NORM DEPENDENCE OF THE COEFFICIENT MAP
ON THE WINDOW SIZE

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ABSTRACT: For sparse exponent sequences \((λ_k)_{−∞}^{∞}\), satisfying a suitable ‘separation condition’ defined by an auxiliary sequence \(ψ\), one has a ‘coefficient map’ \(C_δ\) giving \((c_k)_{−∞}^{∞}\) from observation of \(f = \sum_{k=−∞}^{∞} c_k e^{iλ_k t}\) on any arbitrarily small interval \([−δ, δ]\). In terms of \(ψ\), we estimate the norm of \(C_δ : L^2[−δ, δ] \rightarrow ℓ^2\), asymptotically as \(δ \rightarrow 0\). In particular, for \((λ_k)_{−∞}^{∞} \sim k^p\ (p > 1)\) we get a bound which is exponential in \((1/δ)^{1/(p−1)}\), generalizing an earlier result for the case \(p = 2\).

KEY WORDS: nonharmonic Fourier series, coefficient map, norm, window, asymptotic.

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1. Introduction

Let $\lambda = \{\lambda_k\}$ be a real ‘double sequence’ ($k = 0, \pm 1, \pm 2, \ldots$) and let $\hat{\mathcal{M}} = \hat{\mathcal{M}}(\lambda)$ denote the collection of all finite sums
\begin{equation}
\label{eq:1.1}
    f = \sum_k c_k e^{i\lambda_k t}
\end{equation}

with complex coefficients $c_k$. We will here think of viewing such $f$ through a window $(-\delta, \delta)$ and determining the coefficients $\{c_k\}$ from this, defining a coefficient map
\begin{equation}
\label{eq:1.2}
    \hat{C} = \hat{C}(\lambda) : f = \sum_k c_k e^{i\lambda_k t} \mapsto (c_k)
\end{equation}

for $f \in \hat{\mathcal{M}}$. If we now let $\mathcal{M}_\delta = \mathcal{M}_\delta(\lambda)$ be the closure of $\hat{\mathcal{M}}$ in $L^2(-\delta, \delta)$, then it is classical that $\hat{C}$ extends from $\mathcal{M}_\delta$ to $\mathcal{M}_\delta$ as a continuous linear operator
\begin{equation}
\label{eq:1.3}
    C_\delta = C_\delta(\lambda) : \mathcal{M}_\delta \mapsto \ell^2 : f = \sum_{k=-\infty}^{\infty} c_k e^{i\lambda_k t} \mapsto (c_k)_{-\infty}^{\infty} =: c
\end{equation}

provided the asymptotic density of $\lambda$ is bounded by $\delta/\pi$.

In this paper, we consider sequences $\lambda$ satisfying sparsity conditions of the form
\begin{equation}
\label{eq:1.4}
    |\lambda_{k+m} - \lambda_k| \geq \psi_m \quad (m = 1, 2, \ldots)
\end{equation}

for suitable $\psi = \{\psi_m : m = 1, 2, \ldots\}$. Noting that $m/\psi_m \to 0$ ensures that $C_\delta(\lambda)$ is well defined for all $\delta > 0$, we then investigate the rapidity with which $\|C_\delta(\lambda)\| \to \infty$ as $\delta \to 0$. As a by-product of this analysis, we note that our estimates are uniform over the classes of exponent sequences $\Lambda = \Lambda(\psi)$ satisfying (1.4) for particular admissible sequences $\psi$. Our results are new in this aspect as well as in the consideration of the asymptotics as $\delta \to 0$.

It is clear that $C_\delta(\lambda)$ is made up of the coefficient functionals
\begin{equation*}
    \gamma_k : \mathcal{M}_\delta = \mathcal{M}_\delta(\lambda) \to \ell^2 : f \mapsto c_k
\end{equation*}

and that each of these functionals can be represented as
\begin{equation*}
    \gamma_k : f \mapsto c_k = \langle f, g_k \rangle
\end{equation*}

for some $g_k \in L^2(-\delta, \delta)$. There is some arbitrariness in the determination of $g_k$ since (1.5) constitutes an extension of $\gamma_k$ from $\mathcal{M}_\delta$ to all of $L^2(-\delta, \delta)$; this also gives an extension $\hat{C}_\delta$ of $C_\delta(\lambda)$ to $L^2(-\delta, \delta)$.

Since we are working with exponentials, it is then convenient to construct the Fourier transforms to obtain $g_k$ and we actually will work with the adjoint of $\hat{C}_\delta$,
\begin{equation}
\label{eq:1.6}
    \hat{C}_\delta^* : (a_k) \mapsto \sum_k a_k g_k : \ell^2 \to L^2(-\delta, \delta),
\end{equation}

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\end{equation}

for some $g_k \in L^2(-\delta, \delta)$.
to estimate \( \|C_\delta\| \leq \|\tilde{C}_\delta\| = \|\tilde{C}_\delta^*\| \).

We will be able to treat conditions (1.4) for real sequences \( \psi = \{\psi_m : m = 1, 2, \ldots\} \) for which

\[
0 < \psi_1 \leq \psi_2 \leq \ldots \quad \text{and} \quad \sum_1^\infty 1/\psi_k < \infty \tag{1.7}
\]

Note that this already implies that \( m/\psi_m \to 0 \) which precisely corresponds to the condition that \( \Lambda \) have asymptotic density zero.

Our paper falls into three parts:

**First**, considering a sequence \( \lambda \) satisfying (1.4) subject to (1.7), we will apply an important theorem due to Luxembourg and Korevaar ([2]; Theorem 3.1) which we restate here in a relevant form:

**THEOREM K-L:** Let \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) be nondecreasing with \( \omega(t)/t^2 \) integrable at \( \infty \). Then, for any \( \delta > 0 \), there exists a number \( Q > 0 \) and an entire function \( P(\cdot) \) such that:

\[
\begin{align*}
(i) & \quad P \text{ is of exponential type } \delta, \\
(ii) & \quad P \text{ is normalized so } P(0) = 1, \\
(iii) & \quad |P(s)| \leq e^Q e^{-\omega(|s|)} \quad \text{for } s \in \mathbb{R}.
\end{align*}
\]

Clearly, the constant \( Q \) in (iii) will depend on \( \delta \) so \( Q = Q(\delta) = Q(\delta; \omega) \). From this we then obtain an estimate:

\[
\|C_\delta\| \leq A e^{Q(\delta)} \quad \text{for all } \delta > 0 \tag{1.8}
\]

where \( A \) is independent of \( \delta \) and we use a function \( \omega \) depending only on \( \psi \). So far, this is only slightly different from the treatment in [2].

**Second**, and this is the principal technical innovation of the paper, we extend the analysis of Theorem K-L from that of [2], specifically considering the estimation of \( Q(\delta) \) in (1.8) so as to exhibit explicitly its asymptotics as \( \delta \to 0 \) as well as the dependence on \( \psi \) through \( \omega(\cdot) \). From (1.8), this is precisely what is needed to investigate the asymptotic behavior of \( \|C_\delta\| \). For this estimation, we find it necessary to assume \( \omega \in \Omega \) where

\[
\Omega := \left\{ \omega : \mathbb{R}_+ \to \mathbb{R}_+ : \begin{array}{l}
\omega(t) \text{ is nondecreasing while} \\
\omega(t)/t^2 \text{ is decreasing and integrable at } \infty
\end{array} \right\};
\]

this strengthens very slightly the hypotheses above for Theorem K-L in requiring that \( \omega(t)/t^2 \) be decreasing.

**Third**, the combination of the above is applied to obtain specific growth estimates for a number of interesting special cases. In particular, we apply the general analysis to the case \( \psi_m = am^p \) \((a > 0, p > 1)\) which corresponds to \( \lambda_k \sim \pm ak^p \) and obtain for that case the estimate

\[
\log \|C_\delta\| = \mathcal{O}(1/\delta^{1/(p-1)}) \quad \text{as } \delta \to 0 \tag{1.10}
\]

\[\text{Note that one has the geometric characterization } \|\gamma_k\| = 1/\text{[distance from } e^{i\lambda_k t} \text{ to } M(\lambda^k)\}] \text{ where } \lambda^k = \lambda \setminus \{\lambda_k\}. \text{ This gives } \|\gamma_k\| \geq 1/|e^{i\lambda_k t}| = 1/\sqrt{2\delta}, \text{ showing the uselessness of the crude estimate } \|C_\delta f\|^2 = \sum_k |\langle \gamma_k, f \rangle|^2 \leq \left(\sum_k \|\gamma_k\|^2\right) \|f\|^2.\]
(i.e., \(Q(\delta) \leq \mu[1/\delta]^{1/(p-1)}\) in (1.8) for small \(\delta > 0\); we also have an estimate for \(\mu\). We recall that earlier investigation in [4] of the special case
\[
\lambda_k = k^2 \quad (k \geq 0), \quad \lambda_{-k} = -\lambda_k
\]
resulted in an estimate \(\log \|C_\delta\| = \mathcal{O}(1/\delta)\) which was there shown to be sharp (by an example due to Korevaar).

2. The Interpolation Family

Assume that \(\psi\) satisfies (1.7) and that \(\lambda = (\lambda_k)_{-\infty}^\infty\) is in \(\Lambda(\psi)\), i.e., satisfies the separation condition (1.4). With \(\psi\) we associate the function \(\Psi\) given by
\[
\Psi(s) := 2 \sum_{m=1}^\infty \log \left(1 + \frac{s^2}{\psi_m^2}\right).
\]
(2.1)

We will show in the Appendix (Lemma A.1) that this function \(\Psi(\cdot)\) is, indeed, in \(\Omega\). Given \(\lambda \in \Lambda(\psi)\), we next define a family of functions \((\eta_k)_{-\infty}^\infty\) by first defining
\[
\mu_k(z) := \prod_{j \neq k} \left(\frac{z - \lambda_j}{\lambda_k - \lambda_j}\right) \quad (z \in \mathbb{C})
\]
and then setting
\[
\eta_k(z) := \mu_k(z)\mu_k(2\lambda_k - z) = \prod_{j \neq k} \left[1 - \left(\frac{z - \lambda_k}{\lambda_k - \lambda_j}\right)^2\right].
\]
(2.3)

**LEMMA 2.1:** Let \(\psi, \Psi\) be as above and define \(\eta_k\) for \(k \in \mathbb{Z}\) as in (2.3). Then one has
\[
\eta_k(\lambda_j) = \delta_{j,k} \quad (j, k \in \mathbb{Z})
\]
(2.4)

and each \(\eta_k(\cdot)\) is an entire function of exponential type 0 with
\[
|\eta_k(\lambda_k + s)| \leq e^{\Psi(|s|)} \quad \forall s \in \mathbb{R}.
\]
(2.5)

**Proof:** For each \(k\) and any \(N > 0\), there is some \(M = M_{k,N}\) such that
\[
|\mu_k(z)| \leq \left(\prod_{0 < |j-k| \leq N} \left|\frac{z - \lambda_j}{\lambda_k - \lambda_j}\right|\right) \exp \left([|z| + |\lambda_k|] \sum_{|j-k| > N} \frac{1}{|\lambda_j - \lambda_k|}\right)
\]
\[
\leq M(1 + |z|)^{2N+1} \exp \left[|z| \sum_{m > N} \frac{1}{\psi_m}\right]
\]

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for all \( z \in \mathbb{C} \). This estimate ensures suitable convergence of the product in (2.2) to have \( \mu_k \) entire. Further, the sum in the exponential can be made arbitrarily small by taking \( N \) large since \( \{1/\psi_m\} \) is summable by assumption whence each \( \mu_k \) (and so each \( \eta_k \)) is of exponential type 0. The property (2.4) is obvious from \( \mu_k(\lambda_j) = \delta_{j,k} \). Finally, for real \( s \) we have

\[
|\eta_k(\lambda_k + s)| \leq \prod_{j \neq k} \left[ 1 + \left( \frac{s}{\lambda_k - \lambda_j} \right)^2 \right] \leq \prod_m \left( 1 + \frac{s^2}{\psi_m^2} \right) = e^{\Psi(|s|)}
\]

so one has (2.5) as desired. \( \blacksquare \)

Selecting any \( \gamma \in \Omega \) such that \( e^{-\gamma} \) is integrable, we take \( \omega = \Psi + \gamma \) which is in \( \Omega \) by Lemma A.1; then, fixing \( \delta > 0 \), we let \( P(\cdot) \) and \( Q = Q(\delta) \) be as in Theorem K-L. In terms of this \( P \), we define the family of functions

\[
G_k(z) := \eta_k(z)P(z - \lambda_k) \quad (z \in \mathbb{C}).
\]

Our first principal result of this section is the following.

**THEOREM 2.2:** We have:

(i) Each \( G_k \) is an entire analytic function of exponential type \( \delta \),

(ii) For \( j, k \in \mathbb{Z} \) we have \( G_k(\lambda_j) = \delta_{j,k} := \{1 \text{ for } j = k; \ 0 \text{ else }\} \),

(iii) Each \( G_k \), considered on the reals, is in \( L^1(\mathbb{R}) \) with

\[
|G_k(\lambda_k + s)| \leq e^{Q(\delta)}e^{-\gamma(|s|)} \tag{2.7}
\]

(iv) Each \( G_k \) is in \( L^2(\mathbb{R}) \) and one has

\[
|\langle G_j, G_k \rangle| \leq \left[ 4e^{2Q(\delta)} \int_0^\infty e^{-\gamma(s)} ds \right] e^{-\gamma(\psi_m/2)} \tag{2.8}
\]

for any \( j = k \pm m \) (i.e., \( m = |k - j| \)).

**Proof:** The assertion (i) follows on combining Lemma 2.1 (for \( \eta_k \)) and Theorem K-L with \( \omega = \Psi + \gamma \). As noted in Lemma 2.1, we have \( \eta_k(\lambda_j) = \delta_{j,k} \); hence, since \( P(0) = 1 \), we have (ii). The estimate (2.7) is immediate from (2.5) combined with Theorem K-L (iii) so we have (iii).

Finally, to prove (iv) we assume, with no loss of generality, that \( \lambda_j \leq \lambda_k \) and set \( \lambda := (\lambda_j + \lambda_k)/2 \). Note that we then have \( \lambda_j = \lambda - \tau \) and \( \lambda_k = \lambda + \tau \) with \( \tau := (\lambda_k - \lambda_j)/2 \geq \psi_m/2 \) by (1.4) so \( \gamma(\tau) \geq \gamma(\psi_m/2) \). Note also that for \( t \leq \lambda \) one has \( t - \lambda_j =: s \leq \tau \) so
$2\tau - s \geq \tau$ and $\gamma(|2\tau - s|) \geq \gamma(\tau)$; for $t \geq \lambda$ we set $s := t - \lambda_k \geq -\tau$ and $\gamma(|2\tau + s|) \geq \gamma(\tau)$. Thus, using (2.7),

$$|\langle G_j, G_k \rangle| \leq \int_{-\infty}^{\infty} |G_j(t)||G_k(t)| \, dt = \int_{-\infty}^{\lambda} + \int_{\lambda}^{\infty}$$

$$= \int_{-\infty}^{\tau} |G_j(\lambda_j + s)||G_k(\lambda_k - [2\tau - s])| \, ds$$

$$+ \int_{\tau}^{\infty} |G_j(\lambda_j + [2\tau + s])||G_k(\lambda_k + s)| \, ds$$

$$\leq e^{2Q(\delta)} \left[ \int_{-\infty}^{\tau} e^{-\gamma(s)}e^{-\gamma(\tau)} \, ds + \int_{\tau}^{\infty} e^{-\gamma(\tau)}e^{-\gamma(|s|)} \, ds \right]$$

$$\leq e^{2Q(\delta)}e^{-\gamma(\psi_m/2)} \left[ 2 \int_{-\infty}^{\infty} e^{-\gamma(|s|)} \, ds \right]$$

which is just (2.8). In particular, for $j = k$ this shows $G_k \in L^2(\mathbb{R})$. \hfill \blacksquare

Depending on the choice of $\gamma(\cdot)$, this construction will determine the ‘constant’ $Q(\delta)$ of (2.7) as a function of $\delta > 0$. Also depending on the choice of $\gamma(\cdot)$, but now not on $\delta$, we set

$$A^2 := \frac{2}{\pi} \int_{0}^{\infty} e^{-\gamma(s)} \, ds \left[ e^{-\gamma(0)} + 2 \sum_{m=1}^{\infty} e^{-\gamma(\psi_m/2)} \right].$$

Noting that (1.7) gives $\psi_m \geq cm$ for some $c > 0$ so $\gamma(\psi_m/2) \geq \gamma(cm)$, we may compare the sum to the integral $\int_{0}^{\infty} e^{-\gamma(cm)} \, ds$ and observe that the integrability of $e^{-\gamma}$ ensures finiteness of $A$. Our other principal result of this section is the following.

**THEOREM 2.3:** Let $\lambda \in \Lambda(\psi)$ for some sequence $\psi$ satisfying (1.7). Then for any $\delta > 0$ the coefficient map $C_\delta = C_\delta(\lambda)$ defined by (1.3) satisfies

$$\|C_\delta\| \leq Ae^{Q(\delta)} \quad (\delta > 0)$$

with $Q(\delta)$ as in (2.7) and $A$ as in (2.9), independent of $\delta$.

**Proof:** The argument is here quite similar to that in [4]. We use the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\mathcal{F} : g \mapsto G \quad G(z) = \int_{-\infty}^{\infty} e^{-izt}g(t) \, dt$$

and note that, with a factor of $2\pi$, this is an isometric isomorphism:

$$\langle g, \tilde{g} \rangle := \int_{-\infty}^{\infty} \overline{g(t)}\tilde{g}(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{G(t)}\tilde{G}(t) \, dt = \frac{1}{2\pi} \langle G, \tilde{G} \rangle.$$
Initially, let us consider $f \in \mathcal{M}$ (so $f = \sum c_k e^{i\lambda_k t}$ is a finite sum) and view this through the window as $f \in L^2(-\delta, \delta)$. Then, for each $k \in \mathbb{Z}$, noting that $\text{supp} (g_k) \subset [-\delta, \delta]$, we have

$$
\langle f, g_k \rangle := \int_{-\delta}^{\delta} \left( \sum_{j} c_j e^{i\lambda_j t} \right) g_k(t) \, dt
$$

$$
= \sum_{j} c_j \int_{-\delta}^{\delta} e^{-i\lambda_j t} g_k(t) \, dt = \sum_{j} G_k(\lambda_j) c_j.
$$

By Theorem 2.2 (ii) we thus have, as in (1.5),

$$
(2.13) \quad c_k = \langle \hat{f}, g_k \rangle \quad (k \in \mathbb{Z}, f \in \mathcal{M}).
$$

Now consider the Gramian matrix $G$ with entries $\langle g_j, g_k \rangle$. Since we continue to consider the (fixed) function $f \in \mathcal{M}$ as a finite sum, we may take $G$ to be a finite matrix, considering only the indices $k$ for which $c_k \neq 0$; thus there are no convergence problems but we seek estimates independent of this restricted index set. As a Gramian matrix, $G$ is positive definite so the $\ell^2$-induced matrix norm $\|G\|_2$ is just the largest eigenvalue of $G$. Hence,

$$
(2.14) \quad \|G\|_2 \leq \|G\|_\infty := \max_j \left\{ \sum_k |\langle g_j, g_k \rangle| \right\}
$$

since $\|G\|_\infty$ is itself the $\ell^\infty$-induced matrix norm. Thus we have

$$
\left\| \sum_k a_k g_k \right\|_{L^2(-\delta, \delta)}^2 = \sum_{j,k} \langle g_j, g_k \rangle c_j a_k \leq \|G\|_\infty \|a\|^2
$$

for (finite) vectors $a = (a_k) \in \ell^2$. Hence, using (2.13),

$$
\|c\|^2 = \sum_k |c_k|^2 = \sum_k \langle f, g_k \rangle c_k = \langle f, \sum_k c_k g_k \rangle
$$

$$
\leq \|f\| \|\sum_k c_k g_k\| \leq \|f\| \left( \|G\|_\infty \|c\|^2 \right)^{1/2}
$$

so for $f \in \mathcal{M}$ we have the estimate

$$
(2.15) \quad \|c\|_2 \leq (\|G\|_\infty)^{1/2} \|f\|_{L^2(-\delta, \delta)}.
$$

We now use (2.12) and (2.8) to estimate $\|G\|_\infty$ from (2.14). Fixing $j$, we consider $k \in \mathbb{Z}$ and set $m := |k - j|$ so

$$
|\langle g_j, g_k \rangle| = \frac{1}{2\pi} \langle G_j, G_k \rangle \leq 4e^{2Q(\delta)} \left[ \frac{1}{2\pi} \int_{0}^{\infty} e^{-\gamma(s)} \, ds \right] e^{-\gamma(\psi_m/2)}.
$$

Summing over $k \in \mathbb{Z}$ then gives

$$
\sum_k |\langle g_j, g_k \rangle| \leq A^2 e^{2Q(\delta)}
$$
for each \( j \) so \( \|G\|_\infty \leq (Ae^{Q(\delta)})^2 \). Combining this with (2.15) gives
\[
\|C_\delta f\| = \|c\| \leq Ae^{Q(\delta)}\|f\|
\]
for all \( f \in \hat{\mathcal{M}} \). By the density of \( \hat{\mathcal{M}} \) in \( \mathcal{M}_\delta \), this gives precisely the desired estimate (2.10).

3. The Mollifier

Our next object is to re-examine Theorem K-L so as to introduce the ‘mollifier’ \( P(\cdot) \) with a reasonably explicit estimate for \( Q = Q(\delta) \). To this end, given \( \omega \in \Omega \) we set
\[
(3.1) \quad v(s) = \frac{\omega(s)}{s^2}, \quad dq = -s^2 dv.
\]
Note that the definition (1.9) of \( \Omega \) ensures that \( q \) is an unbounded increasing function of \( s \) and that \( \omega(\alpha)/\alpha \to 0 \) as \( \alpha \to \infty \). For each \( \alpha \) in \((0, \infty)\) we can then set
\[
(3.2) \quad \delta(\alpha) := \frac{1 + 2\omega(\alpha)}{\alpha} + 2 \int_\alpha^\infty \frac{\omega(s)}{s^2} ds = \frac{1}{\alpha} + 2 \int_\alpha^\infty \frac{dq}{s}.
\]

**Lemma 3.1:** Fix \( \omega \in \Omega \) and let \( \delta(\cdot) \) be defined by (3.2). Then \( \delta(\alpha) \) is nonincreasing on \((0, \infty)\) and \( \delta(\alpha) \to 0 \) as \( \alpha \to \infty \) so for each \( \delta > 0 \) there exists an \( \alpha := \alpha(\delta) \) such that \( \delta(\alpha) \leq \delta \). Further, fixing \( \delta > 0 \), there is a sequence \((a_j)\) such that
\[
(3.3) \quad \sum_{0}^{\infty} a_j \leq \delta(\alpha) \leq \delta
\]
\[
(3.4) \quad \sum_{a_j|s|\leq1} [a_j]^2 \geq \frac{2\omega(|s|) - 1}{s^2} \text{ for } |s| > \alpha.
\]

**Proof:** Deferred to the Appendix.

We can now state our revised form of Theorem K-L, including the estimate of \( Q(\delta) \).

**Theorem 3.2:** For any \( \delta > 0 \), define \( P(z) \) by
\[
(3.5) \quad P(z) := \prod_{j=1}^{\infty} \cos(a_j z) \quad (z \in \mathbb{C}),
\]
using the sequence \((a_j)\) of Lemma 3.1. Then \( P(\cdot) \) is an even entire function of exponential type \( \delta \) with \( P(0) = 1 \). Further, one has
\[
(3.6) \quad |P(s)| \leq e^{Q(\delta)} e^{-\omega(|s|)}
\]
for all \( s \in \mathbb{R} \), where Lemma 3.1 is used to define

\[
Q(\delta) := 1/2 + \omega(\alpha(\delta)).
\]

**Proof:** We know that \( \cos(\cdot) \) is even and entire of exponential type 1. By (3.3), it follows that \( P \) is a well-defined even, entire function of exponential type \( \delta \). Observing that

\[
|\cos s| \leq \exp\left[-s^2/2\right] \quad \text{for} \ |s| \leq 1,
\]

it follows from (3.4) that for \( |s| > \alpha \) we have

\[
|P(s)| \leq \prod\{|\cos(a_j s)| : |a_j s| \leq 1\}
\]

\[
\leq \exp\left[-s^2/2 \sum_{a_j |s| \leq 1} a_j^2\right] \leq e^{1/2}e^{-\omega(|s|)}.
\]

Since \( |P(s)| \leq 1 \leq e^{\omega(\alpha)}e^{-\omega(|s|)} \) for any \( s \), we have (3.6) for all \( s \in \mathbb{R} \). \( \square \)

4. **Examples**

We now specialize our work to treat some particular cases more explicitly. In each case, we take \( \omega = (1 + \varepsilon)\Psi \), i.e., \( \gamma := \varepsilon \Psi \). A principal point, here, is that the asymptotics of \( Q(\delta) \) as \( \delta \to 0 \) are (almost) determined by the asymptotics of \( \psi_m \) as \( m \to \infty \). In the first two examples, we also note the convenience of taking \( \psi_m = \psi(m) \) for a suitable function \( \psi(\cdot) \), giving an integral version of (2.1) for the asymptotically correct determination of \( \Psi(\cdot) \).

**EXAMPLE 1:** We first suppose \( \psi(x) = ax^p \) \((a > 0, p > 1)\); when \( p = 2 \) this is the case considered in [4]. It is easily seen that \( \psi := \{\psi_m\} \) satisfies (1.7). To simplify the explicit computation of various quantities, we deal with the integral version of (2.1), namely,

\[
\Psi(s) := 2 \int_0^\infty \log \left(1 + \frac{s^2}{\psi(x)^2}\right) dx
\]

\[
= 2 \int_0^\infty \log \left(1 + \frac{s^2}{a^2 x^{2p}}\right) dx
\]

\[
= 2s^{1/p} \left[ \frac{1}{a^{1/p}} \int_0^\infty \log \left(1 + \frac{1}{u^{2p}}\right) du \right] =: \beta(p) s^{1/p}
\]

where (cf., e.g., [1] p.114) \( \beta(p) = \frac{2\pi}{a^{1/p} \sin \frac{\pi}{2p}} \). Now, let \( \varepsilon > 0 \) and let \( \omega(s) := (1 + \varepsilon)\Psi(s) \). From (3.2),

\[
\delta(\alpha) = \frac{1 + 2\omega(\alpha)}{\alpha} + 2 \int_\alpha^\infty \frac{\omega(s)}{s^2} ds
\]

\[
= \frac{1 + 2(1 + \varepsilon)\beta(p) \alpha^{1/p}}{\alpha} + \int_\alpha^\infty \frac{2(1 + \varepsilon)\beta(p) s^{1/p}}{s^2} ds
\]

\[
= \frac{1}{\alpha} + \frac{\beta}{\alpha^{1/q}}
\]
where \( \vartheta := \vartheta(\varepsilon, p) = 2(1 + q)(1 + \varepsilon)\beta(p) \) with \( pq = p + q \). Since \( \delta(\alpha) \leq (1 + \varepsilon)\vartheta/\alpha^{1/q} \) for large \( \alpha \), we see that \( \alpha(\delta) \leq [(1 + \varepsilon)\vartheta/\alpha]^{q} \) for large \( \alpha \), i.e., for small \( \delta \). By (3.7),

\[
Q(\delta) := \frac{1}{2} + \omega(\alpha) \leq \frac{1}{2} + (1 + \varepsilon)\beta(p) \left( \frac{(1+\varepsilon)\vartheta}{\delta} \right)^{q/p}.
\]

Thus, (2.10) becomes

\[
(4.1) \quad \|C_{\delta}\| \leq A e^{\frac{1}{2}\frac{\varepsilon}{\delta^{2}}} \exp \left[ B(\log \frac{1}{\delta})^{q/p} \right]
\]

where, with a corresponding constant \( B > B_{0} \) arbitrary with

\[
B_{0} := \beta(p)\vartheta^{q/p} = 2^{2q-1} \left( \frac{1+q}{a} \right)^{q/p} \left( \frac{\pi}{\sin \pi/2p} \right)^{q}.
\]

Since \( \frac{2}{p} = \frac{1}{p-1} \), we have the promised estimate (1.10).

**EXAMPLE 2:** We next consider sequences which are even more sparse: \( \psi(x) = ce^{\beta x} \) with \( c, \beta > 0 \), indicating how various quantities can be computed. We now have

\[
\Psi(s) = 2 \int_{0}^{\infty} \log \left( 1 + \frac{s^2}{\psi(x)^2} \right) dx = \frac{2}{\beta} \int_{0}^{\infty} \log \left( 1 + \frac{s^2}{c^2 e^{2\beta x}} \right) \beta dx
\]

\[
= \frac{1}{\beta} \int_{-\sigma}^{\infty} \log(1 + e^{-r}) dr \quad \text{(where } \frac{s^2}{x^2} = e^\sigma, r = 2\beta x - \sigma) \]

\[
= \frac{1}{\beta} \int_{0}^{\infty} \log(1 + e^{-r}) dr + \frac{2}{\beta} \int_{0}^{\sigma} \log(1 + e^r) dr
\]

\[
\sim \frac{2}{\beta} [\log s]^2 \quad \text{as } s \to \infty.
\]

Asymptotically, \( \omega(s) \sim (1 + \varepsilon)\frac{2}{\beta} [\log s]^2 \), so one has \( \delta(\alpha) \sim \frac{8(1+\varepsilon)}{\beta\alpha} [\log \alpha]^2 \) by a simple computation and \( \delta(\alpha) \leq (1/\alpha)^{1/p} \) for arbitrary \( p > 1 \) and large \( \alpha \). Hence \( \alpha(\delta) \leq 1/\delta^{p} \) for small \( \delta \). Thus,

\[
Q(\delta) := \frac{1}{2} + \omega(\alpha) \leq \frac{1}{2} + \frac{2}{\beta}(1 + \varepsilon)p^{2}[\log 1/\delta]^{2}
\]

and one has, therefore,

\[
(4.2) \quad \|C_{\delta}\| \leq A \exp \left[ B(\log 1/\delta)^{2} \right]
\]

for any \( B > B_{0} := 2/\beta \) and a suitable constant \( A \).

**EXAMPLE 3:** In this example, we consider the ultimate asymptotic sparsity: a finite sequence \( \{\lambda_j\} \) of \( L+1 \) distinct real numbers. Taking these in increasing order and setting \( c := \min\{|\lambda_k - \lambda_j| : j \neq k\} > 0 \), we then automatically have the condition (1.4) with \( \psi_m := mc \) for \( m = 0, \ldots, L \) and (formally) \( \psi_m := \infty \) for \( m > L \), giving

\[
\Psi(s) := 2 \sum_{m=1}^{L} \log \left( 1 + \frac{s^2}{c^2 m^2} \right) = 4L \log s + O(1).
\]
We now have $\omega(s) \sim 4L(1+\varepsilon)\log s$ so $\delta(\alpha) \sim 16L(1+\varepsilon)\frac{1}{\alpha}\log \alpha$ for large $\alpha$. Hence, as in the previous example, $\alpha(\delta) \leq 1/\delta^p$ for arbitrary $p > 1$ and small $\delta$ so

$$Q(\delta) := \frac{1}{2} + \omega(\alpha) \leq \frac{1}{2} + 4p(1+\varepsilon)L\log 1/\delta.$$ 

One therefore has algebraic growth in $1/\delta$ for the norm in this case:

$$\|C_\delta\| \leq A\delta^{-\nu} \tag{4.3}$$

for any $\nu > \nu_0 := 4L$ with a corresponding constant $A$.

5. Appendix

**Lemma A.1:** Let $\psi$ be any increasing positive sequence satisfying (1.7). Then (2.1) defines a function $\Psi$ on $\mathbb{R}^+$ such that

(i) $\Psi$ is continuous and unbounded on $[0, \infty)$ with $\Psi(0) = 0$,

(ii) $\Psi$ is $C^1$ and (strictly) increasing on $\mathbb{R}^+$,

(iii) $\Psi(s)/s^2$ is decreasing on $(0, \infty)$,

(iv) $\Psi(s)/s^2$ is integrable at $\infty$,

(v) $e^{-\Psi}$ is integrable on $\mathbb{R}^+$.

**Proof:** Since $\psi = \psi_k \to \infty$ as $k \to \infty$ so $\log(1+1/\psi) \sim 1/\psi$, we see from (1.7) that the sum in (2.1) is well defined and finite for each $s \geq 0$. Further, each term in that sum is (strictly) increasing in $s$ and continuous. By the Weierstrass M-test, the series converges uniformly on any closed and bounded interval in $\mathbb{R}^+$ so $\Psi(s)$ is continuous. Similarly, $\Psi' = 4\sum_{\nu=1}^{\infty} \frac{s}{(s^2+\psi_m^2)^{\nu}}$ which is finite by (1.7) and positive on $\mathbb{R}^+$. Thus we have (5.1-1,ii). To see (iii), we observe that

$$\Psi(s)/s^2 = 2\sum_{\nu=1}^{\infty} \frac{\rho([s/\psi_m]^2)}{\psi_m^2},$$

with $\rho(u) := \log(1+u)/u$ and note that $\rho$ is strictly decreasing for $u > 0$.

To get (iv), we observe that $\Psi(s)/s^2$ will be integrable at $\infty$ if and only if the series $\{\int_1^{\infty}(1/s^2)\log(1 + s^2/\psi_k^2)\, ds\}$ is summable. From the identity

$$\int \frac{\log(1 + u^2)}{u^2} = 2\tan^{-1} u - \frac{\log(1 + u^2)}{u},$$

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we get
\[ \int_{1}^{\infty} \frac{1}{s^2} \log \left( 1 + \frac{s^2}{\psi^2} \right) ds = \frac{\pi}{\psi} - 2 \frac{1}{\psi} \tan^{-1} \frac{1}{\psi} + \log \left( 1 + \frac{1}{\psi^2} \right). \]

Using (1.7) and (i), we get (iv). Statement (v) is obvious.

Finally, we provide the promised proof of Lemma 3.1.

**Proof** [of Lemma 3.1]: As already noted, the definition (1.9) of Ω ensures that q is increasing, so the right hand side of (3.2) is (strictly) decreasing to 0 as \( \alpha \to \infty \) by the integrability of \( \omega/s^2 \). Thus, \( \delta(\cdot) \) is invertible with \( \alpha(\delta) \) defined for (small) \( \delta > 0 \).

Now, fixing \( \delta > 0 \) and so \( \alpha = \alpha(\delta) \), we use (3.1) to define

(5.2) \[ a_j := \frac{1}{q^{-1}(z_j)} \text{ with } z_j := q(\alpha) + j/2 \]

for \( j = 0, 1, \ldots \). An integral comparison, noting that the function \( 1/q^{-1}(\cdot) \) is decreasing and that \( z_{j+1} - z_j \equiv 1/2 \), gives

\[ \sum_{0}^{\infty} a_j = \frac{1}{\alpha} + \sum_{1}^{\infty} \frac{1}{q^{-1}(z_j)} \leq \frac{1}{\alpha} + 2 \int_{q(\alpha)}^{\infty} \frac{dz}{q^{-1}(z)} \]

which precisely gives (3.3) on using (3.2) for \( z = q(s) \).

For \( |s| > \alpha \), we now note that \( j_* \geq 1 \) where \( j_* = j_*(s, \alpha) \) is the smallest \( j \) for which \( a_j |s| \leq 1 \); hence, \( 0 \leq z_{j_*} - q(|s|) < \frac{1}{2} \). An argument similar to the above then gives

\[ \sum_{a_j |s| \leq 1} [a_j]^2 = \sum_{j_*} \frac{1}{[q^{-1}(z_j)]^2} \geq 2 \int_{z_{j_*}}^{\infty} \frac{dz}{[q^{-1}(z)]^2} \]

\[ \geq 2 \int_{q(|s|)}^{\infty} \frac{dz}{[q^{-1}(z)]^2} - \frac{1}{s^2} \]

(since \( q^{-1}(z) \geq |s| \) for \( s \leq z \leq z_{j_*} \))

\[ = 2v(|s|) - 1/s^2 \]

which is just (3.4). ■

**References**


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*Although (3.1) only defines \( q(\cdot) \) to within an additive constant, the formula (5.2) suffices to specify \( (a_j) \).*