‘Optimal Filtering’ for some ill-posed problems [1]

Thomas I. Seidman
Department of Mathematics and Statistics
University of Maryland Baltimore County
Baltimore, MD 21228, USA
e-mail: ⟨seidman@umbc1.umbc.edu⟩ or ⟨seidman@umbc.bitnet⟩

ABSTRACT: We describe an approach to a class of ill-posed problems in which the determination of a ‘filter’ for obtaining approximate solutions is obtained by means of an optimization process. In Hilbert space settings a fairly explicit computation may be possible and this is presented. It is noted that, under certain conditions the resulting filter is, indeed, optimal in the sense of realizing a minimal uniform error bound.

KEY WORDS: *ill-posed problems, filtering, approximation, regularization, optimization.*

[1]This research has been partially supported by the U.S. Air Force Office of Scientific Research under the grant AFOSR-87-0190 and by the National Science Foundation under the grant ECS-8814788.

1. INTRODUCTION

An approach, which we have called 'optimal filtering', was introduced in [12] for the determination of approximate solutions for the 'sideways heat equation'. The approach provided a 'filter' in two senses: (a) its implementation took the form of a multiplication operator on a Fourier transform of the data and (b) its purpose was to extract as much useful information as possible while rejecting 'noise' in the data. The name then came both from the use of an optimization procedure for the determination of the appropriate filter to produce the approximate solution as well as from the fact that this particular filtering procedure did result in an optimal error bound.

The context for that approach involved a semigroup of 'smoothing' operators whose approximate inversion was the assigned task and it is anticipated that a basic context for generalizations of this approach would be a *parametrized family* of operators for which one wished to 'march backward' with respect to the parameter. Numerous applications having this general structure come to mind. Here, in order to present the method in the clearest fashion, we consider the approach only as applied in the context of a *single* ill-posed problem although some comments will be made about the application to families of problems; see Remarks 4.4, 4.5. Also, in order to present the approach most clearly as 'a thing in itself' we treat the method abstractly here and make no attempt to work out any specific application, geophysical or otherwise.

Thus, let $x \in \mathcal{X}$ be an 'unknown of interest' and suppose we are given the following two pieces of information:

$$(1.1) \qquad \begin{array}{ll} .1) & x = \mathbf{B}z \text{ with } \|z - \varphi_0\| \le \varepsilon_0 \\ .2) & \|\mathbf{A}x - \varphi_1\| \le \varepsilon_1 \end{array}$$

which we assume are consistent. Our primary attention is devoted to the case of continuous *linear* operators $\mathbf{A} : \mathcal{X} \to \mathcal{Y}$ and $\mathbf{B} : \mathcal{Z} \to \mathcal{X}$ with $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ uniformly convex Banach spaces, usually a Hilbert space $\mathcal{H}$ — although we note that some of our considerations also apply, more generally, to nonlinear maps and/or more general spaces. It is convenient to set

$$(1.2) \qquad \begin{aligned} \mathcal{U}_0 = \mathcal{U}_0[\varphi_0, \varepsilon_0] &:= \{x \in \mathcal{X} : x = \mathbf{B}z \text{ with } \|z - \varphi_0\| \le \varepsilon_0\}, \\ \mathcal{U}_1 = \mathcal{U}_1[\varphi_1, \varepsilon_1] &:= \{x \in \mathcal{X} : \|\mathbf{A}x - \varphi_1\| \le \varepsilon_1\} \end{aligned}$$

so the set of solutions of (1.1) is

$$\mathcal{U} = \mathcal{U}[\varphi_0, \varepsilon_0; \varphi_1, \varepsilon_1] := \mathcal{U}_0[\varphi_0, \varepsilon_0] \cap \mathcal{U}_1[\varphi_1, \varepsilon_1].$$

Consistency is just the requirement that $\mathcal{U}$ be nonempty. If we assume that $\mathbf{A}$ is injective, then exact knowledge of $\bar{\varphi} := \mathbf{A}\bar{x}$ would uniquely determine $\bar{x}$ from (1.1.2) with $\varepsilon_1 = 0$ even without (1.1.1). On the other hand, we will assume that $\mathbf{A}$ does not have a *continuous* inverse so the problem is ill-posed and the uncertainty inherent in (1.1.2) as stated would be catastrophically amplified by any attempt at (approximately) inverting $\mathbf{A}$ without some suitable stabilization — here to be provided by (1.1.1). The stabilizing effect of such a condition had been noted long ago, cf., e.g., [4].

For our interpretation, we view the first condition (1.1.1) as *a priori* information and view (1.1.2) as corresponding to obtaining a value φ_1 by a 'noisy measurement', characterized by the operator $\mathbf{A}$ and the accuracy bound ε_1. Indeed, a standard form of analysis would be to assume existence of a 'true' solution $x = \bar{x}$ satisfying (1.1.1) and giving $\mathbf{A}\bar{x} = \bar{\varphi}_1$ (with $\bar{\varphi}_1$ 'unknown'). There is then no suggestion that the 'regularizing bound' ε_0 should be 'small' but with $\varepsilon_1 \to 0$, one imagines a (potential) *sequence of increasingly accurate measurements*: $\varphi_1 = \varphi_1(\varepsilon_1)$ satisfying $\|\varphi_1 - \bar{\varphi}\| \leq \varepsilon_1$. One then seeks a way of using (1.1) to select— presumably by a computationally implementable procedure from knowledge of $\mathbf{A}, \mathbf{B}$ and the available data quadruple $[\varphi_0, \varepsilon_0; \varphi_1, \varepsilon_1]$ — some specific 'approximate solution' $v = v(\varepsilon_1)$ for which one would then wish to demonstrate that $v \to \bar{x}$ (as rapidly as possible) as $\varepsilon_1 \to 0$. Compare [11]. This procedure is a form of 'filtering', eliminating the effect of the noise, which is complicated in our present analysis by the ill-posedness-induced error amplification. To the extent that it is reasonable to consider $\varepsilon_0, \varepsilon_1$ not merely as bounds but as fairly sharp estimates of the 'signal strength' $\|z\|$ and of the 'noise' (measurement error), respectively, we may interpret the scaled parameter

$$\nu := \varepsilon_1 / \varepsilon_0$$

as a *noise–to–signal ratio.*

In this setting the *data* for the problem, apart from the specification of the operators involved, consists of the quadruple $[\varphi_0, \varepsilon_0; \varphi_1, \varepsilon_1]$. While one could ask to know the *set* of solutions $\mathcal{U}$, one normally would only seek to specify a pair $[v, \varepsilon]$ such that

$$\|x - v\| \leq \varepsilon \text{ for all } x \in \mathcal{U} = \mathcal{U}[\varphi_0, \varepsilon_0; \varphi_1, \varepsilon_1] \tag{1.3}$$

so ε is an error bound for the 'approximate solution' v of (this particular instance of) (1.1). We would, of course, like ε to be as small as possible and, to the extent that this is the case, will refer to an approximation procedure for computing the pair $[v, \varepsilon]$ as 'optimal'.

Rather than considering an approximate solution v for (1.1) in isolation for a specific data pair $[\varphi_0, \varphi_1]$, we seek a map

$$\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_{[\varepsilon_0,\varepsilon_1]} : \mathcal{Z} \times \mathcal{Y} \to \mathcal{X} : [\varphi_0, \varphi_1] \mapsto v$$

giving an approximate solution $v = \tilde{\mathbf{F}}[\varphi_0, \varphi_1]$ for *each* data pair. We then refer to $\bar{\varepsilon}$ as a (realizable) **uniform bound** and to the map $\tilde{\mathbf{F}}$ as a **filter** realizing this bound if (1.3) holds with $\varepsilon = \bar{\varepsilon}$ for all instances of (1.1), i.e., if for all[2] $[\varphi_0, \varphi_1]$ one has

$$(1.4) \qquad \|x - \tilde{\mathbf{F}}[\varphi_0, \varphi_1]\| \le \bar{\varepsilon} \quad \text{for all } x \text{ satisfying (1.1).}$$

If this $\bar{\varepsilon}$ is a **minimal uniform bound (MUB)** — i.e., if the infimum of all such realizable uniform bounds is itself realizable — then any filter $\tilde{\mathbf{F}}$ which does, indeed, realize $\bar{\varepsilon}$, giving (1.4), will be called an **optimal filter** for (1.1).

The operator $\mathbf{B}$ of the 'regularizing condition' (1.1.1) is typically linear and, in this case, we note a simplifying reduction of the problem: set

$$(1.5) \qquad \begin{array}{rcl} \tilde{\varphi} & := & (\varphi_1 - \mathbf{A}\mathbf{B}\varphi_0)/\varepsilon_0 \\ \tilde{\mathbf{A}} : \mathcal{X} \to \mathcal{Y} & : & \tilde{x} \mapsto [\mathbf{A}(\mathbf{B}\varphi_0 + \varepsilon_0 \tilde{x}) - \mathbf{A}\mathbf{B}\varphi_0]/\varepsilon_0 \end{array}$$

and consider the reduced problem

$$(1.6) \qquad \begin{array}{ll} .1) & \tilde{x} = \mathbf{B}\tilde{z} \text{ with } \tilde{z} \in \mathcal{B} := \{z \in \mathcal{Z} : \|z\| \le 1\} \\ .2) & \|\tilde{\mathbf{A}}\tilde{x} - \tilde{\varphi}\| \le \nu \end{array}$$

with solution set

$$\tilde{\mathcal{U}}_\nu(\tilde{\varphi}) := \{\tilde{x} \in \mathbf{B}\mathcal{B} : \|\tilde{\mathbf{A}}\tilde{x} - \tilde{\varphi}\| \le \nu\}.$$

A simple computation shows the effective equivalence of (1.6) to (1.1) in the sense that x is a solution of (1.1) if and only if $\tilde{x} := (x - \mathbf{B}\varphi_0)/\varepsilon_0$ is a solution of (1.6) with the 'target' and operator modified as in (1.5) — i.e., one has

$$(1.7) \qquad \tilde{\mathcal{U}}_\nu(\tilde{\varphi}) = \frac{1}{\varepsilon_0}\left(\mathbf{B}\varphi_0 + \mathcal{U}[\varphi_0, \varepsilon_0; \varphi_1, \varepsilon_1]\right).$$

[2]While this statement reads 'for *all*', it is clear that the implication is nonempty only subject to consistency.

In the context of this reduced problem (1.6) we call $\varepsilon > 0$ a (realizable) *uniform bound* if there is a *filter*, i.e., a map $\mathbf{F} = \mathbf{F}_\nu : \mathcal{Y} \to \mathcal{X} : \tilde{\varphi} \mapsto \tilde{v}$ such that

$$\text{(1.8)} \qquad \left[x \in \tilde{U}_\nu(\tilde{\varphi}) \;\Rightarrow\; \|x - \mathbf{F}_\nu \tilde{\varphi}\| \le \varepsilon\right] \qquad \text{for all } \tilde{\varphi} \in \mathcal{Y}.$$

We will denote by $\varepsilon_{\#} = \varepsilon_{\#}(\nu)$ a **minimal uniform bound** in this sense (with corresponding **optimal filter** $\mathbf{F}_\nu$). We state the following lemma; proof obvious.

LEMMA 1.1: *A filter $\mathbf{F}_\nu$ realizes a bound ε for (1.6) as in (1.8) if and only if — for arbitrary $\varepsilon_0, \varepsilon_1$ giving $\nu = \varepsilon_1/\varepsilon_0$ and using (1.5) — the filter*

$$\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_{[\varepsilon_0,\varepsilon_1]} : [\varphi_0, \varphi_1] \mapsto \mathbf{F}_\nu \tilde{\varphi}$$

realizes the uniform bound $\varepsilon_0 \varepsilon$ for (1.1) as in (1.3). Thus, $\bar{\varepsilon}$ is a MUB for (1.1) if and only if $\bar{\varepsilon} = \varepsilon_0 \varepsilon_{\#}(\nu)$ with $\varepsilon_{\#}(\nu)$ a MUB for (1.6); further, the notions of optimal filter correspond as noted. ■

2. GENERAL RESULTS

For the problem (1.6) our task will be the development of an optimal (or suboptimal) filter $\mathbf{F}_\nu : \mathcal{Y} \to \mathcal{X}$ satisfying (1.8) and the principal result of this paper, to be presented in the next section, is a computation in the special case of commuting normal operators on a Hilbert space. We begin, however, by noting some results of a less explicit character but in more general settings.

THEOREM 2.1: *Suppose $\mathcal{X}$ is uniformly convex and $\mathcal{U}$ is a bounded set in $\mathcal{X}$. Then there is a unique pair $[v_*, \varepsilon_*]$ minimizing $\varepsilon = \varepsilon_*$ subject to (1.3); we have $v_* \in \overline{\mathrm{co}}\,\mathcal{U}_0$, the closed convex hull of $\mathcal{U}_0$.*

Further, if $\mathcal{U}$ has the form[3] *$\{x \in \mathcal{U}_0 : \|\mathbf{A}x - \varphi_1\| < \varepsilon_1\}$ with $\mathcal{U}_0$ precompact and $\mathbf{A}$ continuous, then this $[v_*, \varepsilon_*]$ depends continuously on the pair $[\varphi_1, \varepsilon_1]$ with $\varepsilon_1 > 0$.*

PROOF: Consider a sequence $\{[v_k, \varepsilon_k]\}$, each satisfying (1.3), for which

[3]Observe that this does not quite fit the framework of (1.1.2) in that it effectively replaces "$\le \varepsilon_1$" by "$< \varepsilon_1$". Using nonlinear $\mathbf{A}$, one can easily construct counterexamples to the continuity without this modification although the modification seems irrelevant in the linear case. We also note that if $\mathbf{A}$ is injective then continuity in φ_1 for the 'unmodified' case with $\varepsilon_1 = 0$ is a classical theorem of topology.

we may assume $\varepsilon_k \searrow \varepsilon_* :=$ inf and (possibly extracting a subsequence by weak compactness, as $\{v_k\}$ is necessarily bounded) $v_k \rightharpoonup v_*$. As the norm is weakly lsc, this $[v_*, \varepsilon_*]$ satisfies (1.3) so the minimum is attained. One has uniqueness of v_* as otherwise $\overline{\mathcal{U}}$ would be contained in the intersection of two balls with the same minimal radius ε_* but distinct centers, hence, by uniform convexity, in a strictly smaller ball with center at their average — which contradicts minimality. Similarly, if we did not have $v_* \in \overline{\mathrm{co}}\,\mathcal{U}$ then replacing v_* by the nearest point in $\overline{\mathrm{co}}\,\mathcal{U}$ would contradict minimality. (Cf., [5].)

Now supposing, as above, that $\mathcal{U} \subset \mathcal{U}_0$ with $\mathcal{U}_0$ precompact, then we have $\overline{\mathrm{co}}\,\mathcal{U}$ compact. Consider a sequence $[\varphi_1^{[k]}, \varepsilon_1^{[k]}] \to [\varphi_1^*, \varepsilon_1^*]$ and the corresponding $[v_*^{[k]}, \varepsilon_*^{[k]}]$. Extracting a subsequence if necessary, we may assume that $v_*^{[k]} \to \bar{v}_*$ and $\varepsilon_*^{[k]} \to \bar{\varepsilon}_*$. If $x \in \mathcal{U}[\varphi_1^*, \varepsilon_1^*]$ then also $x \in \mathcal{U}[\varphi_1^{[k]}, \varepsilon_1^{[k]}]$ for large enough k so $\|x - v_*^{[k]}\| \le \varepsilon_*^{[k]}$. Going to the limit gives (1.3) and a similar argument shows minimality. The uniqueness above shows that this is independent of the extraction of a subsequence. ■

It is clear that the definition above of $[v_*, \varepsilon_*]$ gives the best information about 'the solution' of any particular instance of (1.1) which could possibly be provided in such a form — an approximant with a minimal error bound — and an algorithm which would determine v_* would certainly be truly optimal. For our present analysis, however, we are content to seek an optimal filter in the sense described above — for which it is the *uniform* bound which is to be minimal; compare, e.g., the formulation in [3].

THEOREM 2.2: *Let the Banach space $\mathcal{X}$ be reflexive and $\mathbf{B}$ linear. Then for every $\varepsilon_0, \varepsilon_1 > 0$ there is always an optimal filter $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_{[\varepsilon_0,\varepsilon_1]}$ realizing a minimal uniform bound $\bar{\varepsilon}$ as in (1.4).*

PROOF: We actually prove the result somewhat more generally than stated — we need not assume any linearity of $\mathbf{A}, \mathbf{B}$ but only that $\mathcal{X}$ is reflexive and that (for given $\varepsilon_0, \varepsilon_1 > 0$) the solution sets $\mathcal{U}[\varphi_0, \varepsilon_0; \varphi_1, \varepsilon_1]$ all have finite diameter, bounded uniformly as $[\varphi_0, \varphi_1]$ varies over $\mathcal{Z} \times \mathcal{Y}$. This is already the case if $\mathbf{B}$ is uniformly Lipschitzian, much less linear.

Under these circumstances realizable uniform bounds exist since one can obviously always select a center v for a ball large enough to contain each solution set; such selections define a 'filter'. Now let $\bar{\varepsilon}_k \searrow \bar{\varepsilon}$ be a minimizing sequence with corresponding filters $\mathbf{F}_k$. For any particular instance of (1.1), set $v_k := \mathbf{F}_k[\varphi_0, \varphi_1]$. Note that (unless $\mathcal{U}$ is empty so one could arbitrarily set $v_k = 0$) the sequence $\{v_k\}$ must be bounded. Extracting a subsequence if necessary, we can assume $v_k \rightharpoonup v =: \mathbf{F}[\varphi_0, \varphi_1]$. Having (1.3) for each $[v_k, \bar{\varepsilon}_k]$, the lower semicontinuity of the norm with respect to the weak topology now ensures (1.3) for $[v, \bar{\varepsilon}]$. Doing this for each pair $[\varphi_0, \varphi_1]$ realizes the minimal $\bar{\varepsilon}$ by the optimal filter $\mathbf{F}$. ∎

THEOREM 2.3: *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be uniformly convex Banach spaces; let $\mathbf{A} : \mathcal{X} \to \mathcal{Y}$ and $\mathbf{B} : \mathcal{Z} \to \mathcal{X}$ be continuous linear operators with $\mathbf{B}$ compact and $\mathbf{A}$ injective. Then the minimal uniform bound $\varepsilon_{\#}(\nu)$ for the reduced problem (1.6) satisfies $\varepsilon_{\#}(\nu) \to 0$ as $\nu \to 0$.*

PROOF: We will use a lemma from [10]:[4]

> **LEMMA:** *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces with $\mathcal{Z}$ reflexive; let $\mathbf{B} : \mathcal{Z} \to \mathcal{X}$, $\mathbf{C} : \mathcal{Z} \to \mathcal{Y}$ be continuous linear operators with $\mathbf{B}$ compact and $\mathcal{N}(\mathbf{C}) \subset \mathcal{N}(\mathbf{B})$. Then, for any $\delta > 0$ there exists C_δ such that, with the appropriate norms,*
>
> $$\|\mathbf{B}z\| \leq \delta\|z\| + C_\delta\|\mathbf{C}z\|$$
>
> *for all $z \in \mathcal{Z}$.*
>
> PROOF: Suppose not. Then there is a sequence $z_n \in \mathcal{Z}$ with $\|z_n\| = 1$, so we may assume $z_n \rightharpoonup \tilde{z}$, such that $\|\mathbf{B}z_n\| > \delta + n\|\mathbf{C}z_n\|$. Then $\mathbf{C}z_n \rightharpoonup 0$ and $\mathbf{C}\tilde{z} = 0$ whence, as $\mathcal{N}(\mathbf{C}) \subset \mathcal{N}(\mathbf{B})$, one has $\mathbf{B}\tilde{z} = 0$. As $\mathbf{B}$ is compact we have strong convergence $\mathbf{B}z_n \to \mathbf{B}\tilde{z}$ which, as $\|\mathbf{B}z_n\| > \delta$, gives $\|\mathbf{B}\tilde{z}\| \geq \delta > 0$ — a contradiction. ∎

Now let $\mathcal{K}_0 := \{\mathbf{A}x : x \in \mathcal{U}_0\}$, $\mathcal{K} := \{\mathbf{A}x : x \in \overline{\mathcal{U}_0}\}$; as $\mathbf{A}$ is continuous, $\mathcal{K}$ is closed. Setting $\mathcal{B}_1 := \{y \in \mathcal{Y} : \|y\| \leq \varepsilon_1\}$, let $\mathcal{V}_1 = \mathcal{V}_1(\varphi_1) := \mathcal{K} \cap [\varphi_1 - \mathcal{B}_1]$ so $\mathcal{U} = \{x \in \mathcal{U}_0 : \mathbf{A}x \in \mathcal{V}_1\}$. Note that $\mathcal{K}$ is convex so, given φ_1 outside $\mathcal{K}$, we may find a 'nearest point' $\tilde{\varphi}_1$ in $\mathcal{K}$ and have $\mathcal{V}_1 \subset \tilde{\mathcal{V}}_1 := \mathcal{V}_1(\tilde{\varphi}_1)$; thus we need only consider $\varphi_1 \in \mathcal{K}$.

For any given $\varphi_1 \in \mathcal{K}$, the density of $\mathcal{K}_0$ enables us to find $\tilde{\varphi}_1 \in \mathcal{K}_0$ with $\|\varphi_1 - \tilde{\varphi}_1\| \leq r$. By definition we have $\tilde{\varphi}_1 = \mathbf{C}\tilde{z}$ for some $\tilde{z} \in \mathcal{B} := \{z \in \mathcal{Z} : \|z\| \leq 1\}$ with $\mathbf{C} := \mathbf{AB}$ and we set $v := \mathbf{B}\tilde{z}$. Note that this necessarily

[4] Note, also, the quite similar lemma in [14].

gives v in the compact set $\overline{\mathcal{U}_0}$. For any $x \in \mathcal{U}$, we have $x = \mathbf{B}z$ with $z \in \mathcal{B}$ (so $\|z - \tilde{z}\| \le 2$) and note that $x - v = \mathbf{B}(z - \tilde{x})$ and

$$\|\mathbf{C}(z - \tilde{z})\| \le \|\mathbf{A}x - \varphi_1\| + \|\varphi_1 - \tilde{\varphi}_1\| \le 2r.$$

We may then apply the Lemma to obtain

$$\|x - v\| \le \varepsilon := 2\delta\varepsilon_0 + 2rC_\delta \tag{2.1}$$

(with $\delta > 0$ arbitrary) since compactness of $\mathbf{B}$ makes $\mathbf{C} = \mathbf{AB}$ compact and injectivity of $\mathbf{A}$ gives $\mathcal{N}(\mathbf{C}) = \mathcal{N}(\mathbf{B})$. The estimate (2.1) gives (1.3) with ε uniform for all φ_0, φ_1 once one chooses δ. This bound ε need not be minimal but if (1.3) is satisfied for a sequence $[v_k, \varepsilon_k]$ with $\{\varepsilon_k\}$ a minimizing sequence and $v_k \in \overline{\mathcal{U}_0}$ as above, then we may extract a subsequence giving $v_k \to \tilde{v}$ and $\varepsilon_k \to \varepsilon_\# := \inf$ and have (1.3) satisfied using $[\tilde{v}, \varepsilon_\#]$. Since this $\varepsilon_\#$ is bounded by the $\varepsilon = \varepsilon_\delta$ presented in (2.1), we note that it can be made small by choosing δ small and then, having fixed C_δ, considering small enough r. ■

3. PRINCIPAL RESULTS

At this point we turn to the Hilbert space computation which is our principal result. We will assume that $\mathbf{A}, \mathbf{B}$ are commuting normal operators on a Hilbert space $\mathcal{H}$ ($\mathcal{X}, \mathcal{Y}, \mathcal{Z} = \mathcal{H}$) and that $\mathbf{A}$ is injective. We are, of course, primarily concerned with situations in which $\mathbf{A}$ is not boundedly invertible, i.e., 0 is in the spectrum $\sigma(\mathbf{AB})$ but is not an eigenvalue of $\mathbf{A}$.

Before even stating the theorem we note that in treating (1.6) it is always irrelevant to consider any $\nu > \bar{\nu} := \|\mathbf{AB}\|$ since one could always then replace the data pair $[\tilde{\varphi}, \nu]$ in (1.6.2) by $[0, \bar{\nu}]$ on the basis of (1.6.1) alone. It will also be convenient to recall a standard[5] spectral theory result:

> *One may, to within a unitary isomorphism, take $\mathcal{H}$ to be $L^2_\mu(\Omega)$ for some measure space*[6] (Ω, Σ, μ) *in such a way that* $\mathbf{A}$, $\mathbf{B}$ *simultaneously become multiplication operators:*
>
> $$\mathbf{A}x = \alpha(\cdot)x(\cdot), \qquad \mathbf{B}x = \beta(\cdot)x(\cdot) \tag{3.1}$$

[5] For this see, e.g., Theorem 9.1 of [9] which gives this for the self-adjoint case only. By looking separately at real and imaginary parts and noting Theorem 12.16 of [8], one sees that it can also be applied to commuting normal operators as here.

[6] If either $\mathbf{A}$ or $\mathbf{B}$ is compact, then this is necessarily discrete and $\mathcal{H}$ is effectively the sequence space ℓ^2.

for $x = x(\cdot) \in \mathcal{H} = L^2_\mu(\Omega)$.

This is somewhat more convenient than spectral measures or eigenfunction expansions in permitting us to work with the pair of operators. For a variety of (constant coefficient) differential operators, of course, this becomes just the usual 'frequency domain': the spectral resolution (3.1) is obtained by taking $\omega \in \Omega$ to be the 'Fourier transform variable'; the typical 'engineering' notion of a filter would be a multiplication operator in this representation. In any case, we shall here assume that this resolution is effectively available.

The key to our approach here is a function $\psi : \Sigma^+ \to [0, \|\mathbf{B}\|]$ where[7]

$$\Sigma^+ := \operatorname{ess\,ran}(|\alpha\beta|) = \{t = |\gamma| : \gamma \in \sigma(\mathbf{AB}) = \operatorname{ess\,ran}(\alpha\beta)\}$$

measuring the size of $|\beta|$ in terms of $|\alpha\beta|$. We want, sharply,

$$|\beta| \leq \psi(|\alpha\beta|) \qquad \text{ae on } \Omega; \tag{3.2}$$

roughly, we want $\psi(t) = \sup\{|\beta| : |\alpha\beta| = t\}$. This would, indeed, be an adequate definition for 'nice' α, β but is technically inadequate for merely measurable α, β. What we need, besides (3.2), is that, for any $t \in \Sigma^+$, there should be a set $\mathcal{S}_t$ of positive measure in Ω, on which one has α, β simultaneously (approximately) constant: $|\alpha\beta| \approx t$ with $|\beta| \approx \psi(t)$ (so $|\alpha| \approx t/\psi(t)$). Thus we define

$$\psi(t;\delta) := \operatorname*{ess\,sup}_{\omega}\{|\beta| : |\,|\alpha\beta| - t| < \delta\}, \quad \psi(t) := \lim_{\delta \searrow 0} \psi(t;\delta). \tag{3.3}$$

Note that the set $\{\omega : |\,|\alpha\beta| - t| < \delta\}$ has positive measure for each $\delta > 0$ precisely when $t \in \Sigma^+$. It is clear that we then have (3.2) and that there is necessarily a set $\mathcal{S}_{t;\delta}$ of positive measure such that

$$|\,|\alpha\beta| - t| < \delta \text{ and } |\,|\beta| - \psi(t)| < \delta \text{ on } \mathcal{S}_{t;\delta}. \tag{3.4}$$

In terms of these definitions we can state our result.

THEOREM 3.1: *Let* $\mathbf{A}, \mathbf{B}$, *etc., be as above. Then (for the reduced problem as in Lemma 1.1 for any* $\nu > 0$*) the minimal uniform bound* $\varepsilon_\#(\nu)$ *satisfies*

$$\hat{\varepsilon}_-(\nu) \leq \varepsilon_\#(\nu) \leq \hat{\varepsilon}_+(\nu) \qquad \textit{where} \tag{3.5}$$

[7] Note: $\operatorname{ess\,ran}(f)$ is the (closed) set of r for which $meas[f^{-1}(\mathcal{N})] > 0$ for every neighborhood $\mathcal{N}$ of r.

$$(3.6) \qquad \begin{cases} \hat{\varepsilon}_-(\nu) := & \sup\{\psi(t) : \nu \geq t \in \Sigma^+\} \\ \\ \hat{\varepsilon}_+(\nu) := & \min_{s>0}\{\nu s + \sup_{t\in\Sigma^+}\{\psi(t) - st\}\}. \end{cases}$$

The uniform bound $\hat{\varepsilon}_+(\nu)$ is realizable by a linear filter $\mathbf{F}$ (also commuting with $\mathbf{A}, \mathbf{B}$), explicitly computable by an optimization.[8] *Further, assuming $\Sigma^+ = \operatorname{ess\,ran}(|\alpha\beta|)$ is an interval, one can improve (3.5) when ψ is an increasing concave function: one then has*

$$(3.7) \qquad \hat{\varepsilon}_-(\nu) = \varepsilon_\#(\nu) = \hat{\varepsilon}_+(\nu) = \psi(\nu)$$

so the filter defined here is an optimal filter.

PROOF: We wish to construct a linear filter for the reduced problem as a multiplication operator:

$$(3.8) \qquad \mathbf{F} = \mathbf{F}_\nu : \varphi_1 \mapsto v(\cdot) := \vartheta(\cdot)\varphi_1(\cdot)$$

for a suitably chosen 'filter factor' $\vartheta = \vartheta_\nu(\cdot)$. We begin with the observation that any solution of the problem has the form $x = \beta z$ with $z \in \mathcal{B} := \{y \in \mathcal{H} : \|y\| \leq 1\}$ by (1.6.1) and

$$(3.9) \quad \text{error} := x - \mathbf{F}\varphi_1 = (1 - \vartheta\alpha)\beta z + \nu\vartheta\zeta \quad \text{with } \zeta := [\alpha\beta z - \varphi_1]/\nu$$

where $\zeta \in \mathcal{B}$ by (1.6.2).

We will explicitly construct ϑ for (3.8) so as to realize the bound $\hat{\varepsilon}_+(\nu)$ of (3.6). From (3.8), (3.9) we have

$$(3.10) \qquad \begin{aligned} \|x - \mathbf{F}\varphi_1\| &\leq \|(1 - \vartheta\alpha)\beta z\| + \nu\|\vartheta\zeta\| \\ &\leq \sup_\Omega\{|1 - \vartheta\alpha||\beta|\} + \nu\sup_\Omega\{|\vartheta|\} \end{aligned}$$

as we have $\|z\|, \|\zeta\| \leq 1$. This suggests the key step of the method:

(3.11) **optimization problem:**

1. given $s > 0$, choose $\vartheta(\omega)$ to minimize $|1 - \vartheta\alpha||\beta|$ over $\vartheta \in \mathbb{C}_s := \{\vartheta \in \mathbb{C} : |\vartheta| \leq s\}$;
2. choose s to minimize $\sup_\Omega$ of the result.

[8] In some sense, it is just the use of such optimization to define this linear filter which justifies our use of the term 'method of optimal filtering' — as well, of course, as the possibility that this may be an optimal filter.

For the first step, choosing each $\vartheta(\omega)$ by a constrained minimization, it is elementary 'geometry' in $\mathbb{C}$ to obtain

$$\vartheta_\nu(\omega) := \begin{cases} 1/\alpha & \text{when } 1/|\alpha| \le s \\ s\bar{\alpha}/|\alpha| & \text{when } s \le 1/|\alpha|. \end{cases} \tag{3.12}$$

pointwise on Ω. This then gives $\nu \sup_\Omega\{|\vartheta|\} \le \nu s$ and, for each $\omega \in \Omega$, gives $|1-\vartheta\alpha| \le (1-s|\alpha|)_+$ where $(\cdot)_+$ here indicates the 'positive part'. Note that we need only consider $\omega \in \Omega$ for which $|\alpha\beta| = t \in \Sigma^+$; then, using (3.2), one has

$$|1-\vartheta\alpha||\beta| \le (|\beta| - s|\alpha\beta|)_+ \le (\psi(t) - st)_+ \tag{3.13}$$

so the right hand side of (3.10) is bounded by $\sup_{t\in\Sigma^+}\{(\psi(t)-st)_+\}$; of course we have the bound for arbitrary choice of s. We observe that in taking the $\sup_t = \sup_\omega$ we also need never consider ω for which $1/|\alpha(\omega)| < s$, making $\psi(t) - st$ negative; thus the 'sup' remains the same if one omits taking the positive part. This expression depends lower semicontinuously on s and, on adding the coercive term νs, one need only consider s in a compact set so one does, indeed, have an attained minimum in the definition of $\hat{\varepsilon}_+(\nu)$ in (3.6); let $s_* = s_*(\nu)$ be such a minimizer. This shows that $\hat{\varepsilon}_+(\nu)$ is, indeed, a uniform bound realized by the linear filter given by (3.8) with (3.12), using $s = s_*$.

Using Lemma 1.1, we have now established the existence of $\varepsilon_\#$ and the upper bound in (3.5). We now obtain the lower bound: $\varepsilon_\# \ge \hat{\varepsilon}_-$. To this end, for any $\delta > 0$ we consider any $t_* \in \Sigma^+$ with $t_* \le \nu$. For (1.6) with $\varphi_1 = 0$ and $\varepsilon_0 = \bar{\nu} := t_*$ we now consider

$$z(\cdot) := \frac{t_*}{t_* + \delta}\,\frac{\chi(\cdot)}{\sqrt{\operatorname{meas}\mathcal{S}}}$$

where $\mathcal{S} = \mathcal{S}_{t_*;\delta}$ is as in (3.4) and χ is its characteristic function. Clearly, $\|z\| < 1$ and, since

$$\|\mathbf{AB}z\| = \|\alpha\beta z\| \le \sup_{\mathcal{S}}\{|\alpha\beta|\}\|z\|$$

and (3.4) gives $|\alpha\beta| < t_* + \delta$ on $\mathcal{S}$ (i.e., where $z \ne 0$), we have $\|\mathbf{AB}z\| \le \bar{\nu}$ whence $x(\cdot) := \beta z$ is a solution. Since (3.4) also gives $|\beta| > [\psi(t_*) - \delta]$ where $z \ne 0$, we have $\|z\| \le \|x\|/[\psi(t_*) - \delta]$. Now we have (1.3) for *both* the solutions x and $-x$ so we must have $2\varepsilon \geq \|x-(-x)\|$ whence

$$\varepsilon_{\#}(t_*) \geq \frac{t_*}{t_*+\delta}[\psi(t_*)-\delta].$$

Combining this with our choice of t_* and the obvious fact that $\varepsilon_{\#}(\bar{\nu}) \leq \varepsilon_{\#}(\nu)$ (since it is clear from (1.1) that $\varepsilon_{\#}(\cdot)$ is a nondecreasing function), we have the desired lower bound in the limit as $\delta \searrow 0$.

Finally, we consider the case of $\Sigma^+ = [0, \|\mathbf{AB}\|]$ with ψ increasing and concave. For increasing ψ defined on an interval, the definition of $\hat{\varepsilon}_-$ immediately gives $\hat{\varepsilon}_-(\nu) = \psi(\nu)$. Rewriting the argument of the min sup in defining $\hat{\varepsilon}_+$ as $[\psi(t)+s(\nu-t)]$, we recognize this definition as giving

$$\hat{\varepsilon}(\nu) := -[-\psi]^{**}(\nu)$$

where the $[\cdot]^*$ indicates Legendre-Fenchel conjugation. When ψ is concave so $-\psi$ is convex we have duality: $[-\psi]^{**} = -\psi$ (cf., e.g., [7]) so $\hat{\varepsilon} = \psi$ then. From (3.5) we now have

$$\psi(\nu) = \hat{\varepsilon}_-(\nu) \leq \varepsilon_{\#}(\nu) \leq \hat{\varepsilon}_+(\nu) = \psi(\nu)$$

which gives (3.7). With this, it is not difficult to see that the minimum occurs with $s = s_* := \psi'(\nu)$ when ψ is differentiable at ν and one has a comparable geometric characterization (by 'support') otherwise; this determines the $s = s_*$ to be used in (3.12).

Since the filter $\mathbf{F}_\nu$ defined above was shown to realize $\hat{\varepsilon}_+(\nu)$ which is here the *minimal* uniform bound, this construction gives an optimal filter in this setting. ∎

4. FURTHER REMARKS

REMARK 4.1: We now assert that Theorem 3.1 gives $\varepsilon_{\#}(\nu) \to 0$ as $\nu \to 0$ precisely when $t \to 0$ in Σ^+ implies $\psi(t) \to 0$. The necessity is immediate from the lower bound in (3.5). To see the sufficiency, we need only note that if we can obtain $\psi(t) < \bar{r}$ by requiring $t < \delta$ then, considering $s = \bar{r}/\delta$ for which $(\psi(t) - st)_+$ vanishes whenever $\psi(t) > \bar{r}$, we see that

$\hat{\varepsilon}_+(\nu) \le \nu\bar{r}/\delta + \bar{r}$ — if we can take this $\bar{r}$ arbitrarily small then $\hat{\varepsilon}_+(\nu)$ can also be made arbitrarily small by, e.g., taking $\nu \le 1/\delta$.

By comparison with Theorem 2.3 we note that it is sufficient (but not necessary) for this that **B** be compact. When **B** is compact so is **AB** and we have a common sequence of eigenvectors $\{e_k\}$ with corresponding eigenvalues α_k and β_k, each converging to 0. Now $\rho(r) = \max\{|\beta_k| : |\alpha_k\beta_k| = r\}$ and, since there are only finitely many $|\beta_k| \ge \delta$ for any $\delta > 0$, we must eventually have only $|\beta_k| < \delta$ as $r \to 0$ eliminates consideration of those. ∎

REMARK 4.2: It is interesting to note that instead of (3.10) one could have estimated by

$$\begin{aligned} \|x - \mathbf{F}\varphi_1\|^2 &\le \int_\Omega (|\beta||1-\vartheta\alpha||z| + |\vartheta|\nu|\zeta|)^2 \\ &\le 2\sup_\Omega \{|\beta|^2|1-\vartheta\alpha|^2 + \nu^2|\vartheta|^2\} \end{aligned} \tag{4.1}$$

and then modified the optimization procedure (3.11) to choose $\vartheta = \vartheta_\nu(\cdot)$ so as to minimize this last expression pointwise on Ω. It is elementary to minimize over $\vartheta \in \mathbb{C}$ and obtain $\vartheta = \lambda/\alpha$ with $0 \le \lambda := c^2/(1+c^2) < 1$ where $c := |\alpha\beta|/\nu$. This gives

$$\vartheta_\nu := \frac{\bar{\alpha}|\beta|^2}{\nu^2 + |\alpha\beta|^2} \tag{4.2}$$

pointwise on Ω so

$$|\beta|^2|1-\vartheta\alpha|^2 + \nu^2|\vartheta|^2 = \frac{2\nu^2|\beta|^2}{\nu^2 + |\alpha\beta|^2}. \tag{4.3}$$

Using this and (3.2) in (4.1), we now obtain

$$\varepsilon_\#(\nu) \le \tilde{\varepsilon}_+(\nu) := \sup_{t\in\Sigma^+} \left\{ \frac{\sqrt{2}\nu\psi(t)}{\sqrt{\nu^2+t^2}} \right\}. \tag{4.4}$$

This, of course, shows that $\tilde{\varepsilon}_+(\nu)$ also is a uniform bound which can be realized by the linear filter given by (3.8) with (4.2) rather than (3.12).

It is not immediately obvious how this result compares in general with the filter considered in Theorem 3.1. We *do* note that we necessarily have

$\tilde{\varepsilon}_+(\nu) \geq \psi(\nu)$ since one can consider $t = \nu$ in the defining 'sup'. Thus, in the setting of (3.7) at least, this filter cannot be better than the filter of Theorem 3.1. For a special case, see Remark 4.3, below. ∎

REMARK 4.3: The setting of (3.7) in Theorem 3.1 is particularly applicable to the interesting situation in which, for some $\lambda > 0$, one has

$$|\mathbf{B}| = |\mathbf{A}|^\lambda, \qquad \text{i.e., } |\beta(\cdot)| = |\alpha(\cdot)|^\lambda \text{ ae on } \Omega. \tag{4.5}$$

Here, assuming for simplicity that $\Sigma^+ = \operatorname{ess\,ran}(|\alpha\beta|)$ is an interval, one can carry out the computations by elementary calculus, without direct reference to (3.7). Observe that (4.5) gives, ae on Ω, that

$$|\beta(\cdot)| = |\alpha(\cdot)\beta(\cdot)|^\mu \qquad \text{with } \mu := \frac{\lambda}{1+\lambda}$$

so we have $\psi(t) = t^\mu$ for $0 \leq t \leq \|\mathbf{AB}\|$. As $0 < \lambda$ gives $0 < \mu < 1$, we note that ψ is here (strictly) increasing and concave. Obviously $\hat{\varepsilon}_-(\nu) = \psi(\nu)$. Elementary calculus and a bit of algebraic manipulation shows that $(t^\mu - st)$ attains its maximum at $t = t_*(s) := (\mu/s)^{1/(1-\mu)}$, giving $qs^{-\lambda}$ with $q := (1-\mu)\mu^\lambda$, and then that $\nu s + qs^{-\lambda}$ attains its minimum at $s = s_* = s_*(\nu) := (\lambda q/\nu)^{(1-\mu)}$, giving $\hat{\varepsilon}(\nu) = (\lambda+1)qs_*^{-\lambda}$. With somewhat more manipulation, one simplifies these expressions to obtain

$$s_*(\nu) = \mu\nu^{1-\mu}, \qquad \hat{\varepsilon}_+(\nu) = \nu^\mu. \tag{4.6}$$

In (3.12) we then use $s = s_*(\nu)$ from (4.6) to obtain the filter factor ϑ. Finally, we confirm that this has given $\hat{\varepsilon}_+(\nu) = \psi(\nu)$, just as asserted by (3.7) and also observe that it gives the 'log convexity' estimate

$$\bar{\varepsilon}(\nu) = \varepsilon_0 \varepsilon_\#(\nu) = \varepsilon_0^{1-\mu}\varepsilon_1^\mu \tag{4.7}$$

for the *original* problem (1.1).

One can similarly carry out the computations for the filter of Remark 4.2. Again elementary calculus suffices for computation of the maximization in (4.4) defining $\tilde{\varepsilon}_+$ and one easily sees that $\tilde{\varepsilon}_+(\nu) = C\nu^\mu$ with

$$C = C_\mu := \max\left\{\frac{\sqrt{2}t^{1-\mu}}{\sqrt{1+t^2}}\right\} = \sqrt{2\mu^\mu(1-\mu)^{1-\mu}}.$$

Note that $C_\mu \geq 1$ with equality only for $\mu = \frac{1}{2}$, corresponding to $\lambda = 1$. Thus, at least for this setting with $\lambda \neq 1$, the original filter is strictly better since it is optimal and we have $\hat{\varepsilon}_+ = \varepsilon_\# = \nu^\mu < C\nu^\mu = \tilde{\varepsilon}_+$. ■

REMARK 4.4: Beside 'Problem 1', given by (1.1), one can consider another setting, 'Problem 2', involving a *parametrized family of such problems*. Thus, one might have a single space $\mathcal{X}$ but an *evolution family* of continuous linear operators[9] $\mathbf{S}(t,s) : \mathcal{X} \to \mathcal{X}$ with

$$(4.8) \qquad \begin{cases} \mathbf{S}(t,t) = \mathbf{I} \\ \mathbf{S}(t,s) \circ \mathbf{S}(s,r) = \mathbf{S}(t,r) \quad \text{for } t_0 \leq r \leq s \leq t \leq t_1. \end{cases}$$

One now has a parametrized family of problems like (1.1): seek a function $x(t) := \mathbf{S}(t,t_0)x_0$ for $t_0 < t < t_1$ such that

$$(4.9) \qquad \begin{array}{ll} .1) & \|x(t_0) - \varphi_0\| \leq \varepsilon_0 \\ .2) & \|x(t_1) - \varphi_1\| \leq \varepsilon_1 \end{array}$$

For each fixed t this has the same form as (1.1) with $\mathbf{A} = \mathbf{S}(t,t_1)$ and $\mathbf{B} = \mathbf{S}(t_0,t)$ and we note consistency (for all t with $t_0 < t < t_1$) provided there is some x_0 such that $\|x_0 - \varphi_0\| \leq \varepsilon_0$ and $\|\mathbf{S}(t_1,t_0)x_0 - \varphi_1\| \leq \varepsilon_1$. We now have the solution set $\mathcal{U}(t) := \mathcal{U}_0(t) \cap \mathcal{U}_1(t)$ where

$$(4.10) \qquad \begin{aligned} \mathcal{U}_0(t) &:= \{x \in \mathcal{X} : x = \mathbf{S}(t,t_0)x_0 \text{ with } \|x_0 - \varphi_0\| \leq \varepsilon_0\}, \\ \mathcal{U}_1(t) &:= \{x \in \mathcal{X} : \|\mathbf{S}(t_1,t)x - \varphi_1\| \leq \varepsilon_1\}. \end{aligned}$$

In the linear case we obviously have the same reduction as in Lemma 1.1 to replace consideration of (4.9) in general by consideration after replacing $[\varphi_0, \varepsilon_0]$ by $[0,1]$ while also replacing $[\varphi_1, \varepsilon_1]$ by $[\tilde{\varphi}, \nu]$ with $\nu := \varepsilon_1/\varepsilon_0$ and with $\tilde{\varphi} := (\varphi_1 - \mathbf{S}(t_1,t_0)\varphi_0)/\varepsilon_0$. We then seek a filter — more precisely, a family of filters parametrized by $t \in [t_0,t_1]$ — as a map

$$\mathbf{F} = \mathbf{F}_\nu(t) : \mathcal{H} \to \mathcal{H} : \tilde{\varphi} \mapsto x_*(t)$$

for which we obtain a uniform bound

$$(4.11) \qquad \begin{aligned} &\|\mathbf{S}(t,t_0)x_0 - \mathbf{F}_\nu(t)\tilde{\varphi}\| \leq \varepsilon(\nu,t) \\ &\qquad \text{for } x_0 \in \mathcal{B} \text{ with } \|\mathbf{S}(t_1,t_0)x_0 - \tilde{\varphi}\| \leq \nu. \end{aligned}$$

[9] One could also, more generally, be treating nonlinear maps $\mathbf{S}(t,s) : \mathcal{X}_s \to \mathcal{X}_t$ with a scale of spaces $\{\mathcal{X}_t : t_0 \leq t \leq t_1\}$.

We call such a filter $\mathbf{F}_\nu(t)$ an **optimal filter** if $\varepsilon(\nu,t) = \varepsilon_\#(\nu,t)$ is minimal (for $0 < \nu$ and $t_0 < t < t_1$).

For the present, we comment on this situation only in the context of a *semigroup* $[\mathbf{S}(t,s) = \mathbf{S}(t-s)]$ of normal linear operators acting on a Hilbert space $\mathcal{H}$. Note that this fits the case of (4.5) since for each t ($t_0 < t < t_1$) one has $\mathbf{A} = \mathbf{S}(t_1 - t)$, $\mathbf{B} = \mathbf{S}(t - t_0)$ and, using semigroup properties, one has $\mathbf{B} = \mathbf{A}^\lambda$ with

$$\lambda = \frac{t - t_0}{t_1 - t} \qquad \text{giving } \mu := \lambda/(1+\lambda) = \frac{t-t_0}{t_1 - t_0}.$$

For notational convenience we now assume t is normalized to give $t_0 = 0$ and $t_1 = 1$ so $\mu = t$. It is also convenient to use an exponential formulation so we write our spectral decomposition in the form:

$$\beta(\cdot) = e^{-t\gamma(\cdot)}, \quad \alpha(\cdot) = e^{-(1-t)\gamma(\cdot)} \quad \text{with } \gamma(\cdot) = g(\cdot) + ih(\cdot)$$

where the function $\gamma(\cdot)$ represents the infinitesimal generator of the semigroup. Note that Σ^+ is now $\exp[-\operatorname{ess\,ran}(g)]$ and that our filter has been shown optimal when this is an interval. Using (4.6) in the form

$$s_* = s_*(\nu,t) = e^{(1-t)\bar{r}} \quad \text{with } \bar{r} = \bar{r}_\nu(t) := \log\nu + \frac{\log t}{1-t}, \tag{4.12}$$

the filter $\mathbf{F}_\nu(t)$ here becomes multiplication by ϑ where

$$\begin{aligned} \vartheta &= \vartheta_\nu(t,\cdot) := e^\rho \qquad \text{with} \\ \rho &= \rho_\nu(t,\omega) \\ &:= i(1-t)h(\omega) + \begin{cases} (1-t)g(\omega) & \text{when } g(\omega) \le \bar{r}_\nu(t), \\ (1-t)\bar{r} & \text{otherwise.} \end{cases} \end{aligned} \tag{4.13}$$

As noted, this is an optimal filter, realizing the minimal uniform bound ν^t (also, see (4.7)). ■

REMARK 4.5: The 'sideways heat equation' problem treated in [12] fits the setting of Remark 4.4 and it is interesting to note that we have obtained the identical result here as there although the derivation was rather different. One may also compare our derivation of (4.7) with [6]. Here, the derivation was through the relation to the 'stationary' problem (for one t at a time, essentially independently). In [12], on the other hand, the emphasis was 'dynamic' — marching with respect to the parameter against the smoothing effect of the semigroup. This meant, essentially, that the same approach as in (3.11) was applied 'differentially' in the context of a problem with 'moving information'. In our present notation we may formulate that as: at each t one was 'given' $[v(t+dt), \nu(t+dt)]$ and considered the problem of finding the pair $[v(t), \nu(t)]$ so as to approximate $x(t)$ satisfying

$$(4.14) \qquad \begin{aligned} .1) \qquad & x(t) = \mathbf{S}(t)\tilde{z} \text{ with } \tilde{z} \in \mathcal{B} := \{z \in \mathcal{Z} : \|z\| \le 1\} \\ .2) \qquad & \|\mathbf{S}(dt)x(t) - v(t+dt)\| \le \nu(t+dt) \end{aligned}$$

with an error bound $\nu(t)$. This then led to a differential inequality for $\nu(\cdot)$ (considered for t progressing 'leftward' from $t = 1$, using the given values $[v(1), \nu(1)] = [\tilde{\varphi}, \nu]$, towards $t \searrow 0$) in which $\bar{r}(t)$ is viewed as a 'control', optimally chosen to make $\nu(\cdot)$ grow as slowly as possible. Thus, $\bar{r}$ becomes a function of $\nu(t)$, much as in (4.12) but modified in view of the differential rescaling of the parameter interval. It turns out that the differential equations can be solved explicitly, leading to explicit formulas for $[v(t), \nu(t)]$ which coincide with those obtained in Remark 4.4. The backwards heat equation can be treated similarly [13].

While the present treatment may appear conceptually simpler for the derivation in this case, it does seem that the differential approach employed in [12] may be effective in the generation of useful (suboptimal) algorithms for cases involving evolutionary families not of semigroup form — so one may not have the commutativity implicit in the spectral resolution used here (or even linearity) but might have these to a good enough approximation differentially, i.e., by linearization, ... ■

References

[1] J.R. Cannon, *A priori estimate for continuation of the solution of the heat equation in the space variable*, Ann. Mat. Pura Appl. **65** (1964), pp. 377–388.

[2] L. Eldén, *Hyperbolic approximations for a Cauchy problem for the heat equation*, Inverse Problems **4** (1988), pp. 59–70.

[3] J.N. Franklin, *On Tikhonov's method for ill-posed problems*, Math. of Comp. **28** (1974), pp. 889–907.

[4] F. John, *Continuous dependence on data for solutions with a prescribed bound*, Comm. Pure Appl. Math. **13** (1960), pp. 551–585.

[5] V. Klee, in Math. Rev. **13** (1952), p. 661.

[6] H.A. Levine, *Continuous data dependence, regularization, and a three lines theorem for the heat equation with data in a space like direction*, Ann. Mat. Pura Appl. (IV) **CXXXIV** (1983), pp. 267–286.

[7] R.T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[8] W. Rudin, *Functional Analysis*, McGraw–Hill, New York, 1973.

[9] I.E. Segal and R.A. Kunze, *Integrals and Operators*, McGraw–Hill, New York, 1968.

[10] T.I. Seidman, *An inverse eigenvalue problem with rotational symmetry*, Inverse Problems **4** (1988), pp. 1093–1115.

[11] T.I. Seidman, *Some 'complexity' issues for ill-posed problems*, in *Proc. AMS–SIAM Conference on Ill-Posed and Inverse Problems* (D. Colton, R. Ewing, W. Rundell, eds.), SIAM, 1990, pp. 17–35.

[12] T.I. Seidman and L. Eldén, *An 'Optimal Filtering' method for the sideways heat equation*, Inverse Problems **6** (1990), pp. 681–696.

[13] T.I. Seidman and R. Ewing, *'Optimal Filtering' for the backwards heat equation*, to appear.

[14] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.