, integrable. We then get $\partial/\partial s = [1/(1+h)]\partial/\partial t$ so (1.1) becomes

$$\begin{cases} \hat{u}_s = [\hat{D}\hat{u}_x]_x + \hat{f} & \text{on } \hat{\mathcal{Q}} := (0, S^*) \times (0,1) \\ \hat{u} = 0 & \text{at } x = 0, 1 \\ \hat{u} = \mathring{u} & \text{at } s = 0 \end{cases} \tag{3.3}$$

with $S^* = S := s(T;\hat{h})$, $\hat{D} = D\dfrac{dt}{ds}$, etc. — i.e., we have

$$\begin{aligned} \hat{D}(s,x) &= \hat{G}(x, \hat{h}(s)) := \frac{g_0(x) + \hat{h}(s) g_1(x)}{1 + \hat{h}(s)} \\ \hat{f}(s,x) &= \frac{f}{1+h} := \frac{f(\hat{t}(s), x)}{1 + \hat{h}(s)}. \end{aligned} \tag{3.4}$$

Note that, with no estimates at all for $\hat{h}$, (1.4-i) then gives the uniform bounds for $\hat{D}$ which we will need:

$$0 < \nu \le \hat{D} \le \bar{\nu}, \qquad \|\hat{D}_x\| \le M_g. \tag{3.5}$$

We also note that

$$\int_0^{S^*} (1 + \hat{h}) \|\hat{f}\|^2 \, ds = \|f\|_{\mathcal{Q}}^2 \le M_f^2 \tag{3.6}$$

where, if necessary, we take $\hat{f}(s,x) := 0$ for $s > S$ in defining $\hat{f}$ on $[0, S^*]$.

We conclude this section by obtaining an *a priori* estimate for the integrability of h in terms of a function $\tilde{\varphi}_0$ such that

$$|\rho| \leq r \quad \Longrightarrow \quad \varphi(\rho) \leq \tilde{\varphi}_0(r). \tag{3.7}$$

Note that $\int_{t_1}^{t_2}(1+h)\,dt = s(t_2) - s(t_1) =: s_2 - s_1 = \int_{s_1}^{s_2} 1\,ds$.

LEMMA 1: *Suppose $\hat{h}(\cdot) := \varphi(\hat{\zeta}(\cdot))$ on $[0, S^*]$ and $s(\cdot)$, $\hat{t}(\cdot)$ are defined as in (3.2). Then*

$$s_2 - s_1 \leq [1 + \tilde{\varphi}_0(r)](t_2 - t_1) + \left([1/r]\|\hat{\zeta}\|_{L^4(0,S^*)}\right)^4 \tag{3.8}$$

for any $r > 0$.

PROOF: We integrate 1 from s_1 to s_2, noting that

$$1 \leq \frac{1 + \tilde{\varphi}_0(r)}{1 + \hat{h}} + \frac{|\hat{\zeta}|^4}{r^4}$$

since, by (3.7), we have $1 + \hat{h} = 1 + \varphi(\hat{\zeta}) \leq 1 + \tilde{\varphi}_0(r)$ when $|\hat{\zeta}| \leq r$ while otherwise $1 \leq (|\hat{\zeta}|/r)^4$. ∎

Taking, e.g., $r = 1$ in (3.8) for $s_1 = 0 = t_1$ and $t_2 = T$, we see that a bound of the form $\|\hat{\zeta}\|_{L^4(0,S^*)} \leq C$ (as in (4.1-*iii*)) will ensure that use of

$$S^* := [1 + \tilde{\varphi}_0(1)]T + C^4 \tag{3.9}$$

makes $\hat{t}(S^*) \geq T$. Such a bound also ensures that the set of functions $\{s(\cdot; h)\}$ associated with a set of nonlinearities $\{\varphi\}$ will be equicontinuous if the same function $\tilde{\varphi}_0$ may used in (3.7) for each such φ: we get a modulus of continuity by, e.g., selecting $r = [C/2\varepsilon]^{1/4}$ to get

$$|t_2 - t_1| \leq \delta(\varepsilon) := \frac{\varepsilon}{2[1 + \tilde{\varphi}_0(r)]} \Longrightarrow |s_2 - s_1| \leq \varepsilon. \tag{3.10}$$

4. Existence and well-posedness

Our objective in this section is to demonstrate existence of weak solutions of (1.1) under the fairly weak set of hypotheses (1.4) and then to obtain well-posedness under a slight strengthening of this, primarily to require φ to be (uniformly) Lipschitzian. We remark that the proof of existence could have been simpler had we imposed a growth condition $\varphi(r) = \mathcal{O}(|r|)$ in (1.4-ii) — which would, of course, have followed had we imposed the Lipschitz condition at the beginning.

LEMMA 2: *Assume (1.4), choose $S^* > 0$, and let $\hat{\zeta}_0$ be any measurable function on $[0, S^*]$. Set $\hat{h} := \varphi(\hat{\zeta}_0(\cdot))$ and define $\hat{t} = \hat{t}(\cdot; \hat{\zeta}_0)$ on $[0, S^*]$ by (3.2) and then define $\hat{D}, \hat{f}$ on $\hat{\mathcal{Q}}$ by (3.4), using this $\hat{h}, \hat{t}$ and taking $\hat{f}(s, x) := 0$ if $\hat{t}(s) > T$. Then there exists a unique weak solution $\hat{u}$ of the (linear) partial differential equation (3.3) and one has estimates for $\hat{u}$, $\hat{z} := \hat{u}_x$, and $\hat{\zeta} := \hat{z}\Big|_{x=0}$ of the forms:*

$$(4.1) \qquad \begin{aligned} &i) && \|\hat{u}(s)\|,\ \|\hat{z}\|_{\hat{\mathcal{Q}}} &&\leq C \\ &ii) && \|\hat{z}(s)\|,\ \|\hat{z}_x\|_{\hat{\mathcal{Q}}} &&\leq C \\ &iii) && \|\hat{\zeta}\|_{L^4(0,S^*)} &&\leq C. \end{aligned}$$

Finally, there is a compact, convex set $\mathcal{Z}_1 \subset L^2([0, S^*] \to \mathcal{C}[0, 1])$, such that $\hat{z} \in \mathcal{Z}_1$ and, similarly, one has $\hat{u} \in \mathcal{Z}_0 =$[compact, convex in $\mathcal{C}(\hat{\mathcal{Q}})$] and $\hat{\zeta} \in \mathcal{Z}_* =$[compact in $L^2(0, S^*)$] — with $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_*$ depending only on T and the parameters of (1.4).

PROOF: The weak form of (3.3) is

$$(4.2) \qquad \langle w, \hat{u}_s \rangle + \langle \hat{D} w_x, \hat{z} \rangle = \langle w, \hat{f} \rangle \qquad \text{for } w \in L^2([0, S^*] \to H_0^1(0, 1)).$$

Taking $w = 2\hat{u}$ on [0,s], one gets, as usual,

$$\|\hat{u}(s)\|^2 + 2\underline{\nu} \int_0^s \|\hat{z}(\sigma)\|^2 d\sigma \leq \| \mathring{u} \|^2 + 2 \int_0^s \|\hat{u}\| \|\hat{f}\| \, d\sigma.$$

Note that $\hat{u}\hat{f} = \left[\hat{u}/\sqrt{1 + \hat{h}}\right] \cdot \left[\sqrt{1 + \hat{h}} \hat{f}\right]$ since $\hat{f} = \chi \hat{f}$ with $\chi = \chi(\sigma; T)$ defined as $\{1$ if $\hat{t}(\sigma) \leq T;\ 0$ else$\}$. Using (3.6), we get

$$\|\hat{u}(s)\|^2 \leq C + \int_0^s \|\hat{u}(\sigma)\|^2 \chi \frac{d\sigma}{1 + \hat{h}(\sigma)} \leq C \exp\left[\int_0^s \frac{\chi \, d\sigma}{1 + \hat{h}}\right] \leq C e^T$$

by the Gronwall Inequality — noting that (3.2), etc., give $\int_0^s \chi \, d\sigma/(1 + \hat{h}) = \min\{\hat{t}(s), T\}$. This gives (4.1-$i$); we emphasize that the estimate does not depend on the choice of S^*.

Turning next to (4.1-ii), we observe that (formally) we may differentiate (3.3) with respect to x to get an equation for z:

$$
\left\{\begin{array}{ll}
\hat{z}_s = \left[(\hat{D}\hat{z})_x + \hat{f}\right]_x & \text{on } \hat{\mathcal{Q}} \\
\left[(\hat{D}\hat{z})_x + \hat{f}\right] = 0 & \text{at } x = 0,1 \\
\hat{z} = \overset{\circ}{u}_x & \text{at } t = 0.
\end{array}\right. \tag{4.3}
$$

The boundary condition comes by noting that the original condition $\hat{u}\Big|_{x=0,1} \equiv 0$ can be formally differentiated to obtain $0 \equiv \hat{u}_s = [(\hat{D}\hat{z})_x + \hat{f}]\Big|_{x=0,1}$. The weak form of (4.3) is then

$$
\langle w, \hat{z}_s\rangle + \langle \hat{D}w_x, \hat{z}_x\rangle = -\langle \hat{D}_x w_x, \hat{z}\rangle - \langle w_x, f\rangle \tag{4.4}
$$

and integrating this on $[0,s]$ with $w = 2\hat{z}$ gives

$$
\begin{array}{rl}
\multicolumn{2}{l}{\|\hat{z}(s)\|^2 + 2\underline{\nu}\int_0^s \|\hat{z}_x\|^2} \\
\leq & \|\overset{\circ}{u}_x\|^2 + 2\int_0^s \|\hat{z}\|_{(1)}\left[KM_g\|\hat{z}\| + \|\hat{f}\|\right] \\
\leq & C + \underline{\nu}\int_0^s \|\hat{z}_x\|^2 + C\int_0^s \|\hat{z}\|^2
\end{array}
$$

where we have estimated $\|\hat{D}_x\hat{z}\|$ by $\|\hat{D}_x\|\|\hat{z}\|_\infty \leq KM_g\|\hat{z}\|_{(1)}$, etc. Subtracting the term $\underline{\nu}\int \|\hat{z}_x\|^2$ and using (4.1-i) to bound $\int \|\hat{z}\|^2$, one obtains a bound on $[0,S^*]$ for $[\|\hat{z}(s)\|^2 + \underline{\nu}\int_0^s \|\hat{z}_x\|^2]$ and thence obtains (4.1-ii).

Since $\hat{\zeta}^2 = \hat{z}^2(s,x) - 2\int_0^x \hat{z}\hat{z}_x\,dx$ for arbitrary $x \in (0,1)$, one can integrate over $x \in (0,1)$ to get $\hat{\zeta}^2 \leq \|\hat{z}\|[\|\hat{z}\| + 2\|\hat{z}_x\|]$. Using (4.1-$ii$) to bound the first factor $\|\hat{z}(s,\cdot)\| \leq C$ then gives $|\hat{\zeta}|^4 \leq C[\|\hat{z}\|^2 + \|\hat{z}_x\|^2]$ and integrating this on $[0,S^*]$ gives (4.1-iii) by (4.1-i,ii). Again we emphasize that the estimate does not depend on the choice of S^*.

Finally, we observe that (4.1-i) bounds $(\hat{D}\hat{z})$ in $L^2(\hat{\mathcal{Q}})$ so, also noting (3.6), $\hat{u}_s = [(\hat{D}\hat{z})_x + \hat{f}]$ will be bounded in $L^2(\to H^{-1}(0,1))$ whence $\hat{z}_s = [\hat{u}_s]_x$ will be bounded in $L^2(\to H^{-2}(0,1))$. As (4.1-ii) bounds $\hat{z}$ in $L^2(\to H^1(0,1))$ and $H^1(0,1))$ embeds compactly into $\mathcal{C}[0,1]$, the Aubin Compactness Theorem [1] gives existence of $\mathcal{Z}_1 =$[compact, convex in $L^2(\to \mathcal{C}[0,1])$], as desired; then, continuity of the trace: $L^2(\to \mathcal{C}[0,1]) \to L^2(0,S^*)$ gives existence of $\mathcal{Z}_* =$[compact, convex in $L^2(0,S^*)$]. The existence of the desired $\mathcal{Z}_0 =$[compact in $\mathcal{C}(\to \mathcal{C}[0,1])$] for $\hat{u}$ follows by [13] from the bound on $\hat{u}_s$ above with the bound for $\hat{u}$ in $L^\infty(\to H^1(0,1))$ implied by (4.1-ii). ■

At this point we fix the constant C of (4.1-iii) to use in (3.9) and so fix the choice of S^* without *a priori* knowledge of $\hat{\zeta}_0(\cdot)$ — or of $\varphi(\cdot)$, other than (1.4-ii) and (3.7) for $r = 1$. Note that (4.1-i) implies uniqueness of solutions of (3.3) so the construction of Lemma 2, restricted to $\mathcal{Z}_*$, determines a well-defined map

$$\boldsymbol{\mathcal{F}} : \hat{\zeta}_0 \mapsto \hat{z} \mapsto \hat{\zeta} : \mathcal{Z}_* \to \mathcal{Z}_* \,. \tag{4.5}$$

LEMMA 3: *Assume (1.4). Then* $\boldsymbol{\mathcal{F}}$*, defined as above through Lemma 2, is continuous.*

PROOF: Let $\hat{\zeta}_n \to \hat{\zeta}_0$ in $\mathcal{Z}_* \subset L^2(0, S^*)$. Let $\hat{u}_n$ be the corresponding solutions of $(3.3)_n$ and $\hat{z}_n = (\hat{u}_n)_x$, etc. It is sufficient to work with suitable subsequences so, without loss of generality, we may assume that $\hat{\zeta}_n \xrightarrow{ae} \hat{\zeta}_0$ from which it follows that $h_n := \varphi(\hat{\zeta}_n) \xrightarrow{ae} h_0$; by compactness, we may also assume that $\hat{u}_n \to \hat{u}$ and that $\hat{z}_n \to \hat{z} = \hat{u}_x$. For use in (4.2) we fix a test function w such that w, w_x are bounded. Note that, using the Dominated Convergence Theorem in (3.2), we have $\hat{t}_n(s) \to \hat{t}_0(s)$ for each s whence, as the functions $\hat{t}_n = \hat{t}(\cdot; \hat{\zeta}_n)$ are uniformly Lipschitzian, we then have uniform convergence $\hat{t}_n \to \hat{t}_0$ on $[0, S^*]$.

We have observed that $\{(\hat{u}_n)_s\}$ is bounded in $L^2(\to H^{-1}(0,1))$, hence has a weakly convergent subsequence there and, by the closedness of the differentiation operator, $(\hat{u}_n)_s \rightharpoonup \hat{u}_s$ so $\langle w, (\hat{u}_n)_s \rangle \to \langle w, \hat{u}_s \rangle$.

We have $\hat{D}_n \xrightarrow{ae} \hat{D}_0$ so $\hat{D}_n w_x \xrightarrow{ae} \hat{D}_0 w_x$. Since $\{\hat{D}_n w_x\}$ is uniformly bounded on $\hat{\mathcal{Q}}$, we may apply the Dominated Convergence Theorem to $\|\hat{D}_n w_x - \hat{D}_0 w_x\|^2 = \int_{\hat{\mathcal{Q}}} |\hat{D}_n w_x - \hat{D}_0 w_x|^2 \, dx \, dt$ and conclude that $\hat{D}_n w_x \to \hat{D}_0 w_x$ strongly in $L^2(\hat{\mathcal{Q}})$. Since $\hat{z}_n \to \hat{z}$, we then get $\langle \hat{D}_n w_x, \hat{z}_n \rangle \to \langle \hat{D}_0 w_x, \hat{z} \rangle$.

One easily verifies that setting $\hat{g}_n(s, x) := g(\hat{t}_n(s), x)/[1 + \hat{h}_n(s)]$ gives $\hat{g}_n \xrightarrow{ae} \hat{g}_0$ for an elementary step function g (i.e., $g(t, x) := a_j(x)$ for $t_{j-1} < t < t_j$ with each a_j bounded) whence, since $\{\hat{g}_n\}$ is uniformly bounded on $\hat{\mathcal{Q}}$, we may argue, much as above, that $\hat{g}_n \to \hat{g}_0$ strongly in $L^2(\hat{\mathcal{Q}})$. These elementary step functions are dense so we can approximate f by such a $g = g^\varepsilon$ such that $\|f - g^\varepsilon\|_{\mathcal{Q}} \le \varepsilon$ and use (3.6) to show that $\|\hat{g}_n - \hat{f}_n\| \le \varepsilon$ uniformly in $n = 1, 2, \ldots, 0$. It then follows that $\hat{f}_n \to \hat{f}_0$ so $\langle w, \hat{f}_n \rangle \to \langle w, \hat{f}_0 \rangle$.

Going to the limit for each term of $(4.2)_n$, we see that $\hat{u}$ satisfies $(4.2)_0$

for each such w; these are dense so $\hat{u} = \hat{u}_0$, $\hat{z} = \hat{z}_0$ and, as desired,

$$\lim_{n\to\infty} \boldsymbol{\mathcal{F}}(\hat{\zeta}_n) = \lim_{n\to\infty} \hat{z}_n(\cdot,0) = \hat{z}_0(\cdot,0) =: \boldsymbol{\mathcal{F}}(\hat{\zeta}_0). \qquad \blacksquare$$

THEOREM 1: *The hypotheses (1.4) suffice to ensure existence of at least one weak solution u of the problem (1.1) with*

$$u \in \mathcal{C}(\bar{\mathcal{Q}}), \quad u_{xx} \in L^2(\mathcal{Q}), \quad u_x\big|_{x=0} \in L^4(0,T). \tag{4.6}$$

If the hypotheses are strengthened to have g_1 bounded and φ uniformly Lipschitzian on $\mathbb{R}$, then the solution is unique and depends uniformly Lipschitz continuously on the data $\{\mathring{u}, f\}$ in the indicated spaces of (4.6).

PROOF: By the lemmas, (1.4) ensures that $\boldsymbol{\mathcal{F}}$ satisfies the hypotheses of the Schauder Theorem. Thus, we obtain a fixpoint $\hat{\zeta}$ giving $\hat{u}$ by (3.3), (3.4) for which one has $\hat{\zeta} = [\hat{u}_x]\big|_{x=0}$. For this fixpoint we necessarily would have (4.1) so $s(\cdot) := [\hat{t}(\cdot;\hat{z})]^{-1}$ is well-behaved and, by the choice of S^* from (3.9), would be defined for all $t \in [0,T]$. Reversing our time rescaling, one easily sees that setting $u(t,x) := \hat{u}(s(t),x)$ on $\mathcal{Q}$ then provides the desired solution of (1.1). We observe that (4.1-iii) now gives

$$\int_0^T [1+\varphi(\zeta)]\,|\zeta|^4\,dt \le C \tag{4.7}$$

and Lemma 1 shows continuity of $s(\cdot) = s(\cdot;\zeta)$ whence the continuity of $\hat{u}$ means that $u \in \mathcal{C}(\bar{\mathcal{Q}})$; in general, the estimates of (4.1) translate into corresponding estimates for u, etc., as in (4.6).

Now assume the strengthened hypotheses and let u^j ($j = 1,2$) be solutions, respectively, of $(1.1)^j$, using $\{\mathring{u}^{\,j}, f^j\}$; correspondingly, let $z^j := [u^j]_x$, etc., and set $f := f^2 - f^1$, $u := u^2 - u^1$, etc., so $u\Big|_{t=0} = \mathring{u} := \mathring{u}^{\,2} - \mathring{u}^{\,1}$ and

$$\begin{aligned} u_t - [D^1 u_x]_x &= f + (h^2 - h^2)(g_1 z^2)_x \\ z_t - [D^1 z_x]_x &= [D^1_x z + f + (h^2 - h^2)(g_1 z^2)_x]_x \end{aligned} \tag{4.8}$$

since $D^2 - D^1 = (h^2 - h^1)g_1 = [\varphi(\zeta^2) - \varphi(\zeta^1)]g_1$. Using $2z$ as test function for the weak form of the second equation in (4.8) gives

$$\begin{aligned}
&\|z(t)\|^2 + 2\underline{\nu}\textstyle\int_0^t (1+h^1)\|z_x\|^2 \\
&\qquad \le \; \| \overset{\circ}{u}_x \|^2 + 2\textstyle\int_0^t [(1+h^1)M\|z\|_\infty + \|f\| + L|\zeta|\|g_1 z^2\|]\|z\|_{(1)} \\
&\qquad \le \; \| \overset{\circ}{u}_x \|^2 + (1/\varepsilon)\|f\|_{\mathcal{Q}}^2 \\
&\qquad\qquad + C\textstyle\int_0^t (1+h^1)[\|z\|_\infty + \varepsilon\|z\|_{(1)} + |\zeta|]\|z\|_{(1)}
\end{aligned}$$

where, in our hypotheses, $|g_1(x)| \le M$, the Lipschitz constant for φ is L, and we have used the fact that we have an *a priori* bound for $\|z^2\|$. By compactness of the embedding: $H^1(0,1) \hookrightarrow \mathcal{C}[0,1]$, we may estimate

$$\|z\|_\infty, |\zeta| \le \varepsilon\|z\| + K_\varepsilon\|z_x\|$$

so, taking $\varepsilon = \underline{\nu}/3$ above and subtracting the integral term for $\|z_x\|^2$, we get

$$\begin{aligned}
&\|z(t)\|^2 + \underline{\nu}\textstyle\int_0^t (1+h^1)\|z_x\|^2 \\
&\qquad \le \; \left[\| \overset{\circ}{u}_x \|^2 + (3/\underline{\nu})\|f\|_{\mathcal{Q}}^2\right] + C\textstyle\int_0^t (1+h^1)\|z\|^2.
\end{aligned} \tag{4.9}$$

Application of the Gronwall Inequality then gives $\|z(t\|^2 \le a^2 e^{CS^*}$ with $a^2 := \left[\| \overset{\circ}{u}_x \|^2 + (3/\underline{\nu})\|f\|_{\mathcal{Q}}^2\right]$ and $\int_0^T (1+h^1) \le S^*$ estimated by (3.9). ■

In view of the earlier comment in Section 2 regarding symmetry, we note the following:

COROLLARY: *Suppose g_0, g_1, $f, \overset{\circ}{u}$ are all symmetric in x — i.e., $g_0(x) = g_0(1-x)$, etc. Then the solution u will also be symmetric.*

PROOF: We know that u corresponds to the solution $\hat{u}$ of (3.3)-(3.4) for the appropriate $\hat{h}(\cdot)$. The assumed symmetry ensures that $\hat{D}, \hat{f}$ in (3.3) will be defined symmetrically by (3.4) so the reflection $\hat{u}^*$, given by $\hat{u}^*(t,x) := \hat{u}(t, 1-x)$, will also satisfy (3.3). Uniqueness for the linear equation (3.3) ensures that $\hat{u}^* = \hat{u}$ so u must be symmetric. ■

We emphasize that this does not depend on uniqueness for (1.1)–(1.2) so only (1.4) is needed for this corollary.

We note that further regularity for u can be obtained by imposing stronger regularity conditions on (1.2) and the data. This rapidly gets quite complicated and we only state here (without proof) one such result.

Suppose we now require a bound on g_j'' rather than on g_j' as before, that $\mathring{u}$ be in $H^2(0,1)$ rather than only in $H_0^1(0,1)$, and not only that $f \in L^2(\mathcal{Q})$ but that $f_x \in L^2(\mathcal{Q})$. This last already implies that the trace $\alpha := f\Big|_{x=0}$ is well-defined in $L^2(0,T)$ but we now additionally require that $d\alpha/dt$ be integrable. It can then be shown, in addition to (4.6), that $z_{xx} = u_{xxx}$ is in $L^2_\rho(\mathcal{Q})$, defined by use of the weighted L^2-norm $\sqrt{\int_{\mathcal{Q}} |\cdot|^2 \rho}$ with, in this case, $\rho := (1-x)^2$ — this choice of weight allows for some possible loss of regularity near $x = 1$, where we have imposed no special condition on f_t.

5. A comparison theorem

Our objective in this section is to bound ζ away from 0 for solutions u of (1.1) with $f \geq 0$ and initial data for which $\zeta(0) > 0$ in a suitable weak sense. This weak sense is selected to permit comparisons involving only values of u itself, without actually requiring differentiation: we observe that, for a smooth function $\varphi \geq 0$ satisfying $\varphi(0) = 0$, a sufficient condition to have $\varphi'(0) \geq \beta > 0$ is to have $\varphi(x) \geq \beta\omega(x)$ for some (fixed) function ω with $\omega(0) = 0$, $\omega'(0) = 1$ while, conversely, if $\varphi(0) = 0$ and $\varphi'(0) = \bar{\beta} > \beta > 0$, then there will be some interval $[0, x_*]$ on which $f(x) \geq \beta x$. We begin with a lemma.

LEMMA 4: *Fix $x_* \in (0,1)$, $0 < \underline{\nu} < \bar{\nu}$, $M > 0$. Then there is a function ω and a number $\lambda \geq 0$ such that:*

(5.1)

- *i)* $\omega \in W^{2,\infty}([0,1] \to \mathbb{R}_+)$;
- *ii)* $\omega(0) = 0, \quad \omega'(0) = 1, \quad supp\, \omega \subset [0, x_*]$;
- *iii) For every function Γ with $\underline{\nu} \leq \Gamma \leq \bar{\nu}$ and $|\Gamma'| \leq M$ on $[0,1]$, one has the inequality:* $\quad (*) \quad (\Gamma\omega')' + \lambda\omega \geq 0.$

PROOF: Set $\alpha = M/\underline{\nu}$ and choose x_1, x_2 such that $0 < x_1 < x_2 < x_*$ with

$(x_* - x_2) < 1/\alpha$. We then define

$$(5.2) \qquad \omega(x) := \begin{cases} [e^{\alpha x} - 1]/\alpha & \text{for } 0 \le x \le x_1 \\ \tilde{\omega}(x) & \text{for } x_1 \le x \le x_2 \\ (x_* - x)^2 & \text{for } x_2 \le x \le x_* \\ 0 & \text{for } x_* \le x \le 1 \end{cases}$$

where we take $\tilde{\omega}$ to be any C^2 function on $[x_1,\, x_2]$ which correctly matches values and first derivative values across x_1 and x_2 while keeping

$$(5.3) \qquad \tilde{\omega}(x) \ge \underline{\omega} := \min\{\omega(x_1),\, \omega(x_2)\} > 0;$$

this is clearly possible.

This function ω will be C^1 and piecewise C^2 with $\omega > 0$ on $(0, x_*)$ so we have *i*) of the Lemma; *ii*) is equally clear and we need only verify *iii*). We begin with the observation that, on each of the intervals $[0,\, x_1]$ and $[x_2,\, x_*]$, we have $\omega \ge 0$, $\omega'' > 0$, and

$$(5.4) \qquad \underline{\nu}\, \omega'' \ge M|\omega'|.$$

Now let $M_2 := \max_{[x_1,\, x_2]}\{\bar{\nu}|\tilde{\omega}''| + M|\tilde{\omega}'|\}$ and then set $\lambda := M_2/\underline{\omega}$ so

$$(5.5) \qquad \bar{\nu}|\tilde{\omega}''| + M|\tilde{\omega}'| \le \lambda\tilde{\omega} \quad \text{on } [x_1,\, x_2].$$

Next, considering any function Γ as in (5.1-*iii*), we verify $(*)$ on each of the four subintervals, writing $(\Gamma\omega')' = \Gamma\omega'' + \Gamma'\omega$. On $[0,\, x_1] \cup [x_2,\, x_*]$ we have

$$(\Gamma\omega')' + \lambda\omega \ge \underline{\nu}\omega'' - M\omega' \ge 0,$$

using (5.4), while on $[x_1,\, x_2]$ we have

$$(\Gamma\omega')' + \lambda\omega \ge \lambda\underline{\omega} - [\bar{\nu}|\tilde{\omega}''| + M|\tilde{\omega}'|] \ge 0$$

using (5.5); finally, $(*)$ holds trivially on $[x_*,\, 1]$. ■

To apply this, we strengthen the hypotheses (1.4) to:

(5.6)
- *i*) $\quad 0 < \underline{\nu} \le g_j(\cdot) \le \bar{\nu}, \;\; |g_j'(x)| \le M = M_g$ for $j = 0, 1,\; x \in [0, 1]$;
- *ii*) $\quad \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous;
- *iii*) $\quad \mathring{u} \in H_0^1([0, 1] \to \mathbb{R}_+)$ with $\mathring{u}(x) \ge \beta_0 x$ on $[0, x_*]$ for some $\beta_0 > 0$, some $x_* \in (0, 1)$;
- *iv*) $\quad f \in L^2(\mathcal{Q} \to \mathbb{R}_+)$.

THEOREM 2: *Assume (5.6). Then, for a solution u of the problem (1.1) on $\mathcal{Q}$, the function $\zeta = u_x|_{x=0}$ is bounded away from 0 — i.e., one has $u_x|_{x=0} \geq 1/C$ on $[0,T]$.*

PROOF: Extending φ to $\mathbb{R}$ by setting $\varphi(r) := \varphi(0)$ for $r < 0$, we note that $(5.6) implies (1.4)$ so (1.1) has a solution u by Theorem 1 with all the regularity corresponding to (4.6).

Now let $\omega(\cdot)$ and λ be as in Lemma 4, with the parameters taken from (5.6-i, iii). Clearly there is some $\beta > 0$ (e.g., $\beta = \beta_0/\max_x\{\omega(x)/x\}$) such that

$$u(0,\cdot) = \overset{\circ}{u} \geq \beta\omega \geq 0 \quad \text{on } [0,1] \tag{5.7}$$

and we will show that

$$\hat{u}(s,x) \geq \beta e^{-\lambda s}\omega(x) \text{ on } \hat{\mathcal{Q}} \tag{5.8}$$

from which it will then follow that

$$\zeta(t) = \hat{\zeta}(s) \geq \beta e^{-\lambda S^*} =: 1/C > 0 \text{ for } 0 \leq t \leq T \tag{5.9}$$

with $S^* \geq s(T)$ given by (3.9).

To obtain (5.8), we show that $\hat{v}(s,x) := \left[\hat{u}(s,x) - \beta e^{-\lambda s}\omega(x)\right]$ remains positive, i.e., that $w := \hat{v}_- := \min\{0, \hat{v}\}$ is identically 0. By a theorem of Stampacchia [14] we note that $w \in H^1$ for $v \in H^1$ and that (a.e.)

$$w_s = \begin{cases} v_s \\ 0 \end{cases} \qquad w_x = \begin{cases} v_x & \text{where } w \neq 0 \\ 0 & \text{else} \end{cases} \tag{5.10}$$

so (a.e.) $2wv_s = (w^2)_s$ and $w_x v_x = |w_x|^2$. From (3.3),

$$\begin{aligned} \hat{v}_s &= \hat{u}_s + \lambda\beta e^{-\lambda s}\omega \\ &= \left[\hat{D}\left(\hat{v}_x + \beta e^{-\lambda s}\omega'\right)\right]_x + \hat{f} + \lambda\beta e^{-\lambda s}\omega \\ &= \left[\hat{D}\hat{v}_x\right]_x + \hat{f} + \beta e^{-\lambda s}\left[(\hat{D}\omega')' + \lambda\omega\right]. \end{aligned} \tag{5.11}$$

We take this $w = \hat{v}_-$ in the weak form of (5.11), observing that $w\Big|_{x=0,1} = 0$, and use (5.10) to get

$$\begin{aligned} &\frac{d\,\|w\|^2}{ds} + \textstyle\int_0^1 \hat{D}|w_x|^2\,dx \\ &\qquad = 2\langle w, v_s\rangle + 2\langle \hat{D}w_x, v_x\rangle \\ &\qquad = 2\langle w, \hat{f}\rangle + 2\beta e^{-\lambda s}\langle w, [(\hat{D}\omega')' + \lambda\omega]\rangle \leq 0. \end{aligned} \tag{5.12}$$

The final inequality follows from the facts that $w \le 0$ by definition, that $\hat{f} \ge 0$ by assumption, and that $[(\hat{D}\omega')_x + \lambda\omega] \ge 0$ by Lemma 4 since, for each s, $\hat{D}(s,\cdot)$ satisfies the hypotheses on Γ in (5.1-*iii*) by (5.6-*i*). Our choice of β gives $\hat{v}\Big|_{t=0} \le 0$ so $\|w(0)\|^2 = 0$. Then integrating (5.12) gives $\|w\|^2 \equiv 0$, i.e., (5.8).

For a smooth function q one would have

$$\begin{aligned}
[q(0) = 0,\ q(x) \ge c\omega(x)] \quad &\Longrightarrow \\
q'(0) \ &= \lim_{x\to 0+} q(x)/x \\
&= \lim_{x\to 0+} q(x)/\omega(x) \cdot \lim_{x\to 0+} \omega(x)/x \\
&\ge c \lim_{x\to 0+} \omega(x)/x = c.
\end{aligned}$$

so $[(5.8) \Longrightarrow (5.9)]$ for smooth functions. The set of $\hat{u}$ satisfying (5.8) is closed in $L^2(\hat{\mathcal{Q}})$, hence closed *a fortiori* in $L^2(\to H^2(0,1))$ from which the embedding $\hat{u} \mapsto [\hat{u}_x]_{x=0}$ is continuous to $L^2 = L^2(0,S^*)$. Thus the implication is valid also for $\hat{u} \in L^2(\to H^2(0,1)) \mapsto \hat{\zeta} \in L^2$. ∎

COROLLARY: *Assume (5.6-i, iii, iv) with $\varphi(r) := \sqrt{r}$ on $\mathbb{R}^+$ as in the motivating example (1.3). Then (1.1) has a unique solution u, depending continuously on $\mathring{u}, f$.*

PROOF: By the Theorem, it is only relevant to consider $\varphi(r) := \sqrt{r}$ for $r \ge 1/C > 0$ where C is as in (5.9). Thus, one could extend φ to $(-\infty, 1/C]$ fairly arbitrarily, e.g., to make $\varphi(\cdot)$ uniformly Lipschitzian, without affecting relevant solutions. The result then follows by applying Theorem 1. ∎

6. Stability of the steady state

In this section we consider existence and stability of the steady state solution of the nonlinear partial differential equation

$$\begin{gathered}
u_t = \left(\hat{D}u_x\right)_x + f(x) \\
u\Big|_{x=0} = 0, \qquad u_x\Big|_{x=1/2} = 0 \\
\text{where } \hat{D} := \nu + \xi g(x) \ \text{ with } \xi := \sqrt{u_x\Big|_{x=0}},
\end{gathered} \tag{6.1}$$

corresponding to (1.1)–(1.3) with symmetry.

THEOREM 3: *Assume that (5.6) is satisfied with symmetry on* $[0,1]$ *and with* $0 \not\equiv f \geq 0$*. Then there is a unique steady state solution* v *of (6.1) and (6.1) is locally exponentially stable in a neighborhood of this* v*.*

PROOF: In steady state we have (with the $\cdot_x = \partial/\partial x$ replaced by $\cdot' = d/dx$, as there is now no t-dependence)

$$(6.2)\qquad \begin{gathered} (Dv')' + f(x) = 0 \\ v(0) = 0 \qquad v'(0) = 0 \\ \text{where } D := \nu + \eta g(x) \ \text{ with } \eta := \sqrt{v'(0)}. \end{gathered}$$

From (6.2) we get

$$\varphi(x) := \int_x^{1/2} f = -\int_x^{1/2} (Dv')' = -Dv'\Big|_x^{1/2} = [Dv'](x)$$

whence

$$(6.3)\qquad v' = \frac{\varphi(x)}{\nu + \eta g(x)} \qquad v(x) = \int_0^x v'.$$

Setting $x = 0$ in (6.3) gives a cubic equation for the constant η in (6.2)

$$(6.4)\qquad \eta^2\left(\nu + \eta g(0)\right) = \varphi(0).$$

Since we have assumed $f \geq 0$ (whence $\varphi(0) > 0$), (6.4) can be solved uniquely for $\eta > 0$ and using this in (6.3) gives v', hence v, uniquely satisfying (6.2). We have shown existence of a unique steady state.

Next consider the linearization of (6.1) around the steady state v, i.e., for small perturbations ω we set

$$(6.5)\qquad u = v + \omega \ \ [+ \textit{higher order}\text{: to be neglected}].$$

Setting $\vartheta := \omega_x\Big|_{x=0}$, we observe that

$$(6.6)\qquad \xi := \sqrt{u_x\Big|_{x=0}} \approx \sqrt{[v' + \omega_x]\Big|_{x=0}} = \sqrt{\eta^2 + \vartheta} \approx \eta + \vartheta/2\eta$$

and using this in (6.1), neglecting the higher order terms and noting (6.2), gives

$$(6.7)\qquad \begin{gathered} \omega_t = (D\omega_x)_x + \vartheta\psi \qquad \left(\vartheta := \omega_x\Big|_{x=0}\right) \\ \omega\Big|_{x=0} = 0 = \omega_x\Big|_{x=1/2} \end{gathered}$$

with $D(x) := \nu + \eta g(x)$ as in (6.2) and, using (6.3),

$$\psi(x) \ := q' := \frac{1}{2\eta}\,(gv')' = \frac{1}{2\eta}\left[\left(\frac{g}{\nu+\eta g}\right)\varphi\right]'. \tag{6.8}$$

We now introduce the operators

$$\begin{gathered}\mathbf{A} : z \mapsto (Dz')', \quad \mathbf{B} : z \mapsto z'(0)\psi \qquad \text{for} \\ z \in H^2_{BC} := \{z \in H^2(0,\tfrac{1}{2}) : \ z(0) = 0 = z'(\tfrac{1}{2})\}.\end{gathered} \tag{6.9}$$

where $D := \nu + \eta g$ is obtained through (6.4) and ψ is obtained through (6.8). It is standard that $\mathbf{A}$ is self-adjoint (as an unbounded closed operator on $L^2(0,1/2)$) with compact resolvent and that $\sigma(\mathbf{A}) \subset \{\lambda \in \mathbb{R} : \lambda \leq \lambda_0 < 0\}$. One easily sees that the rank one operator $\mathbf{B}$ is a relatively compact perturbation to $\mathbf{A}$ so (cf., [8]) the equation (6.7) — whose abstract form is just $\dot{\omega} = (\mathbf{A}+\mathbf{B})\omega$ — corresponds to an analytic semigroup whose infinitesimal generator has compact resolvent (hence discrete point spectrum) and for which we have the 'spectrum-determined growth condition'. We will embed the eigenvalue equation $(\mathbf{A}+\mathbf{B})z = \lambda z$ in a homotopy $(\mathbf{A}+s\mathbf{B})z = \lambda z$ with $0 \leq s \leq 1$ and will proceed to show that, for all $s \in [0,1]$, the equation

$$\lambda z = (Dz')' + s\zeta\psi \ \ [\zeta := z'(0)] \qquad \text{with } 0 \neq z \in H^2_{BC} \tag{6.10}$$

has only real solutions λ with $\lambda \leq -\delta$ for some $\delta > 0$. We will thus have exponential stability for the linearized equation (6.7).

Observe, first, that the eigenvalues λ of $\mathbf{A} + s\mathbf{B}$ are always simple: otherwise we could find an eigenfunction $z \neq 0$ with $\zeta = z'(0) = 0$ and could then re-interpret (6.10) as an initial value problem: $(Dz')' - \lambda z = 0$ with $z(0) = z'(0) = 0$ to get $z \equiv 0$ — a contradiction.

From this, it follows that the eigenvalues λ of $\mathbf{A} + s\mathbf{B}$ are always real. To see this, note first that — by a result due to Keldysh (cf., e.g., [7]) — the eigenfunctions obtained by continuation along the homotopy form a complete set so it is sufficient to work with these. We note, from [8], that the eigenvalues of $\mathbf{A}+s\mathbf{B}$ will be (locally) analytically dependent on s — except for the possibility of bifurcation if we could have multiplicity greater than one, which is impossible by our previous observation. Since the eigenvalue equation (6.10) is real, non-real eigenvalues necessarily occur in conjugate pairs which could connect to the spectrum for $s = 0$ only by a bifurcation which we have already observed is impossible.

Next we note that we cannot have $\lambda = 0$ in (6.10). To see this, note that $\psi = q'$ by (6.8) with $q := g\varphi/2\eta D$, so $\lambda = 0$ would give $(Dz' + \zeta q)' = 0$ and, as $z(\frac{1}{2}) = 0 = q(\frac{1}{2})$, we could integrate this from $\frac{1}{2}$ and evaluate at 0 to get $z'(0)\,[D(0) + sq(0)] = 0$. This, however, would give $z'(0) = 0$ — so $z \equiv 0$, a contradiction — since $D(0) > 0$, $q(0) \geq 0$ whence $[D(0) + sq(0)] \neq 0$ for $s \geq 0$.

Starting from the largest eigenvalue $\lambda_0 < 0$ of $\mathbf{A}$, the continuation varies (smoothly) in $\mathbb{R}\backslash\{0\}$ as s varies in the compact interval $[0, 1]$ and so remains bounded below some $-\delta < 0$. As we cannot have multiple eigenvalues, the remainder of the spectrum (continuation starting from more negative eigenvalues of $\mathbf{A}$) must remain to the left of this and so also satisfies $\lambda \leq -\delta < 0$, as promised.

This completes the proof of linearized stability around the steady state for (6.1): the spectral condition shows that the linear semigroup corresponding to (6.7) decays exponentially and it then follows — see, e.g., Theorem 9.1.2 of [11] — that one also has local stability (in a neighborhood of the steady state v) for the nonlinear equation. ■

7. Comparison of predictions with experiment

Experimental data for certain 'fully developed flows' in a duct are available form Comte-Bellot [6]. These data correspond to the steady state setting for the parabolic equation (2.5)–(2.7), i.e., to the solution v discussed in the preceding section. Following the treatment in [9], we present here a comparison between these experimental data and the computed solution of the steady state equation (6.2). Note that the data of [6] correspond to effective Reynolds numbers 57000, 120000, 230000 with (modified) pressure gradient $f = \mathit{constant} > 0$ in each case.

For computation, the function $g(\cdot)$ appearing in (6.2) was obtained as an empirical fit, following Van Driest [10], [15]; we refer to [5] (Fig. 4.16) for a typical shape. Physically, this description is efficient in that g is independent of the Reynolds number, without need for further parameter adjustment. Moreover, this model represents the mean velocity field effectively in the entire duct, even in the so-called 'logarithmic zone' where other models break down. Indeed, this model seems quite robust: even with a simplified cubic approximation to g one already obtains accuracy comparable to that of the experimental data.

The discretization used in [9] of (6.2) was by finite elements ($P1$, using 20 points), with solution of the resulting nonlinear system obtained via conjugate gradients with optimization of the descent parameter. This permitted an adaptive treatment of the wall zone, where both g and the solution v (velocity) vary rapidly.

The figure here shows the comparison between the experimental data of [6] and the numerical results for the Reynolds number $Re = 57000$. The other experimental cases ($Re = 120000$ and $Re = 230000$) have also been checked numerically and give correspondingly satisfactory agreement.

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