GENERALIZED LINEAR COMPLEMENTARITY PROBLEMS

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It has been shown by Lemke that if a matrix is copositive plus on $\mathbb{R}^n$, then feasibility of the corresponding Linear Complementarity Problem implies solvability. In this article we show, under suitable conditions, that feasibility of a *Generalized* Linear Complementarity Problem (i.e., defined over a more general closed convex cone in a real Hilbert Space) implies solvability whenever the operator is copositive plus on that cone. We show that among all closed convex cones in a finite dimensional real Hilbert Space, polyhedral cones are the *only* ones with the property that every copositive plus, feasible $GLCP$ is solvable. We also prove a perturbation result for Generalized Linear Complementarity Problems.

Key Words: Linear Complementarity Problems, polyhedral cones, copositive plus, thin cone.

Abbreviation Title: Complementarity Problem.

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1 Introduction

There are many existence results concerning Complementarity Problems in both finite dimensional and infinite dimensional settings. See, e.g., Habetler and Price [10], Karamardian [12–15], Moré [22,23], Parida and Roy [25], Saigal [26], Allen [1], Borwein [3,4], Isac [11].

By a *Generalized Linear Complementarity Problem* (GLCP) on a real Hilbert Space $\mathcal{H}$ we mean: Find an $x \in \mathcal{K}$ such that

$\langle \mathbf{T}x + q, k \rangle \geq 0 \ \ (\forall\, k \in \mathcal{K})$ and $\langle \mathbf{T}x + q, x \rangle = 0$

where we are given a closed convex cone $\mathcal{K}$ in $\mathcal{H}$, a continuous linear operator $\mathbf{T}$ on $\mathcal{H}$, and a q in $\mathcal{H}$. We denote this problem by $GLCP(\mathbf{T}, \mathcal{K}, q)$. The phrase 'generalized linear complementarity problem' was introduced (in the finite dimensional setting) by Habetler and Price [10]. The generalization there is to the use of a cone other than the non-negative orthant $\mathcal{K} = (\mathbb{R}_+)^n \subset \mathbb{R}^n$ — the setting for the (classical) Linear Complementarity Problem which has been extensively studied. Here, we include the generalization to infinite dimensional Hilbert spaces (although we must note that only our main existence result, Theorem 4.1, deals with a general Hilbert space; the rest is set in $\mathbb{R}^n$) and also a further generalization to consider linear operators $\mathbf{T}$ which are copositive plus but which need not be strictly copositive.

In this article, we prove existence and perturbation results for Generalized Linear Complementarity Problems. Our main result, which says that under suitable conditions the solution set of a Generalized Linear Complementarity Problem is nonempty and weakly compact, extends several results of Borwein [3]. Our proof of the main result is based on approximating the given cone by polyhedral cones and applying the (classical) result of Lemke (that *for copositive plus matrices, feasibility implies solvability*) to each of the polyhedral cones. This result can also be proved by using Ky Fan's min-max inequality (Allen [1], Isac [11]); such a proof (for a nonlinear mapping on a locally compact cone in a locally convex space) will appear elsewhere.

This paper is organized as follows. In Sec. 3, we state an existence result of Lemke type for polyhedral cones. The main existence result appears in Sec. 4. A perturbation result of Mangasarian-Doverspike type is given in Sec. 6. Sec. 7 deals with a characterization of polyhedral cones in finite dimensional spaces. Finally we state some open problems and indicate a generalization to reflexive Banach spaces.

2 Preliminaries

Let $\mathbf{T}$ be a bounded linear operator on a real Hilbert space $\mathcal{H}$. The adjoint $\mathbf{T}^*$ of $\mathbf{T}$ is the (unique) bounded linear operator on $\mathcal{H}$ defined by $\langle \mathbf{T}^* x, y\rangle = \langle x, \mathbf{T}y\rangle (x, y \in \mathcal{H})$; when $\mathcal{H}=\mathbb{R}^n$, $\mathbf{T}^*$ is just the transpose of $\mathbf{T}$. If $\mathbf{T}^* = \mathbf{T}$, we say that $\mathbf{T}$ is *self-adjoint*. For a closed *convex cone* $\mathcal{K} \subset \mathcal{H}$ (i.e., one has $(\alpha x + \beta y) \in \mathcal{K}$ for $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$) we define the *polar* of $\mathcal{K}$ by

$$\mathcal{K}^* := \{x \in \mathcal{H} : \langle x, k\rangle \geq 0 \text{ for all } k \in \mathcal{K}\}.$$

Given such $\mathbf{T}$, $\mathcal{K}$ and an element $q \in \mathcal{H}$, we say that $GLCP(\mathbf{T}, \mathcal{K}, q)$, as defined in the previous section, is *feasible* provided there exists some x in $\mathcal{K}$ such that $(\mathbf{T}x + q)$ is in $\mathcal{K}^*$. In this case we also call such an x *feasible for* $GLCP(\mathbf{T}, \mathcal{K}, q)$. We say that $GLCP(\mathbf{T}, \mathcal{K}, q)$ is *solvable* if there is an x which is feasible for $GLCP(\mathbf{T}, \mathcal{K}, q)$ and for which $\langle \mathbf{T}x + q, x\rangle = 0$. We say that $\mathbf{T}$ is *copositive plus on* $\mathcal{K}$ if

(a) $k \in \mathcal{K}$ implies $\langle \mathbf{T}k, k\rangle \geq 0$,

(b) $k \in \mathcal{K}$ and $\langle \mathbf{T}k, k\rangle = 0$ imply $(\mathbf{T} + \mathbf{T}^*)k = 0$.

We say that $\mathbf{T}$ is *monotone* on $\mathcal{K}$ if $\langle \mathbf{T}x - \mathbf{T}y, x - y\rangle \geq 0$ for $x, y \in \mathcal{K}$. A set $\mathcal{E}$ in $\mathcal{H}$ is *separable* if $\mathcal{E}$ contains a countable subset which is dense in $\mathcal{E}$. The *weak topology* on $\mathcal{H}$ is the weakest topology on $\mathcal{H}$ making the functional: $x \mapsto \langle x, y\rangle$ continuous for each $y \in \mathcal{H}$ and the *weak-closure* of a set $\mathcal{E}$ in $\mathcal{H}$ is the closure of $\mathcal{E}$ in the weak topology of $\mathcal{H}$.

A cone $\mathcal{K}$ is said to be *thin* if $\mathcal{K}$ is separable and 0 is not in the weak-closure of $\{k \in \mathcal{K} : \|k\| = 1\}$. (Note that: Every cone in a finite dimensional subspace of $\mathcal{H}$ is thin. For any $\alpha > 0$ and a fixed non-zero element e in a separable Hilbert space $\mathcal{H}$, the cone $\{x \in \mathcal{H} : \langle x, e\rangle \geq \alpha\|x\|\|e\|\}$ is thin. It can easily be shown, using the reflexivity of $\mathcal{H}$, that a thin cone is weakly locally compact). A cone $\mathcal{K}$ is said to be *polyhedral* if there exists a finite set $\{a_1, a_2, \ldots, a_n\} \subset \mathcal{K}$ such that

$$\mathcal{K} = \{x \in \mathcal{H} : x = \sum_{m=1}^{n} \lambda_m a_m, \lambda_m \geq 0\}.$$

We note that polyhedral cones are always thin.

For $q_1, q_2 \in \mathcal{H}$, $q_2 \otimes q_1$ denotes the linear operator on $\mathcal{H}$ defined by

$$(q_2 \otimes q_1)(x) = \langle q_1, x\rangle q_2$$

A real valued function $\varphi(x)$ is *lower semi-continuous* (*lsc* for short) on $\mathcal{K}$, with respect to the weak topology, if $\lim\inf\varphi(y) \geq \varphi(x)$ whenever $y \to x$ weakly in $\mathcal{K}$. For $\mathcal{E} \subset \mathcal{H}$, we write: Span $\mathcal{E}$ for the smallest linear subspace of $\mathcal{H}$ containing $\mathcal{E}$, int $\mathcal{E}$ for the *interior* of $\mathcal{E}$ in $\mathcal{H}$, $\partial\mathcal{E}$ for its *boundary*, and $\mathcal{E}^\perp$ for the set $\{x \in \mathcal{H} : \langle x, e\rangle = 0 \text{ for all } e \in \mathcal{E}\}$. For an operator $\mathbf{T}$, we write: Ran $\mathbf{T} = \{\mathbf{T}x : x \in \mathcal{H}\}$ for the *range* and Ker $\mathbf{T} = \{x \in \mathcal{H} : \mathbf{T}x = 0\}$ for the *kernel.*

3 An Existence Result for Polyhedral Cones

Theorem 3.1 *Let $\mathcal{K}$ be polyhedral and let $\mathbf{T}$ be copositive plus on $\mathcal{K}$. If $GLCP(\mathbf{T}, \mathcal{K}, q)$ is feasible then it is solvable.*

Proof: If $\dim \mathcal{H} < \infty$, then, by using an inner product preserving transformation, we can assume that $\mathcal{H} = \mathbb{R}^n$ for some n; using the usual inner product on $\mathbb{R}^n$. Since $\mathcal{K}$ is polyhedral, there is a positive integer m and an $n \times m$ matrix $\mathbf{B} : \mathbb{R}^m \to \mathbb{R}^n$ such that $\mathbf{B}(\mathbb{R}^m_+) = \mathcal{K}$. It follows easily that $\hat{\mathbf{T}} := \mathbf{B}^*\mathbf{T}\mathbf{B}$ is copositive plus on $\mathbb{R}^m_+$. Since $GLCP(\mathbf{T}, \mathcal{K}, q)$ is feasible, there is some $x_0 \in \mathcal{K}$ such that $\langle \mathbf{T}x_0 + q, k\rangle \geq 0$ for $k \in \mathcal{K}$. For any element in $\hat{x}_0 \in \mathbb{R}^m_+$ such that $\mathbf{B}\hat{x}_0 = x_0$ one then has

$$\langle \hat{\mathbf{T}}\hat{x}_0 + \mathbf{B}^*q, x\rangle = \langle \mathbf{B}^*(\mathbf{T}x_0 + q), x\rangle = \langle \mathbf{T}x_0 + q, \mathbf{B}x\rangle \geq 0$$

for each $x \in \mathbb{R}^m_+$. Thus, $GLCP(\hat{\mathbf{T}}, \mathbb{R}^m_+, \mathbf{B}^*q)$ is feasible. By Lemke's Theorem [16], there exists an $\hat{x} \in \mathbb{R}^m_+$ such that

$$\langle \hat{\mathbf{T}}\hat{x} + \mathbf{B}^*q, x\rangle \geq 0 \ \ (x \in \mathbb{R}^m_+) \quad \text{and} \quad \langle \hat{\mathbf{T}}\hat{x} + \mathbf{B}^*q, \hat{x}\rangle = 0. \tag{1}$$

Simplification of (3.1) leads to

$$\langle \mathbf{T}(\mathbf{B}\hat{x}) + q, \mathbf{B}x\rangle \geq 0 (x \in \mathbb{R}^m_+) \text{ and } \langle \mathbf{T}(\mathbf{B}\hat{x}) + q, \mathbf{B}\hat{x}\rangle = 0$$

and we observe that $\mathbf{B}\hat{x}$ solves $GLCP(\mathbf{T}, \mathcal{K}, q)$.

For the general case, let $\mathcal{X}$ be a finite dimensional subspace of $\mathcal{H}$ containing $\mathcal{K}$. Let $\mathbf{P}$ be the (orthogonal) projection from $\mathcal{H}$ into $\mathcal{X}$ and set $\mathbf{S} := \mathbf{PT}$. Then $\mathbf{S} : \mathcal{X} \to \mathcal{X}$ is copositive plus on $\mathcal{K}$, and $\langle \mathbf{S}x_0 + \mathbf{P}q, k\rangle = \langle \mathbf{T}x_0 + q, k\rangle \geq 0$ for each $k \in \mathcal{K}$; thus $GLCP(\mathbf{S}, \mathcal{K}, \mathbf{P}q)$ is feasible in $\mathcal{X}$. By the previous case, there exists $x \in \mathcal{K}$ such that

$$\langle \mathbf{S}x + \mathbf{P}q, k\rangle \geq 0 (k \in \mathcal{K}), \text{ and } \langle \mathbf{S}x + \mathbf{P}q, x\rangle = 0. \tag{2}$$

Since $\langle \mathbf{S}x + \mathbf{P}q, k\rangle = \langle \mathbf{T}x + q, k\rangle$ for each $k \in \mathcal{K}$, (3.2) shows that this x is a solution of $GLCP(\mathbf{T}, \mathcal{K}, q)$. ∎

4 Existence Results for a General Cone

Theorem 4.1 *Suppose that*

(i) $\mathbf{T}$ *is copositive plus on* $\mathcal{K}$*,*

(ii) the map $x \mapsto \langle \mathbf{T}x, x\rangle$ *is weak-lsc on* $\mathcal{K}$*,*

(iii) $\mathcal{K}$ *is thin,*

(iv) $\{k \in \mathcal{K} : \mathbf{T}k \in \mathcal{K}^*,\ \langle \mathbf{T}k, k\rangle = 0, \text{ and } \langle q, k\rangle = 0\} = \{0\}$*.*

If $GLCP(\mathbf{T}, \mathcal{K}, q)$ *is feasible, then the solution set of* $GLCP(\mathbf{T}, \mathcal{K}, q)$ *is non-empty and weakly compact.*

Proof: We first show existence of a solution. Let x_0 be feasible for $GLCP(\mathbf{T}, \mathcal{K}, q)$. We can assume that $x_0 \neq 0$ and $\mathcal{K} \neq \{0\}$; else 0 is the desired solution. Then the set $\{x \in \mathcal{K} : \|x\| = 1\}$ is non-empty and separable by (iii). Let $\{\xi_0, \xi_1, \ldots\}$ be a dense subset of $\{x \in \mathcal{K} : \|x\| = 1\}$ with $\xi_0 := x_0/\|x_0\|$ and let $\mathcal{K}_n$ be the closed convex cone generated by $\{\xi_0, \xi_1, \ldots, \xi_n\}$; each $\mathcal{K}_n$ is polyhedral and contains x_0. Since $K_n \subset \mathcal{K}$, $\mathbf{T}$ is copositive plus on $\mathcal{K}_n$ for each $n = 1, 2, \ldots$ by (i). Now fix an n. Since x_0 is feasible for $GLCP(\mathbf{T}, \mathcal{K}_n, q)$, Theorem 3.1 gives an $x_n \in \mathcal{K}_n$ such that

$$\langle \mathbf{T}x_n + q, k\rangle \geq 0\ (k \in \mathcal{K}_n), \quad \text{and} \quad \langle \mathbf{T}x_n + q, x_n\rangle = 0. \tag{3}$$

We claim that the sequence $\{x_n\}$ is bounded as $n \to \infty$. If not, then we would have (for a subsequence) $\|x_n\| \to \infty$ so, from (4.1),

$$\langle \mathbf{T}u_n + q_n, u_n\rangle = 0 \quad (n = 1, 2, \ldots) \tag{4}$$

with $u_n := x_n/\|x_n\|$ and $q_n := q/\|x_n\| \to 0$. Since $\{u \in \mathcal{K} : \|u\| \leq 1\}$ is closed, convex, and bounded in the Hilbert Space $\mathcal{H}$, $\{u_n\}$ (or a subsequence) converges weakly to some element $d \in \mathcal{K}$. By (ii), and (4.2), we get $\langle \mathbf{T}d, d\rangle \leq 0$ which implies, by (i),

$$\mathbf{T}d = -\mathbf{T}^* d. \tag{5}$$

Now (4.1) implies that $\langle q, x_n\rangle = -\langle \mathbf{T}x_n, x_n\rangle \leq 0$ (by (i)). This gives

$$\langle q, d\rangle \leq 0. \tag{6}$$

Since any $k \in \mathcal{K}$ can be written as $\lim_{n\to\infty} k_n$ with $k_n \in \mathcal{K}_n$, we have from (4.1) that $\langle \mathbf{T}d, k\rangle \geq 0$ for every $k \in \mathcal{K}$, that is,

$$\mathbf{T}d \in \mathcal{K}^*. \tag{7}$$

Now the feasibility of $GLCP(\mathbf{T}, \mathcal{K}, q)$ implies that

$$\langle \mathbf{T}x_0 + q, d\rangle \geq 0$$

whence, using (4.3)and (4.5),

$$\begin{aligned} \langle q, d\rangle &\geq -\langle \mathbf{T}x_0, d\rangle = -\langle x_0, \mathbf{T}^* d\rangle \\ &= \langle x_0, \mathbf{T}d\rangle \geq 0. \end{aligned}$$

From (4.4), we get $\langle q, d\rangle = 0$. Thus, d is in $\{x \in \mathcal{K} : \mathbf{T}x \in \mathcal{K}^*, \langle \mathbf{T}x, x\rangle = 0, \langle q, x\rangle = 0\}$. By (iv) this implies $d = 0$ so $0 \in$ [weak-closure $\{x \in \mathcal{K} : \|x\| = 1\}$] which contradicts (iii). It follows that $\{x_n\}$ (or a subsequence) is bounded. We may then assume that $\{x_n\}$ converges weakly to some element $\bar{x}_0 \in \mathcal{K}$. Now (4.1) gives, by (ii), that $\langle \mathbf{T}\bar{x}_0 + q, \bar{x}_0\rangle \leq 0$. Further, on writing $k = \lim k_n$ with $k_n \in \mathcal{K}_n$, (4.1) shows that, $\langle \mathbf{T}\bar{x}_0 + q, k\rangle \geq 0$ for any $k \in K$. Therefore, $\bar{x}_0$ solves $GLCP(\mathbf{T}, \mathcal{K}, q)$.

Finally, a similar argument shows the solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ cannot be unbounded: otherwise, as before, one could construct an element $d \in \mathcal{K}$ satisfying (4.3), (4.4), (4.5) leading to a contradiction (by (iii)). The weak compactness follows, since the solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is weakly closed in $\mathcal{K}$. ■

Remarks on the Hypotheses of Theorem 4.1

(1) Conditions (i) and (ii) hold, for example, when $\mathbf{T}$ is monotone on $\mathcal{H}$.

(2) Feasibility of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is equivalent to $q \in [\mathcal{K}^* - \mathbf{T}(\mathcal{K})]$. In the presence of (i), $q \in$ int $[\mathcal{K}^* - \mathbf{T}(\mathcal{K})]$ implies that condition (iv) holds (see [3], Prop. 2.3).

(3) If $\text{int}(\mathcal{K}^*)$ is nonempty, then q is in $\text{int}(\mathcal{K}^* - \mathbf{T}(\mathcal{K}))$ if and only if $(\mathbf{T}x + q)$ is in int $\mathcal{K}^*$ for some $x \in \mathcal{K}$ (cf. [3], p.349).

(4) If $\mathcal{K}$ is locally norm compact then condition (iii) holds. In view of Remarks 1 and 2, Theorem 4.1 generalizes the following result of Borwein [3]: If $\mathbf{T}$ is monotone on $\mathcal{H}, \mathcal{K}$ is locally norm compact, and $\mathcal{K}^* - \mathbf{T}(\mathcal{K}) = \mathcal{H}$, then the solution set for $GLCP(\mathbf{T}, \mathcal{K}, q)$ is non-empty and bounded for any q (cf. [3], Theorem 3.1, Remark (v), p.353, and Corollary 2.1).

(5) Without conditions (iii) and (iv) the above result may fail: see Section 5.

(6) When $\mathbf{T}$ is self-adjoint, condition (iv) (in the presence of (i)) becomes $\mathcal{K} \cap$ Ker $T \cap \{q\}^{\perp} = \{0\}$.

(7) When $\dim \mathcal{H} < \infty$, conditions (ii) and (iii) are always true.

Theorem 4.2 *Let* $\dim \mathcal{H} < \infty$ *and suppose that* $\mathbf{T}$*,* $\mathcal{K}$*,* q *are such that*

(i) $\mathbf{T}$ *is self-adjoint,*

(ii) $\mathbf{T}$ *is copositive plus on* $\mathcal{K}$*,*

(iii) $(\mathbf{T} + q \otimes q)(\mathcal{K})$ *is closed.*

If $GLCP(\mathbf{T}, \mathcal{K}, q)$ *is feasible then it is solvable.*

Proof: Let $\mathcal{M}$:=Ker $\mathbf{T} \cap \{q\}^{\top}$. If $\mathcal{M} = \{0\}$, then the result follows immediately from Theorem 4.1 (see Remarks 6 and 7) so assume that $\mathcal{M} \neq \{0\}$. Let $\mathbf{P}$ denote the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}^{\perp}$ =Ran $\mathbf{T}$ + Span$\{q\}$). Since

$\langle \mathbf{T}x + q, k \rangle = \langle \mathbf{TP}x + q, \mathbf{P}k \rangle$ $(x, k \in \mathcal{K})$,

we see that the solvability of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is equivalent to the solvability of $GLCP(\mathbf{T}, \mathbf{P}(\mathcal{K}), q)$. By (iii), the set

$\mathcal{K}$ + Ker $(\mathbf{T} + q\otimes q) = \mathcal{K}$ + Ker $\mathbf{T} \cap \{q\}^{\perp} = \mathcal{K}$ + Ker $\mathbf{P}$

is closed; hence, $\mathbf{P}(\mathcal{K})$ is closed. It follows easily from (ii) that $\mathbf{T}$ is copositive plus on $\mathbf{P}(\mathcal{K})$ and, finally, we have

$\mathbf{P}(\mathcal{K})\cap$ Ker $\mathbf{T} \cap \{q\}^{\perp} \subset$ Ran $\mathbf{P}\cap$ Ker $\mathbf{P} = \{0\}$.

Thus, all the conditions of Theorem 4.1 are met for $GLCP(\mathbf{T}, \mathbf{P}(\mathcal{K}), q)$ so it is solvable. This implies that $GLCP(\mathbf{T}, \mathcal{K}, q)$ is also solvable. ∎

5 Examples

Example 1. This example shows that if condition (iii) is omitted in Theorem 4.1 then $GLCP(\mathbf{T}, \mathcal{K}, q)$ need *not* be solvable. Let

$\mathcal{H} = \ell_2 = \{x = (x_1, x_2, \ldots) : x_n \in \mathbb{R},\ \sum_1^{\infty} x_n^2 < \infty\}$

with inner product given by $\langle x, y\rangle = \sum_1^{\infty} x_n y_n$. Let ℓ_2^+ be the positive cone in ℓ_2 (i.e., each $x_n \geq 0$ for $x = (x_1, x_2, \ldots) \in \ell_2^+ \subset \ell_2$). Set $v = \alpha(1, \frac{1}{4}, \frac{1}{9}, \ldots)$ and $u = \beta(\gamma, \frac{1}{2}, \frac{1}{3}, \ldots)$ where α, β, γ are chosen so that $\alpha, \beta > 0, \|u\| = \|v\| = 1$, and $\langle u, v\rangle = 0$. We define a projection operator $\mathbf{P} : \ell_2 \to \ell_2$ by $\mathbf{P} : x \mapsto \langle x, u\rangle u + \langle x, v\rangle v$. Being a projection, $\mathbf{P}$ is monotone on ℓ_2 and hence copositive plus on ℓ_2^+. Once again, monotonicity of $\mathbf{P}$ implies the weak lower semi-continuity of $x \mapsto \langle \mathbf{P}x, x\rangle$. If $0 \neq x \in$ Ker $\mathbf{P}$, then $\langle x, v\rangle = 0$ so some component of x must be negative. This shows that $\ell_2^+\cap$ Ker $\mathbf{P} = \{0\}$. Although ℓ_2^+ is separable, we do have $0 \in$ weak-closure $\{x \in \ell_2^+ : \|x\| = 1\}$. Let e_n denote the element in ℓ_2^+ with 1 as the n^{th} entry and zero elsewhere. Put $q = -u$. Then, for $n > 1$,

$$\begin{aligned} \mathbf{P}(\tfrac{n}{\beta}e_n) &= \langle \tfrac{n}{\beta}e_n, u\rangle u + \langle \tfrac{n}{\beta}e_n, v\rangle v \\ &= u + \tfrac{\alpha}{\beta}\tfrac{1}{n}v. \end{aligned} \tag{8}$$

Since $v \in (\ell_2^+)^*$ $(= \ell_2^+)$, we have $[\mathbf{P}(\frac{n}{\beta}e_n) + q] \in (\ell_2^+)^*$. Thus $GLCP(\mathbf{P}, \ell_2^+, q)$ is feasible and we have verified all the conditions of Theorem 4.1 except part of the thinness of ℓ_2^+. Suppose, there were an $a \in \ell_2^+$ with $\langle \mathbf{P}a + q, x\rangle \geq 0$ for all $x \in \ell_2^+$, and $\langle \mathbf{P}a + q, a\rangle = 0$. The inequality $\langle \mathbf{P}a + q, \frac{n}{\beta}e_n\rangle \geq 0$ $(n = 1, 2, \ldots)$ gives $\langle a, u\rangle + \frac{\alpha}{\beta}\frac{1}{n}\langle a, v\rangle - 1 \geq 0$ for $n = 2, 3, \ldots$ so $\langle a, u\rangle \geq 1$. Now $\langle \mathbf{P}a + q, a\rangle = 0$ reduces to $|\langle a, u\rangle|^2 + |\langle a, v\rangle|^2 - \langle a, u\rangle = 0$ and, since $\langle a, u\rangle \geq 1$, this implies $\langle a, v\rangle = 0$. Since $a \in \ell_2^+$ and all entries of v are strictly positive, one must then have $a = 0$ whence $\langle -u, x\rangle = \langle q, x\rangle = \langle \mathbf{P}a + q, x\rangle \geq 0$ for

$x \in \ell_2^+$. But $\langle -u, e_2\rangle = -\beta/2 < 0$ and we have a contradiction. Thus, *$GLCP(\mathbf{P}, \ell_2^+, q)$ is not solvable.*

Remark:

(8) Letting $n \to \infty$ in (5.1) we see that $u \in cl[\mathbf{P}(\ell_2^+)]$. However, u is not in $\mathbf{P}(\ell_2^+)$: if one were to have $\mathbf{P}x = \langle x, u\rangle u + \langle x, v\rangle v = u$ for some $x \in \ell_2^+$, then $\langle x, v\rangle = 0$ which implies that $x = 0$ and hence $u = \mathbf{P}x = 0$. Thus, *the image of ℓ_2^+ under $\mathbf{P}$ is not closed.*

Example 2. This example shows that if condition (iv) were to be omitted in Theorem 4.1 then the result need not be true; the same example shows that condition (iii) in Theorem 4.2 cannot be omitted. Let

$\mathcal{H} = \mathbb{R}^3$, $\mathbf{T}(x, y, z) = (x, y, 0)$, $\mathcal{K} = \{(x, y, z) : x, z \geq 0, 2xz \geq y^2\}$.

Put $q = (1, 1, 0)$ and $x_0 = (1, -1, 1)$. Then $\mathcal{K}$ is a closed convex cone with $\mathcal{K}^* = \mathcal{K}$ and $x_0 \in \mathcal{K}$. Since $\mathbf{T}x_0 + q = (2, 0, 0) \in \mathcal{K}^*$, $GLCP(\mathbf{T}, \mathcal{K}, q)$ is feasible. Being a projection, $\mathbf{T}$ is monotone on $\mathbb{R}^3$ and hence copositive plus on $\mathcal{K}$. Now suppose $k = (x, y, z)$ were any solution of $GLCP(\mathbf{T}, \mathcal{K}, q)$. Then $\mathbf{T}k + q = (x + 1, y + 1, 0) \in \mathcal{K}^*$ $(= \mathcal{K})$ whence $y = -1$. Since $2xz \geq y^2 = 1$, one has $x > 0$ so $\langle \mathbf{T}k + q, k\rangle = x(x + 1) + y(y + 1) = x(x + 1) > 0$. Thus $GLCP(\mathbf{T}, \mathcal{K}, q)$ is not solvable. We observe that

$\mathcal{K} \cap$ Ker $\mathbf{T} \cap \{q\}^\perp = \{(0, 0, z) : z \in \mathbb{R}\} \neq \{0\}$,

$(\mathbf{T} + q \otimes q)\mathcal{K} = \{(2x + y, x + 2y, 0) : x = y = 0 \text{ or } x > 0\}$.

This set is not closed, since $(1, 2, 0)$ is a limit point of $(\mathbf{T} + q \otimes q)\mathcal{K}$ which is not in $(\mathbf{T} + q \otimes q)\mathcal{K}$. Thus, neither condition (iv) in Theorem 4.1 nor condition (iii) in Theorem 4.2 is satisfied.

6 A Perturbation Result

Theorem 6.1 *Let $\dim \mathcal{H} < \infty$ and suppose that $\mathbf{T}$ is copositive plus on $\mathcal{K}$. Then the following are equivalent:*

(a) The solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is nonempty and compact.

(b) $\{x \in \mathcal{K} : \mathbf{T}x \in \mathcal{K}^*, \langle \mathbf{T}x, x\rangle = 0$ and $\langle q, x\rangle \leq 0\} = \{0\}$.

(c) $q \in int(\mathcal{K}^* - \mathbf{T}(\mathcal{K}))$, i.e., $GLCP(\mathbf{T}, \mathcal{K}, q')$ is feasible for all q' near q.

(d) $GLCP(\mathbf{T}, \mathcal{K}, q)$ is feasible and

$$\{x \in \mathcal{K} : \mathbf{T}x \in \mathcal{K}^*, \langle \mathbf{T}x, x\rangle = 0, \text{ and } \langle q, x\rangle = 0\} = \{0\}.$$

PROOF:

(a)$\Longrightarrow$(b): Suppose that $GLCP(\mathbf{T}, \mathcal{K}, q)$ has nonempty compact solution set. Following Mangasarian [7, p.159], we show that (b) holds. Suppose there *were* some $\hat{x} \in \mathcal{K}$ such that $\mathbf{T}\bar{x} \in \mathcal{K}^*, \langle \mathbf{T}\bar{x}, \bar{x}\rangle = 0$, and $\langle q, \bar{x}\rangle \leq 0$. Let x_0 be a solution of $GLCP(\mathbf{T}, \mathcal{K}, q)$. Then, for any $t \geq 0$, one has $x_0 + t\bar{x} \in \mathcal{K}$ and $\mathbf{T}(x_0 + t\bar{x}) + q \in \mathcal{K}^*$. Further,

$$\begin{aligned} \langle \mathbf{T}(x_0 + t\bar{x}) &+ q, x_0 + t\bar{x}\rangle \\ &= \langle \mathbf{T}x_0 + q, x_0\rangle + \langle \mathbf{T}x_0 + q, t\bar{x}\rangle + \langle \mathbf{T}(t\bar{x}), x_0 + t\bar{x}\rangle \\ &= 0 + t\langle q, \bar{x}\rangle + t\langle x_0, \mathbf{T}^*\bar{x}\rangle \\ &\qquad + t\langle x_0, \mathbf{T}\bar{x}\rangle + t^2\langle \mathbf{T}\bar{x}, \bar{x}\rangle \\ &\leq 0 \end{aligned}$$

since $\langle q, \bar{x}\rangle \leq 0$, $\mathbf{T}^*\bar{x} + \mathbf{T}\bar{x} = 0$, and $\langle \mathbf{T}\bar{x}, \bar{x}\rangle = 0$. Since $\mathbf{T}(x_0 + t\bar{x}) + q \in \mathcal{K}^*$, we must have $\langle \mathbf{T}(x_0+t\bar{x})+q, x_0+t\bar{x}\rangle \geq 0$. Thus $\langle \mathbf{T}(x_0+t\bar{x})+q, x_0+t\bar{x}\rangle = 0$ whence $x_0 + t\bar{x}$ solves $GLCP(\mathbf{T}, \mathcal{K}, q)$ for all $t \geq 0$. By the boundedness of the solution set, we must then have $\bar{x} = 0$, proving (b).

(b)$\Longrightarrow$(c): Suppose that (b) holds but (c) does not. Then either $q \notin (\mathcal{K}^* - \mathbf{T}(\mathcal{K}))$ or $q \in \partial(\mathcal{K}^* - \mathbf{T}(\mathcal{K}))$. In either case, there exists a sequence $\{q_n\}$ such that $q_n \notin (\mathcal{K}^* - \mathbf{T}(\mathcal{K}))$, and $q_n \to q$ (as $n \to \infty$). Fixing n, a separation theorem [18, p.49] shows that there exists ξ_n with $\|\xi_n\| = 1$ and $\alpha \in \mathbb{R}$ such that

$$\langle q_n, \xi_n\rangle \leq \alpha \leq \langle \xi_n, k^*\rangle - \langle \xi_n, \mathbf{T}k\rangle \qquad (k^* \in \mathcal{K}^*,\ k \in \mathcal{K}).$$

Since $\mathcal{K}^* - \mathbf{T}(\mathcal{K})$ is a cone, we can take $\alpha = 0$. Then

$$\langle q_n, \xi_n\rangle \leq 0 \leq \langle \xi_n, k^*\rangle - \langle \xi_n, \mathbf{T}k\rangle \qquad (k^* \in \mathcal{K}^*,\ k \in \mathcal{K}). \tag{9}$$

Putting $k = 0$, we get $\langle \xi_n, k^*\rangle \geq 0$ $(k^* \in \mathcal{K}^*)$, which shows that $\xi_n \in \mathcal{K}$. Next putting $k^* = 0$, we get $0 \leq -\langle \xi_n, \mathbf{T}k\rangle$ and hence $\langle \mathbf{T}\xi_n, \xi_n\rangle \leq 0$ so $\langle \mathbf{T}\xi_n, \xi_n\rangle = 0$, and $\langle q_n, \xi_n\rangle \leq 0$. Since, $\|\xi_n\| = 1$ $(n = 1, 2, \ldots)$, we may assume that $\{\xi_n\}$ (or a subsequence) converges to some element ξ. Then $\|\xi\| = 1$, $\xi \in \mathcal{K}$, $\mathbf{T}\xi \in \mathcal{K}^*$, $\langle \mathbf{T}\xi, \xi\rangle = 0$, and $\langle q, \xi\rangle \leq 0$. This would contradict (b) so one has (b)$\Longrightarrow$(c).

(c)$\Longrightarrow$(d): Suppose (c) holds. Clearly $GLCP(\mathbf{T}, \mathcal{K}, q)$ is then feasible. Suppose that there is some $\xi \neq 0$ such that $\xi \in \mathcal{K}, \mathbf{T}\xi \in \mathcal{K}^*, \langle \mathbf{T}\xi, \xi \rangle = 0$, and $\langle q, \xi \rangle = 0$. Take any $z \in \mathcal{K}^* - \mathbf{T}(\mathcal{K})$ so $z = k^* - \mathbf{T}k$ where $k^* \in \mathcal{K}^*$ and $k \in \mathcal{K}$. Then,

$$\begin{aligned} \langle \xi, z \rangle &= \langle \xi, k^* \rangle - \langle \xi, \mathbf{T}k \rangle \\ &= \langle \xi, k^* \rangle + \langle -\mathbf{T}^*\xi, k \rangle \\ &\geq 0 + 0 \, (\text{since } \xi \in \mathcal{K} \text{ and } -\mathbf{T}^*\xi = \mathbf{T}\xi \in \mathcal{K}^* \\ &\geq 0 = \langle q, \xi \rangle. \end{aligned}$$

This shows that the hyperplane $\{x \in \mathcal{H} : \langle x, \xi \rangle = 0\}$ separates q from $\mathcal{K}^* - \mathbf{T}(\mathcal{K})$. On the other hand, this is impossible since $q \in \text{int}\, [\mathcal{K}^* - \mathbf{T}(\mathcal{K})]$. Hence $\xi = 0$ and (d) holds. (Note that (c)$\Longrightarrow$(d) also follows from [3]: Prop. 2.3 on p. 349 and (32) on p. 351.)

(d)$\Longrightarrow$(a): Follows from Theorem 4.1. ■

Remarks on Theorem 6.1

(9) When $\mathcal{H} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, the equivalence of (a) and (c) recaptures a result of Mangasarian [19,Theorem 2.] (see also [6]).

(10) Suppose that $\mathcal{K}^*$ has nonempty interior and $\mathbf{T}$ is copositive plus on $\mathcal{K}$. Assume that there is an $x \in \mathcal{K}$ with $(\mathbf{T}x + q) \in \text{int}(\mathcal{K}^*)$. Borwein [3, Theorem 4.1] has shown that $GLCP(\mathbf{T}, \mathcal{K}, q)$ has a compact solution set, possibly empty. Since $(\mathbf{T}x + q) \in \text{int}\ \mathcal{K}^*$ implies $q \in \text{int}\ [\mathcal{K}^* - \mathbf{T}(\mathcal{K})]$, Theorem 6.1 actually shows that $GLCP(\mathbf{T}, \mathcal{K}, q)$ must have a nonempty solution set.

(11) For the finite dimensional case, compactness of the solution set is equivalent to boundedness since the solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is always closed.

(12) In [20], McLinden proves perturbation results for monotone operators defined on reflexive Banach spaces.

7 A Characterization of Polyhedral Cones in Finite Dimensional Spaces

The examples given in section 5 lead one to conjecture that if the image of the cone under the operator were not closed then the corresponding $GLCP$ need not be solvable. This turns out to be correct if the space is finite dimensional. (It seems likely that the same result extends to infinite dimensional spaces but we have not shown this.) The following result shows that among all closed convex cones in $\mathcal{H}$ (dim $\mathcal{H} < \infty$), polyhedral cones are the *only* ones with the property that *every* copositive plus, feasible GLCP is solvable.

Theorem 7.1 *For* $\dim \mathcal{H} < \infty$ *the following are equivalent:*

(a) There is an operator $\mathbf{S} : \mathcal{H} \to \mathcal{H}$ *such that* $\mathbf{S}(\mathcal{K})$ *is not closed.*

(b) There is an (orthogonal) projection $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ *such that* $\mathbf{P}(\mathcal{K})$ *is not closed.*

(c) There is an operator $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ *which is copositive plus on* $\mathcal{K}$*, and a* $q \in \mathcal{H}$ *such that* $GLCP(\mathbf{T}, \mathcal{K}, q)$ *is feasible but not solvable.*

(d) $\mathcal{K}$ *is not polyhedral.*

(e) There is a projection operator $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ *with* $\dim(\mathrm{Ran}\ \mathbf{P}) = 2$ *for which* $\mathbf{P}(\mathcal{K})$ *is not closed.*

PROOF: (a)$\Longrightarrow$(b): Suppose (a) holds. Since $\mathcal{H}/\mathrm{Ker}\ \mathbf{S}$ is isomorphic to Ran $\mathbf{S}$ (recall that $\dim \mathcal{H} < \infty$) and $\Pi : \mathcal{H} \to \mathcal{H}/\mathrm{Ker}\ \mathbf{S}$ is a quotient map, the set $\mathcal{K}+\mathrm{Ker}\ \mathbf{S}$ is not closed in $\mathcal{H}$. Let $\mathbf{P}$ be the orthogonal projection from $\mathcal{H}$ onto $(\mathrm{Ker}\ \mathbf{S})^{\perp}$. Since Ker $\mathbf{P}$ = Ker $\mathbf{S}$, it follows that $[\mathcal{K}+\mathrm{Ker}\ \mathbf{P}]$ is not closed; therefore $\mathbf{P}(\mathcal{K})$ is not closed in $\mathcal{H}$. This gives (b).

(b)$\Longrightarrow$ (c): Let $\mathbf{P}$ be a projection on $\mathcal{H}$ such $\mathbf{P}(\mathcal{K})$ is not closed in $\mathcal{H}$. Let u be a limit point of $\mathbf{P}(\mathcal{K})$ which is not in $\mathbf{P}(\mathcal{K})$. Put $q = -u$ and let I denote the identity operator. We show that $\mathbf{P}(\mathcal{K})^* - \mathbf{P}(\mathcal{K}) = \mathcal{H}$. Let L denote any convex cone (not necessarily $\mathbf{P}(\mathcal{K})$). Suppose, to get a contradiction, that $L^* - L \neq \mathcal{H}$. Let $x \in \mathcal{H}$ with $x \notin (L^* - L)$. Then by a separation theorem (see [18], p.49), there exists $\xi \neq 0$ in $\mathcal{H}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \xi, x \rangle \leq \alpha \leq \langle \xi, w \rangle - \langle \xi, v \rangle \ (v \in L,\ w \in L^*). \tag{10}$$

Since $[L^* - L]$ is a cone, we can take $\alpha = 0$. Putting $w = 0$ in (7.1), we then get $0 \leq -\langle \xi, v\rangle$ $(v \in L)$ so $-\xi \in L^*$. By putting $w = -\xi$, $v = 0$, and recalling that we took $\alpha = 0$, we get from (7.1) that $0 \leq \langle \xi, -\xi\rangle$ so $\xi = 0$, contradicting the assertion $\xi \neq 0$. Therefore $[\mathbf{P}(\mathcal{K})^* - \mathbf{P}(\mathcal{K})] = \mathcal{H}$ whence $GLCP(I, \mathbf{P}(\mathcal{K}), q)$ is feasible. Suppose $GLCP(I, \mathbf{P}(\mathcal{K}), q)$ were solvable. Then there would exist an $a \in \mathbf{P}(\mathcal{K})$ such that $\langle a + q, v\rangle \geq 0$ for $v \in \mathbf{P}(\mathcal{K})$ and $\langle a + q, a\rangle = 0$. Writing $q = -u$, we have, for any $v \in \mathbf{P}(\mathcal{K})$, that

$$\|v - u\|^2 = \|v - a\|^2 + \|a - u\|^2 + 2\langle v - a, a - u\rangle \geq \|a - u\|^2$$

since $\langle v - a, a - u\rangle = \langle v, a - u\rangle - \langle a, a - u\rangle \geq 0$. As $u \in cl[\mathbf{P}(\mathcal{K})] \backslash \mathbf{P}(\mathcal{K})$, this cannot happen; thus $GLCP(I, \mathbf{P}(\mathcal{K}), q)$ cannot have a solution. Since $q \in$ Ran $\mathbf{P}$ and $\mathbf{P}$ is a projection, the feasibility (solvability) of $GLCP(I, \mathbf{P}(\mathcal{K}), q)$ is equivalent to the feasibility (solvability) of $GLCP(\mathbf{P}, \mathcal{K}, q)$. Thus we see that $GLCP(\mathbf{P}, \mathcal{K}, q)$ is feasible but not solvable. Finally, $\mathbf{P}$ is copositive plus on $\mathcal{K}$ since projections are always monotone.

(c)$\Longrightarrow$(d): Follows from Theorem 3.1

(d)$\Longrightarrow$(e): Follows from a result of Mirkil [21].

(e)$\Longrightarrow$(a): Obvious. ■

8 Concluding Remarks and Open Problems

(13) Let $\dim \mathcal{H} < \infty$ and let $\mathbf{T}$ be copositive plus on $\mathcal{K}$. Suppose that $GLCP(\mathbf{T}, \mathcal{K}, q)$ is feasible, i.e., $q \in [\mathcal{K}^* - \mathbf{T}(\mathcal{K})]$. Then our results show that

(i) If $q \in \mathrm{int}[\mathcal{K}^* - \mathbf{T}(\mathcal{K})]$ then the solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is nonempty and compact,

(ii) If $q \in \partial(\mathcal{K}^* - \mathbf{T}(\mathcal{K})), (T + q \otimes q)K$ is closed, and $\mathbf{T}$ is self-adjoint then the solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ is nonempty and unbounded.

We thus ask:

Problem 1: Can self-adjointness of $\mathbf{T}$ be dropped in (ii), i.e., in Theorem 4.2?

(14) Theorem 4.2 can also be proved by minimizing a quadratic functional. This approach is described in [9].

(15) Theorem 6.1 shows that when $\dim \mathcal{H} < \infty$ and $\mathbf{T}$ is copositive plus, the solution set of $GLCP(\mathbf{T}, \mathcal{K}, q)$ being nonempty and compact is equivalent to the existence of an $\varepsilon > 0$ such that $GLCP(\mathbf{T}, \mathcal{K}, q')$ has nonempty (compact) solution set for $\|q' - q\| < \varepsilon$.

Problem 2: Assume that $\dim \mathcal{H} < \infty$ and $\mathbf{T}$ is copositive plus. If $GLCP(\mathbf{T}, \mathcal{K}, q)$ is nonempty and compact then can one find an $\varepsilon > 0$ such that $GLCP(\mathbf{T}', \mathcal{K}, q')$ has a nonempty solution set for all $\mathbf{T}'$ and q' with $\max\{\|\mathbf{T}' - \mathbf{T}\|, \|q' - q\|\} < \varepsilon$? We note that when $\mathcal{H} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, an affirmative answer to Problem 3 is known ([19],[6]).

Problem 3: Is Theorem 7.1 true when $\mathcal{H}$ is infinite dimensional?

(16) Here is a generalization of Theorem 4.1 to reflexive Banach spaces: Let $\mathcal{X}$ be a reflexive Banach space with dual $\mathcal{X}^*$. For $f \in \mathcal{X}^*$ and $x \in \mathcal{X}$, let $\langle f, x \rangle$ denote $f(x)$. Let $\mathcal{K}$ be a closed convex cone in $\mathcal{X}$ and let $q \in \mathcal{X}^*$. Let $\mathbf{T} : \mathcal{X} \to \mathcal{X}^*$ be linear (and continuous) such that $\langle \mathbf{T}x, x \rangle \geq 0$ for all $x \in \mathcal{K}$, and, $x \in \mathcal{K}, \langle \mathbf{T}x, x \rangle = 0$ imply $\langle \mathbf{T}x, k \rangle + \langle \mathbf{T}k, x \rangle = 0$ for all $k \in \mathcal{K}$. Assume further that the map: $x \mapsto \langle \mathbf{T}x, x \rangle$ is weak-lsc on $\mathcal{K}$, that $\mathcal{K}$ is thin, that $\{x \in \mathcal{K} : \mathbf{T}x \in \mathcal{K}^*,\ \langle \mathbf{T}x, x \rangle = 0,\ \langle q, x \rangle = 0\} = \{0\}$. Then $GLCP(\mathbf{T}, \mathcal{K}, q)$ has a nonempty compact solution set whenever it is feasible. The proof of this is obtained by modifying the proofs of Theorems 3.1 and 4.1 and hence we omit it. (While proving Theorem 3.1 for this case, one may want to look at $GLCP(\mathbf{J}^*\mathbf{T}, \mathcal{K}, \mathbf{J}^*q)$ where $\mathbf{J}$ is the inclusion map from $[\mathcal{K} - \mathcal{K}]$ into $\mathcal{X}$).

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