Well-posedness of a dissolution-growth problem on an unbounded domain

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ABSTRACT: We consider a spherical grain which may be growing by accretion or dissolving in a dilute solution of the same substance where one also has reaction and diffusion. The resulting free boundary problem is shown to be well-posed and additional regularity is obtained.

KEY WORDS: *Stefan Problem with surface tension, unbounded domain, Free Boundary Problem, radial geometry, system, parabolic, partial differential equation, reaction/diffusion, well-posedness, regularity.*

1. Introduction

In this paper we study a simple model of a solid/liquid interface which forms a free boundary for the problem. This model is composed of a solid phase of a single substance and an incompressible liquid phase which is a dilute solution of the same substance involving both diffusion and possible chemical reaction. One interpretation of this model is the corrosion of a material in water with both dissolution of the material into the water and accretion of dissolved material to the solid possible so the interface motion need not have a fixed direction. We consider the special setting of radial symmetry in two or three dimensions.

The system evolution consists of two interacting processes: a dissolution/accretion process at the interface and a reaction/diffusion process in the liquid phase, occupying the time-varying spatial region $\Omega = \Omega(t) \subset \mathbb{R}^d$ — i.e., Ω is an unbounded exterior domain and $\partial\Omega = \partial\Omega(t)$ is the interface. The reaction/diffusion process is governed by the usual equation for the concentration C:

$$(1.1) \qquad C_t = \Delta C + F(C) \;\text{ in } \mathcal{Q} := \left\{ (t,x) \in \mathbb{R}^{1+d} : \begin{array}{c} 0 < t < \tau^* \\ x \in \Omega(t) \end{array} \right\}$$

with the conservation boundary condition at the interface[1]

$$(1.2) \qquad \frac{\partial C}{\partial n} = (1-C)V_n \;\text{ on } \Sigma := \left\{ (t,x) \in \mathbb{R}^{1+d} : \begin{array}{c} 0 < t < \tau^* \\ x \in \partial\Omega(t) \end{array} \right\}$$

In (1.2), V_n is the rate of motion of the interface in its normal direction, with both this motion and $\partial C/\partial n$ taken as oriented inward to $\Omega(t)$. The

[1]We are considering the concentration C of a single chemical component and assume this is dilute enough that the reaction does not effectively change any other relevant concentrations so we may avoid working with a coupled reaction/diffusion system. Consideration of such a system would involve only minor modification of our arguments (if the other components remain only in the liquid phase) and is left to the reader.

As $0 < C \ll 1$, one also will find in the literature the boundary condition (1.2) simplified, replacing $(1-C)$ by 1 to get: $\partial C/\partial n = V_n$; not only is (1.2) physically more correct as stated, but we need this form for our analysis, to ensure that C will take only physically meaningful values in the range $[0,1]$; cf., Lemma 3.2.

constitutive law for the interface motion may be expected[2] to take the form

$$V_n = \Lambda\,[C - h(\kappa)] \ \text{ on } \Sigma \tag{1.3}$$

where Λ is a rate constant, C is here the concentration in the adjacent liquid phase, and h is a suitable regular given function of the local mean curvature κ of $\partial\Omega$ (where we take $\kappa > 0$ when, for example, the solid phase would be convex).

The physical derivation originally leading to [1], [2], [6], [9], [13] and [14] was due to M. Cournil [7] and employed the Ostwald-Freundlich Law

$$h(\kappa) = \Gamma^* e^{\gamma\kappa} \tag{1.4}$$

but one might alternatively consider the frequently used exactly linear law

$$h(\kappa) = \Gamma^*(1 + \gamma\kappa)\,. \tag{1.5}$$

These match to first order; each is a reasonable approximation so long as $|\kappa|$ remains small, but certainly requires correction for large $|\kappa|$.

In this paper, we assume radial symmetry so the solid is and remains a spherical 'grain' in dimension 2 or 3; we denote by $R = R(t)$ the radius of the grain (so $\kappa = 1/R$) and by $r := |x|$ the distance from the center. In this radial case, $\Omega(t) = \{x : |x| = r > R(t)\}$ and (1.1), (1.2), (1.3) become

$$C_t = C_{rr} + \frac{d-1}{r} C_r + F(C) \ \text{ in } \mathcal{Q} := \left\{ (t,r) : \begin{array}{c} 0 < t < \tau^* \\ r > R(t) \end{array} \right\} \tag{1.6}$$

with

$$C_r = (1 - C)\dot{R} \qquad \text{at } r = R(t) \tag{1.7}$$

[2]This represents the balance — to first order — of competing reactions: adjacent free molecules in the liquid bind to the solid surface and (in the presence of energy fluctuations exceeding a threshhold) bound molecules become free. The differential in binding energy between surface and interior molecules implies a surface term in the Gibbs free energy, effectively 'surface tension'. From the geometric fact that the derivative of local surface area with respect to normal motion of the surface is precisely the mean curvature, this makes the reaction balance curvature dependent, leading to (1.3) with $0 < \Gamma^* := h(0)$ giving the saturation concentration and $0 < \gamma := h'(0)/h(0) > 0$ proportional to the interfacial 'surface tension' at equilibrium.

and the constitutive law

$$\dot{R} = H(R, \omega) \quad \text{with } \omega := C\Big|_{r=R(t)} \tag{1.8}$$

for the interface kinetics; we have in mind, but do not at this point require, the specific form

$$H(R, \omega) = \Lambda\,[\omega - h(1/R)] \tag{1.9}$$

directly corresponding to (1.3).

Extending [9], our treatment here is related to the analysis of F. Conrad, D. Hilhorst, and T.I. Seidman [6] who considered (1.6) in a bounded domain $0 < R(t) < r < 1$ with (1.7), (1.8) at $r = R(t)$ and with a Dirichlet condition

$$C(t, 1) = \alpha(t) \tag{1.10}$$

specified at the outer boundary $r = 1$. In this setting, the problem was shown to be (locally) well-posed under rather general hypotheses regarding F, H and the data R_0, C_0, α.[3]

In this paper we consider the case of radial geometry with an unbounded domain. One would expect a spatially uniform concentration $\Gamma = \Gamma(t)$ if there were no grain at all so that (1.6) would then become an ODE

$$\dot{\Gamma}(t) = F(\Gamma(t)), \qquad \text{for } t > 0. \tag{1.11}$$

It is classical [8] that (1.11) has global $C^{1,1}$ solutions under the conditions we impose on F and we assume henceforth that $\Gamma(\cdot)$ is a known solution of

[3] In a nonradial setting, the problem { (1.1) with $F \equiv 0$, (1.2), and (1.3) with (1.4) } has been studied in a 2-dimensional bounded region by F. Abergel, D. Hilhorst and F. Issard-Roch [1], by J.F. Scheid [13], [14] in the case where the interface is parametrised as $y = f(x, t)$, and by F. Abergel, D. Hilhorst, F. Issard-Roch and J.F. Scheid [2] when $\Omega(t)$ is a more general annulus. In [1] it is proved that this problem has a unique solution in a neighborhood of the stationary solution $(C, f) = (\Gamma^*, 0)$ while in [2] and [14] a result of local existence in time and uniqueness is established for more general initial data.

A quite similar problem, corresponding to a model of supersaturated solidification, was studied by J. Chadam, S.D. Howison, and P. Ortoleva in spherical symmetry [5] and by J.C. Baillon, M. Bertsch, J. Chadam, P. Ortoleva, and L.A. Peletier for a planar version [4]. These versions omit the reaction term F and impose the Dirichlet boundary condition

$$C = \Gamma^*(1 + \gamma\kappa),$$

which corresponds to using (1.5) in (1.3) and dividing by $\Lambda \to \infty$ to treat the scaled asymptotic evolution of a perturbed shape.

(1.11) which gives the behavior of the concentration in the liquid far away from the grain so $C(r,t) \to \Gamma(t)$ as $r \to \infty$. More precisely, the 'boundary condition at infinity' are given by

$$[C(t,\cdot) - \Gamma(t)] \in H^1(R(t), \infty). \tag{1.12}$$

We also impose appropriate initial conditions:

$$R(0) = R_0 \quad \text{and} \quad C(0,r) = C_0(r) \text{ for } r > R_0. \tag{1.13}$$

The complete problem, then, is the coupled system

$$\{ \ (1.6), (1.12), (1.7), (1.8), (1.13) \ \}. \tag{$\mathbf{P}$}$$

The purpose of this paper is to prove that **(P)** is well-posed. We will study the large time behavior of the growth rate $R(\cdot)$ in a forthcoming paper [10].

Before proceeding further, we pause to state the hypotheses we will impose throughout on the constitutive functions[4] $\{F, H\}$ of the problem:

$$\begin{array}{rl} i) & F \text{ is Lipschitzian on } [0,1] \text{ with } F(1) \le 0 \le F(0); \\ ii) & H \in C^1((0,\infty) \times [0,1]) \text{ with } H(\cdot, 0) \le 0. \end{array} \tag{1.14}$$

and on the data $\{R_0, \Gamma, C_0\}$:

$$\begin{array}{rl} i) & 0 < R_0 < \infty; \\ ii) & \Gamma(\cdot) \text{ satisfies (1.11) with } 0 < \Gamma_0 = \Gamma(0) < 1; \\ iii) & 0 \le C_0 \le 1 \text{ with } [C_0(\cdot) - \Gamma_0] \in L^2(R_0, \infty). \end{array} \tag{1.15}$$

The proof of existence and uniqueness is based on a fixed point theorem for a convenient map $\mathcal{M}$. The argument strategically follows that of [6] but deals with some new technical difficulties arising due to the unbounded region. In Section 2 we treat the reformulation of Problem **(P)** needed to be

[4] We remark that the regularity imposed here could later be relaxed, using a limit argument after the results presented. For example, rather than requiring Lipschitz continuity of $F(\cdot)$, one might ask only that it be continuous with a one-sided estimate: $F(r) - F(s) \le L(r-s)$ for $0 < s < r < 1$. The requirement $H(\cdot, 0) \le 0$ in (1.14-*ii*) means physically that the grain cannot grow when there is no material supply; it would be equally plausible to require that $H(\cdot, 1) \ge 0$ so the grain would not shrink in the presence of an abundant supply but (see the proof of Lemma 3.2) we have no need to impose this explicitly.

able, in Section 3, to show that the map $\mathcal{M}$ is a well-defined contraction (for a suitably created metric) and then conclude the well-posedness argument. Finally, in Section 4 we give some additional regularity.

2. Reformulation of the problem

Much as for ordinary differential equations, we proceed locally in t so, in defining Problem **(P)**, we restrict t to an interval $[0, \tau]$ with $\tau = \tau(R_0)$ chosen to ensure that $R(\cdot)$ will vary in a tractable range:

$$0 < \bar{a} \leq R(t) \leq \bar{b} \qquad \text{for } 0 \leq t \leq \tau \tag{2.1}$$

where, for convenience, we will take $\bar{a} := R_0/2$ and $\bar{b} := 2R_0$. [Later, in subsection 3.3, we will discuss the 'maximal time of existence'.]

It will be convenient to use L for a large enough Lipschitz constant both to apply to $F(\cdot)$ and to give (2.5), below. We also introduce the exponentially weighted inner product and norm for $L^2(0, t)$:

$$\langle \zeta, \hat{\zeta} \rangle_\lambda := \int_0^t \zeta \hat{\zeta}\, e^{-\lambda s}\, ds, \qquad \|\zeta\|_\lambda := \left[\int_0^t \zeta^2\, e^{-\lambda s} ds \right]^{1/2}, \tag{2.2}$$

where $\lambda > 0$ will be determined later and with $t \in [0, \tau]$ defined by context.

Our strategy will be the construction of a map $\mathcal{M}$ whose fixpoint will provide the desired solution. In this section we decompose **(P)** into subproblems **(R)**, **(C)**, whose composition will define $\mathcal{M}$; we also provide an equivalent form **(U)** of **(C)** in which the moving boundary problem **(C)** is transformed into a problem **(U)** on the fixed unbounded domain $\hat{\mathcal{Q}} = \hat{\mathcal{Q}}_\tau := [0, \tau] \times [0, \infty)$.

Note that, while we must eventually have C in the physical range (i.e., $0 \leq C \leq 1$) for the final coupled Problem **(P)**, this need not be the case at the intermediate steps. This requires that $F(\cdot)$ must, temporarily, be extended outside the physical range for its appearance in (2.1.3-i) and, similarly, that extension is needed for the conditions at $r = R(t)$. It will be important for our arguments that these extensions — necessarily nugatory at the conclusion — should here be done in a carefully chosen way. We begin by extending $F(\cdot)$ to $\mathbb{R}$ by setting $F(z) := F(0)$ for $z \leq 0$ and $F(z) := F(1)$ for $z \geq 1$. For the constitutive relation (1.8), we first note that (2.1) will mean we can restrict H from $(0, \infty) \times [0, 1]$, as in (1.14-$ii$), to $[\bar{a}, \bar{b}] \times [0, 1]$ before extending to $[\bar a,\bar b]\times\mathrm{I\!R}$. We wish to replace (1.14) by

$$(2.3)\qquad \begin{array}{ll} i) & F \text{ is uniformly Lipschitzian on } \mathrm{I\!R} \text{ with} \\ & \qquad F(z)\ge 0 \text{ for } z\le 0 \text{ and } F(z)\le 0 \text{ for } z\ge 1; \\ ii) & H\in C^1([\bar a,\bar b]\times\mathrm{I\!R}) \text{ with } H(\cdot,z)\le 0 \text{ for } z\le 0 \\ & \qquad \text{and } H(\cdot,z)\ne 0 \text{ only for } -1<z<2. \end{array}$$

Given (1.14), our extension of F satisfies (2.3-i) and one can easily extend H so as to satisfy (2.3-ii). For the conservation boundary condition (1.7), we define

$$(2.4)\qquad k(z):=\begin{cases}(1-z) & \text{for } z\le 1\\ (1-z)/2 & \text{for } z\ge 1\end{cases}\qquad G(r,z):=k(z)H(r,z).$$

From (2.3-ii) it follows that H and G are each bounded and uniformly Lipschitzian on $[\bar a,\bar b]\times\mathrm{I\!R}$:

$$(2.5)\qquad \begin{aligned} &|H(r,z)|,\ |G(r,z)|\le M,\\ &|H(r_1,z_1)-H(r_2,z_2))|,\ |G(r_1,z_1)-G(r_2,z_2))|\\ &\qquad\qquad \le L[|r_1-r_2|+|z_1-z_2|]. \end{aligned}$$

2.1. The map $\mathcal{M}$

We construct a map $\mathcal{M}$ from ω to a 'new ω' by

$$(2.1.1)\qquad \mathcal{M}:\omega\overset{\mathbf{(R)}}{\longmapsto}R\overset{\mathbf{(C)}}{\longmapsto}C\mapsto C(\cdot,R(\cdot))=\omega_{\text{new}},$$

i.e., we start with a function ω in $L^2(0,\tau)$ and, using the extended H, first solve the ordinary differential equation

$$(2.1.2)\qquad \dot R=H(R,\omega)\ \text{ on } [0,\tau]\quad \text{with } R(0)=R_0$$

and then, using this $R(\cdot)$ so this is not a free boundary problem, solve

$$(2.1.3)\qquad \begin{array}{ll} i) & C_t=C_{rr}+\dfrac{d-1}{r}C_r+F(C)\quad \text{in } \mathcal{Q}:=\left\{(t,r):\ \begin{array}{c}0<t<\tau\\ r>R(t)\end{array}\right\}\\ ii) & C_r=G(R,C)\qquad \text{at } r=R(t)\\ iii) & C(0,r)=C_0(r)\ \text{ for } r>R_0, \end{array}$$

corresponding to (1.6), (1.7), and (1.13) — with (1.12) entering in our later choice of solution space. We will refer to the first of these steps (2.1.2) as ‘ Problem **(R)**: $\omega \mapsto R$ ’ and to the second step (2.1.3) as ‘ Problem **(C)**:$R \mapsto C$ ’. These decoupled problems **(R)**, **(C)** are then coupled by

$$\omega(t) = C(t, R(t)) \quad \text{for } 0 \le t \le \tau. \tag{2.1.4}$$

We note that the form of (2.1.3-*ii*), using G as in (2.4), differs from that of (1.7), but that these will eventually be equivalent, in view of (2.1.2), when C is in the physical range and (2.1.4) holds.

We will show in the next section that $\mathcal{M}$ is, indeed, well-defined by a modification of (2.1.1) (transforming so the moving boundary $r = R(t)$ becomes the line $y = 0$ and we have a fixed domain) and that it has a unique fixpoint, which we will show in the following section provides the desired solution of Problem **(P)**.

2.2. Problem (R) and transformation (C) ≡ (U)

We begin by choosing $\tau = \tau(R_0)$, setting

$$\begin{aligned} \tau \;&:= (\ln 2)/M \quad \text{ where } \bar{a} := R_0/2,\ \bar{b} := 2R_0, \\ M = M(R_0) \;&:= \max\{|H(r,z)|/r : r \in [\bar{a}, \bar{b}],\ z \in \mathbb{R}\}. \end{aligned} \tag{2.2.1}$$

PROPOSITION 2.1: *Given any function $\omega \in L^2(0,\tau)$, there is a unique corresponding solution $R(\cdot)$ of (2.1.2). This solution is Lipschitzian and satisfies (2.1).*

PROOF: From the definition (2.2.1) one easily sees the invariance of the closed set $\mathcal{S} := \{R(\cdot) \in C([0,\tau]) : |\ln R/R_0| \le \ln 2\}$ under the Picard map:

$$R(\cdot) \mapsto R_0 \exp\left[\int_0^t [H(R(s), \omega(s))/R(s)]\, ds\right].$$

By (2.3-*ii*) and (2.5) $R_0 H(r,z)/r$ has a uniform Lipschitz constant L on $[\bar{a}, \bar{b}] \times \mathbb{R}$, so this Picard map is contractive with respect to the norm $|R|_L = \max\{e^{-4Lt}|R(t)| : t \in [0,\tau]\}$. Thus there is a unique fixpoint in $\mathcal{S}$ of the map — which is the desired unique solution of (2.1.2). This gives $R \in C([0,\tau])$ but, as $\dot{R} = H(R,\omega)$ is bounded on this compact set, R is actually Lipschitzian. The fact that $R \in \mathcal{S}$ is equivalent to (2.1). ■

Given $R(\cdot)$, we make the substitution $y := r - R(t)$, transforming $\mathcal{Q}$ into $\hat{\mathcal{Q}} := [0,\tau] \times [0,\infty)$, as indicated earlier. We also set

$$\text{(2.2.2)} \quad \psi(t,y) := \dot{R}(t) + \frac{d-1}{y + R(t)}, \qquad u_0(y) := C_0(y + R_0) - \Gamma_0 .$$

Since the input $R(\cdot)$ will later always be the output of Problem **(R)**, we can assume (2.1) and that $R(\cdot)$ is absolutely continuous with $\dot{R} = H(R,\omega)$ bounded so $\psi(\cdot)$ will be bounded on $\hat{\mathcal{Q}}$.

It is not difficult to see that using this substitution to define $u(t,y) := C(t,r) - \Gamma(t)$ gives a solution u of

$$\text{(2.2.3)} \quad \begin{array}{rl} i) & u_t = u_{yy} + \psi(t,y)u_y + [F(u+\Gamma) - F(\Gamma)] \ \text{ in } \hat{\mathcal{Q}} := [0,\tau] \times [0,\infty) \\ ii) & u_y = G(R, u+\Gamma) \ \text{ at } y = 0 \\ iii) & u(0,y) = u_0(y) := C_0(y+R_0) - \Gamma_0 \ \text{ for } y > 0, \end{array}$$

if and only if $C = u + \Gamma$ is a solution of (2.1.3). We refer to (2.2.3) as 'Problem **(U)**: $R \mapsto u$'.

We have, then, the equivalent construction for the map $\mathcal{M}$

$$\text{(2.2.4)} \qquad \mathcal{M} : \omega \overset{\mathbf{(R)}}{\longmapsto} R \overset{\mathbf{(U)}}{\longmapsto} u \mapsto [u(\cdot,0) + \Gamma(\cdot)].$$

This form will be more convenient for showing solvability and for the computations showing contractivity of $\mathcal{M}$ with respect to $\|\cdot\|_\lambda$.

To have defined the map $\mathcal{M}$, it is necessary to supplement Proposition 2.1 by showing that the step **(U)** is also well-defined. We first consider this in the context of

$$\text{(2.2.5)} \qquad u \in \boldsymbol{U} := L^\infty([0,\tau] \to \mathcal{H}) \cap L^2([0,\tau] \to \mathcal{V}),$$

with $\mathcal{H} = L^2(0,\infty)$ and $\mathcal{V} = \{v \in \mathcal{H} : v_y \in \mathcal{H}\}$ so $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. The Hilbert space norm on $\mathcal{H}$ will be denoted by $\|\cdot\|$. We then consider $\mathcal{H}_* = L^2_\mu(0,\infty)$ with $\mu(y) := (1+y)^{d-1}$ — defined as the closure of $C_0^\infty[0,\infty)$ with respect to the Hilbert space norm (denoted by $\|\cdot\|_*$) obtained from the weighted inner product

$$\langle f, g\rangle_* := \int_0^\infty f(y)g(y)\,(1+y)^{d-1}\,dy$$

— and the corresponding space $\mathcal{V}_* = \{v \in \mathcal{H}_* : v_y \in \mathcal{H}_*\}$. This choice of weight μ gives a norm equivalent to the r^{d-1}-weighted norm coming from the use of polar coordinates in the context of $L^2(\mathbb{R}^d \setminus [\text{grain}])$. For $u_0 \in \mathcal{H}_*$ we will show that

$$u \in \boldsymbol{U}_* := L^\infty([0,\tau] \to \mathcal{H}_*) \cap L^2([0,\tau] \to \mathcal{V}_*). \tag{2.2.6}$$

PROPOSITION 2.2: *Assume (1.14) and let $R(\cdot)$ satisfy (2.1.2), (2.1). Then, for each $u_0 \in \mathcal{H} = L^2(0,\infty)$, there is a unique solution of (2.2.3) as in (2.2.5); further, if $u_0 \in \mathcal{H}_*$, then we have (2.2.6).*

PROOF: Set $f(t,u) := F(u+\Gamma(t)) - F(\Gamma(t))$ and let $\hat{g} = \hat{g}(\cdot,u)$ be the right hand side of (2.2.3-ii) for the given R, i.e.,

$$\hat{g}(t,z) := G(R(t), z+\Gamma(t)) = \begin{cases} H(R(t), z+\Gamma(t))(1-\Gamma(t)-z) & (z \le 1) \\ H(R(t), z+\Gamma(t))(1-\Gamma(t)-z)/2 & (z \ge 1), \end{cases}$$

using (2.4). By our assumptions, there are $L, M > 0$ such that

$$|\psi| \le M, \quad |f(\cdot,z_1) - f(\cdot,z_2)|,\ |\hat{g}(\cdot,z_1) - \hat{g}(\cdot,z_2)| \le L|z_1 - z_2| \tag{2.2.7}$$

for all z_1, z_2. Multiplying (2.2.3-i) by a smooth function φ with compact support, one gets

$$\langle \varphi, u_t \rangle + \langle \varphi_y, u_y \rangle = \langle \psi\varphi, u_y \rangle + \langle \varphi, f(\cdot,u) \rangle - \varphi\hat{g}(\cdot,u)\Big|_{y=0} \tag{2.2.8}$$

with $\langle \cdot,\cdot \rangle$ indicating the inner product of $\mathcal{H}$ — reducing to an integral over $[0,\ell]$ for some $\ell > 0$, depending on the support of φ.

We now apply a fairly standard Galerkin argument for existence. Given $\operatorname{span}\{\varphi_1,\ldots\}$ simultaneously dense in $\mathcal{H}, \mathcal{V}$, we set $\mathcal{H}^\nu := \operatorname{span}\{\varphi_1,\ldots,\varphi_\nu\}$ for $\nu = 1,2,\ldots$ and seek $u = u^\nu$ taking values in $\mathcal{H}^\nu$ such that (2.2.8) holds for all (smooth) $\mathcal{H}^\nu$-valued φ. This is an ordinary differential equation in the finite dimensional space $\mathcal{H}^\nu$, so unique existence of each u^ν is standard. Taking $\varphi = 2u = 2u^\nu$ in (2.2.8), we get an estimate

$$\frac{d}{dt}\left(\|u\|^2\right) + 2\|u_y\|^2 \le a + b\|u\|^2 + c\|u\|\|u_y\| \tag{2.2.9}$$

with a, b, c depending only on L, M of (2.2.7), estimating $u(\cdot, 0)$ by

$$u^2\Big|_{y=0} = -2\int_0^\infty u u_y \, dy \le 2\|u\|\|u_y\|. \tag{2.2.10}$$

Noting that $2c\|u\|\|u_y\| \le \|u_y\|^2 + c^2\|u\|^2$ and subtracting $\|u_y\|^2$ from each side of (2.2.9), we may apply the Gronwall Inequality to get

$$\|u(t,\cdot)\|^2, \int_0^\tau \|u_y\|^2 \, dt \le \quad \text{bound} \ = B \tag{2.2.11}$$

where B is independent of ν, depending only on $L, M, \|u_0\|, \tau$.

At this point we must be somewhat careful, as the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ is not compact for the unbounded interval $[0, \infty)$. We do note that, for any choice of $\ell > 0$, we may restrict our functions spatially to $[0, \ell]$ so (2.2.11) gives uniform bounds for u^ν in $L^2\left([0,\tau] \to H^1(0,\ell)\right)$ whence also, using the differential equation, for u_t^ν in $L^2\left([0,\tau] \to H^{-1}(0,\ell)\right)$. The Aubin Compactness Theorem [3] then gives $\{u^\nu\}$ in a compact subset of $L^2\left([0,\tau] \to H^\sigma(0,\ell)\right)$ for any $\sigma < 1$ and we may thus extract a subsequence which is weakly convergent in $L^2([0,\tau] \to \mathcal{V})$ to some $\bar{u}$ and strongly convergent to $\bar{u}$ in $L^2\left([0,\tau] \to H^\sigma(0,\ell)\right)$ — indeed, we may do this extraction so the same subsequence works for every $\ell > 0$.

Fixing this subsequence, we return to (2.2.8) with $\varphi \in L^2([0,\tau] \to H^{\bar{\nu}})$ for some fixed $\varphi, \bar{\nu}$. Then (2.2.8) holds for each u^ν with $\nu \ge \bar{\nu}$. The spatial support of this φ will be in $[0, \ell]$ for some ν-dependent ℓ so we need only consider (2.2.8) in the sense of $L^2(0,\ell)$ inner products. Note that trace theory gives $u^\nu\Big|_{y=0} \to \bar{u}\Big|_{y=0}$ if we took $\sigma > \frac{1}{2}$ above. Hence, applying Krasnosel'skĭi's Theorem on the continuity of Nemytsky operators to the nonlinearities f and $\hat{g}$, we see that $\bar{u}$ also satisfies (2.2.8) for this φ. As the set of such functions φ is dense in, say, $L^2([0,\tau] \to \mathcal{V})$, we can see that $\bar{u}$ is, indeed, a weak solution.

Uniqueness follows immediately from the estimates in the next subsection (specialized to R, u_0 fixed) so we avoid repetition by omitting this here.

For the case: $u_0 \in \mathcal{H}_*$, we already know unique existence as above, since $\mathcal{H}_* \subset \mathcal{H}$. To see (2.2.6) it is appropriate to use essentially the argument above with $\langle\cdot,\cdot\rangle_*$ now the μ-weighted inner product of $\mathcal{H}_*$ with $\mu = (1+y)^{d-1}$; the only change required is the insertion of an additional term in this weak formulation, coming from the involvement of μ in the integration by parts of

$\int_0^\infty \varphi\, u_{yy}\, \mu\, dy$. We get

$$\begin{aligned}(2.2.12)\quad &\langle \varphi, u_t\rangle_* + \langle \varphi_y, u_y\rangle_* \\ &\quad = \langle \psi\varphi, u_y\rangle_* + \langle \varphi, f(\cdot, u)\rangle_* - \langle \varphi(\mu_y/\mu), u_y\rangle - \varphi G(\cdot, u)\Big|_{y=0}\end{aligned}$$

and, as μ_y/μ is bounded, the rest of the argument is unchanged. ■

3. Well-posedness for (P)

In this section we use the mapping $\mathcal{M}$ just constructed to show well-posedness for Problem **(P)** on the time interval $[0, \tau]$ as in (2.2.1).

THEOREM 3.1: *Assume the hypotheses (1.14) for the constitutive functions F, H and (1.15) for the data R_0, Γ, C_0. Then there exists a unique solution pair (R, C) of Problem* **(P)**= $\{$ (1.6), (1.12), (1.7), (1.8), (1.13) $\}$ *on $[0, \tau]$ with $0 \le C \le 1$ (and $R_0/2 \le R \le 2R_0$) and the map: $(R_0, C_0) \mapsto (R, C)$ is continuous for suitable topologies.*

The proof proceeds by showing contractivity of the map $\mathcal{M}$ with respect to the norm $\|\cdot\|_\lambda$ for some λ. For this we will need estimates for differences of solutions of (2.1.2) and (2.2.3); similar estimates will also be needed for the continuous dependence on initial data. These will be obtained in subsection 3.1; in subsection 3.2 we complete the proof of Theorem 3.1 and the maximal interval of existence $(0, \tau^*)$ is considered in subsection 3.3.

3.1. Estimates for differences

Our immediate objective will be the contractivity of the map $\mathcal{M}$ with respect to the norm $\|\cdot\|_\lambda$ for some choice of λ in (2.2). For this the only relevant datum is ω but, to make more efficient use of our estimation for differences of solutions, we also consider now the variation of the initial data (R_0, u_0) relevant to well-posedness. We keep the functions H, F, Γ fixed throughout and assume that R_0 is limited to some specified interval

$$(3.1.1)\qquad 0 < R_* \le R_0 \le R^*$$

so the Lipschitz constants in (2.3), (2.5) hold uniformly; we then fix τ accordingly so this works uniformly. Similarly, also imposing a bound on $\|u_0\|$

makes the estimate (2.2.11) uniform.

Notation: We will consider two data sets: $(\omega_j,\ R_{0,j}\, u_{0,j})$ for $j = 1, 2$ so correspondingly we have solutions $(R_j,\ u_j)$ of $(2.1.2)_j$ and $(2.2.3)_j$ with $\eta_j := u_j(\cdot, 0)$; note that if $R_{0,1} = R_{0,2}$ and $u_{0,1} = u_{0,2}$, then we would just have $\mathcal{M}(\omega_j) := \eta_j$. It will be convenient to set

$$F_j := F(\cdot, u_j), \quad \psi_j := \dot{R}_j + \frac{d-1}{y + R_j}, \quad \hat{g}_j := G(R_j, \eta_j + \Gamma).$$

For the various functions, we here use the same letter without the superscript index to denote the difference so

$$R := R_1 - R_2, \quad u := u_1 - u_2, \quad \hat{g} := \hat{g}_1 - \hat{g}_2, \quad \text{etc.};$$

in particular, $\eta := \eta_1 - \eta_2 = u(\cdot, 0)$.

As noted, the constants L, M, B of (2.3), (2.5), (2.2.11) are fixed. The letter K will here be used generically to denote a constant (different at different occurrences) which may depend[5] on these, on (3.1.1) and τ, etc., and on prior occurrences of K, but not otherwise on the data or solution and not on any choice of the parameter λ. Finally, we introduce the abbreviation $[\![\,\cdot\,]\!]_\lambda$ for the integral $\int_0^t \cdot\, e^{-\lambda s}\, ds$ with the parameter λ as in (2.2).

First consider $(2.1.2)_j$ for $j = 1, 2$, multiply the difference by $2Re^{-\lambda t}$, and integrate over [0,t]. Noting that $\left(|R|^2 e^{-\lambda t}\right)^{\cdot} = e^{-\lambda t}\left(2R\dot{R} - \lambda |R|^2\right)$, one has

$$|R|^2 e^{-\lambda t} + \lambda [\![|R|^2]\!]_\lambda = |R_0|^2 + [\![2R\dot{R}]\!]_\lambda.$$

Next consider the weak form $(2.2.8)_j$, taking $\varphi = 2ue^{-\lambda t}$. Subtract these for $j = 1, 2$ and then integrate over $[0, t]$ to obtain

$$\begin{aligned} \|u\|^2 e^{-\lambda t} + \lambda [\![\|u\|^2]\!]_\lambda + 2[\![\|u_y\|^2]\!]_\lambda \qquad & \\ = \|u_0\|^2 + 2[\![\langle u, F\rangle]\!]_\lambda - 2[\![\eta \hat{g} \dot{R}_1]\!]_\lambda - 2[\![\eta g_2 \dot{R}]\!]_\lambda & \\ + 2[\![\langle u, \psi_1 u_y + \psi u_{2,y}\rangle]\!]_\lambda. & \end{aligned} \tag{3.1.2}$$

[5] We may temporarily write K_δ to indicate dependence on some $\delta > 0$ not yet chosen (in particular, in writing, e.g., $L\|u\|\|u_y\| \le \delta\|u_y\|^2 + K_\delta\|u\|^2$ where we would actually have $K_\delta = L^2/4\delta$ in this case.

At this point we list our preliminary estimations:

$$
(3.1.3)\quad
\begin{array}{rrl}
i] & |\dot R| & = |H(R^1,\omega^1) - H(R^2,\omega^2)| \le L[|R| + |\omega|] \quad \text{and} \\
ii] & 2\langle u, F\rangle & \le 2L\|u\|^2 \qquad \text{by (2.3)}. \\
iii] & 2\langle u, \psi_1 u_y\rangle & \le 2M\|u\|\|u_y\| \le \delta\|u_y\|^2 + K_\delta\|u\|^2, \\
iv] & |\psi| & \le |\dot R| + [(d-1)/R_*^2]\,|R| \le K[|R| + |\omega|]. \\
v] & 2\langle u, \psi u_{2,y}\rangle & \le K[|R| + |\omega|]\|u\|\|u_{2,y}\| \\
 & & \le K\,[|R|^2 + |\omega|^2] + \beta\|u\|^2; \text{ see below.} \\
vi] & 2\eta\hat g & \le 2L|\eta|[|R| + |\eta|] \qquad \text{by (2.5)} \\
 & & \le K|R|(\|u\|\|u_y\|)^{1/2} + K\|u\|\|u_y\| \\
 & & \le K|R|\|u\| + K|R|\|u_y\| + K\|u\|\|u_y\| \\
 & & \le 2\delta\|u_y\|^2 + K_\delta|R|^2 + K_\delta\|u\|^2.
\end{array}
$$

In $v]$ above we have set $\beta = \beta(t) := \|u_{2,y}(t,\cdot)\|^2$.

With the aid of the estimates (3.1.3) we then obtain:

$$
\begin{aligned}
|R|^2 e^{-\lambda t} + \lambda[|R|^2]_\lambda &\le |R_0|^2 + 2L[|R|\,(|R| + |\omega|)]_\lambda \\
&\le |R_0|^2 + [|\omega|^2]_\lambda + (2L + L^2)[|R|^2]_\lambda
\end{aligned}
$$

whence

$$
(3.1.4)\quad |R|^2 e^{-\lambda t} + \lambda[|R|^2]_\lambda \;\le\; |R_0|^2 + [|\omega|^2]_\lambda \qquad \text{if } \lambda \ge (1+L)^2.
$$

We also obtain

$$
(3.1.5)\quad
\begin{aligned}
&\|u\|^2 e^{-\lambda t} + \lambda[\|u\|^2]_\lambda + 2[\|u_y\|^2]_\lambda \\
&\qquad \le \|u_0\|^2 + 3\delta[\|u_y\|^2]_\lambda + K_\delta[|R|^2 + |\omega|^2]_\lambda \\
&\qquad\qquad + (K_\delta)[\|u\|^2]_\lambda + K[\beta\|u\|^2]_\lambda
\end{aligned}
$$

for arbitrary λ; take $\delta = 1/3$ and this becomes

$$
(3.1.6)\quad
\begin{aligned}
&\|u\|^2 e^{-\lambda t} + \lambda[\|u\|^2]_\lambda + [\|u_y\|^2]_\lambda \\
&\qquad \le K_0 + K\int_0^t (1 + \beta(s))\left(\|u\|^2 e^{-\lambda s}\right) ds \\
&\text{with}\quad K_0 = K_0(t) := \|u_0\|^2 + K[|R|^2 + |\omega|^2]_\lambda\,.
\end{aligned}
$$

Note that (2.2.11) gives $b(t) := K \int_0^t (1 + \beta \leq b(\tau) \leq K\tau + KB = K$.

As $K_0(t)$ in (3.1.6) is nondecreasing, we may consider this on any interval $[0, \sigma]$ with $0 < \sigma \leq \tau$ and replace K_0 by $\bar{K} = K_0(\sigma)$ to get

$$\zeta(t) \leq \bar{K} + \int_0^t b' \zeta \, ds \qquad \text{with } \zeta(t) := \|u\|^2 e^{-\lambda t}$$

so $\zeta(t) \leq \bar{K} e^{b(t)}$ on $[0, \sigma]$ by the Gronwall Inequality. Integrating, we get

$$\bar{K} + \int_0^\sigma b' \zeta \, ds \leq \bar{K} + \bar{K} \left(e^{b(\sigma)} - 1 \right) \leq \bar{K} e^{b(\tau)}$$

so, replacing σ by t here as $\sigma \geq t$ was arbitrary, we see that the right hand side of (3.1.5) is bounded by KK_0 with $K = e^{b(\tau)}$. Using (3.1.4) to estimate $[|R|^2]_\lambda$ in (3.1.6), we have our second fundamental estimate

$$\|u\|^2 e^{-\lambda t} + \lambda [\|u\|^2]_\lambda + [\|u_y\|^2]_\lambda \leq K \left(\|u_0\|^2 + |R_0|^2 + [|\omega|^2]_\lambda \right) \tag{3.1.7}$$

for any $\lambda \geq (1 + L)^2$.

3.2. Well-posedness for (P)

In this subsection we complete the proof of Theorem 3.1. Our principal tool for this will be the estimates (3.1.4), (3.1.7) above, but we first need a Lemma, later used to assure us that the fixpoint of $\mathcal{M}$ provides a solution of Problem **(P)**.

LEMMA 3.2: *Assume the hypotheses (1.14) and (1.15) and, in addition, that the initial data C_0 satisfies*

$$\int_{R_0}^\infty |C_0(r) - \Gamma_0|^2 r^{d-1} \, dr < \infty. \tag{3.2.1}$$

Suppose the map $\mathcal{M}$ has a fixpoint ω to which there corresponds, via (2.1.2), (2.2.3), a solution pair (R, u) and so a solution C of (2.1.3) — inverting our substitution so

$$C(t, r) := u(t, y) + \Gamma(t), \qquad \omega(t) = C(t, R(t)) - \Gamma(t). \tag{3.2.2}$$

Then this function C satisfies $0 \leq C \leq 1$ ae on $\mathcal{Q}$.

PROOF: The assumption (3.2.1) means that $u_0 \in \mathcal{H}_*$ so u satisfies (2.2.6) by Proposition 2.2 and we use (2.2.12), written in terms of r so

$$\langle \varphi, \psi \rangle_* := \int_{R(t)}^{\infty} \varphi(r)\psi(r)\, r^{d-1}\, dr \qquad \left(\|\varphi\|_*^2 := \langle \varphi, \varphi \rangle_* \right).$$

From (2.1.3), this gives

$$\langle \varphi, C_t \rangle_* + \langle \varphi_r, C_r \rangle_* = \langle \varphi, F(C) \rangle_* - \left[\varphi k(C) \dot{R} r^{d-1} \right] \Big|_{r=R(t)} \tag{3.2.3}$$

for suitable φ. Note also the identity

$$\langle \varphi, \varphi_t \rangle_* = \frac{d}{dt} \left[\tfrac{1}{2} \|\varphi\|_*^2 \right] + \tfrac{1}{2} \dot{R} \left[r^{d-1} \varphi^2 \right] \Big|_{r=R(t)}.$$

Now consider: (a) $\varphi = C_- := \min\{C, 0\}$ and (b) $\varphi = (C-1)_+ := \max\{C-1, 0\}$. In either of these cases, by a theorem of Stampacchia [15] and noting (2.3-i), we have *ae*

$$\varphi C_t = \varphi \varphi_t \qquad \varphi_r C_r = |\varphi_r|^2 \qquad \varphi F(C) \leq 0.$$

Thus, in each case we have

$$\begin{aligned} \frac{d}{dt} \left[\tfrac{1}{2} \|\varphi\|_*^2 \right] &+ \|\varphi_r\|_*^2 \\ &= \langle \varphi, F(C) \rangle_* - \dot{R} R^{d-1} \left[\tfrac{1}{2} \varphi^2 + k(C) \varphi \right] \Big|_{r=R} \\ &\leq -\dot{R} R^{d-1} \left[\tfrac{1}{2} \varphi^2 + k(C) \varphi \right] \Big|_{r=R}. \end{aligned} \tag{3.2.4}$$

In case (a), having $\varphi \neq 0$ means $\varphi = C < 0$ so $\dot{R} = H(R, C) \leq 0$ by (2.1.2)=(1.8), (2.3-ii) and we have

$$\tfrac{1}{2}\varphi^2 + k(C)\varphi = \tfrac{1}{2}\varphi^2 + (1-\varphi)\varphi = \varphi - \tfrac{1}{2}\varphi^2 < 0.$$

For the case (b), on the other hand, $\varphi \neq 0$ means $C > 1$ so $k(C) = -\frac{1}{2}\varphi$ here and the right hand side of (3.2.4) vanishes. In either case the right hand side of (3.2.4) is always nonpositive[6] and, as (1.15-iii) gives $\varphi\Big|_{t=0} = 0$, integration

[6] We remark that if we had imposed the further requirement that $H(\cdot, 1) \geq 0$ as part of (1.14-ii), then we could have used the more 'natural' extension $G(r, z) := 0$ for $z \geq 1$ since in case (b) we would have $\dot{R} = H(R, C) \geq 0$ when $\varphi \neq 0$ gives $C > 1$.

in t of the differential inequality (3.2.4) gives $\varphi \equiv 0$. For case (a) this gives $C \geq 0$ and for case (b) it gives $C \leq 1$. ■

With this in hand, we are ready to prove our well-posedness result.

PROOF (of Theorem 3.1): From Propositions 2.1 and 2.2 we know that $\mathcal{M}$ is a well-defined selfmap of $L^2(0,\tau)$ and now wish to show that it is strictly contractive with respect to the norm $\|\cdot\|_\lambda$ — i.e., that

$$\|\mathcal{M}(\omega_1) - \mathcal{M}(\omega_2)\|_\lambda \leq \vartheta \|\omega_1 - \omega_2\|_\lambda \qquad \text{with } \vartheta < 1$$

for arbitrary $\omega_1, \omega_2 \in L^2(0,\tau)$. Since $R_{0,1} = R_{0,2}$ and $u_{0,1} = u_{0,2}$ here, we have $\mathcal{M}(\omega_1) - \mathcal{M}(\omega_2) = \eta = u(\cdot, 0)$ in the notation of the previous subsection. Now suppose λ is taken greater than $(1 + L^2)$ (so (3.1.7) will apply) and also greater than $4K^2$, with K as in (3.1.7), so

$$\begin{aligned} |\eta|^2 &\leq \|u\|\|u_y\| \leq \frac{1}{4K}\|u_y\|^2 + K\|u\|^2 \\ &\leq (\lambda\|u\|^2 + \|u_y\|^2)/\,4K. \end{aligned} \tag{3.2.5}$$

From this and (3.1.7) with $t = \tau$ and noting that, in considering $\mathcal{M}$ the initial data is fixed so $R(0) = 0$ and $u(0, \cdot) = 0$ here, we then have

$$[\![|\eta|^2]\!]_\lambda \leq [\![(\lambda\|u\|^2 + \|u_y\|^2]\!]_\lambda)\big/\,4K \;\leq \tfrac{1}{4}[\![|\omega|^2]\!]_\lambda$$

— which gives the contractivity estimate with $\vartheta = 1/2$ for such λ.

By Banach's Contraction Mapping Principle, there is then a unique fixpoint ω of $\mathcal{M}$ and so a corresponding solution pair (R, u) for (2.1.2), (2.2.3). The construction of the map $\mathcal{M}$ depends on the initial data (R_0, u_0) so, of course, (ω, R, u) will also depend on these. If we consider two sets of initial data: $(R_{0,j}, u_{0,j})$ for $j = 1, 2$, then $\eta_j = \omega_j$ for the fixpoints and, on substituting (3.2.5) into the right hand side of (3.1.7), we get

$$\|u\|^2 e^{-\lambda t} + \lambda[\![\|u\|^2]\!]_\lambda + [\![\|u_y\|^2]\!]_\lambda \leq K\left(\|u_0\|^2 + |R_0|^2\right) \tag{3.2.6}$$

whence, returning to (3.1.4), we also get

$$|R|^2 e^{-\lambda t} + \lambda[\![|R|^2]\!]_\lambda \;\leq K\left(\|u_0\|^2 + |R_0|^2\right). \tag{3.2.7}$$

This shows the continuous dependence on the initial data of the fixpoint ω and of the corresponding solution pairs (R, u).

Note that in obtaining the estimates leading to (3.2.6), (3.2.7) we have implicitly taken $\|\cdot\|$ throughout to indicate, as in (2.2.11), the usual norm of $\mathcal{H} = L^2(0, \infty)$. Following the proof of Proposition 2.2, we note that all the estimates still hold — with some changes of the values of the constants K — if one were instead to have used the weighted norm $\|\cdot\|_*$ of $\mathcal{H}_*$ so we also have continuous dependence in that context. We further note that the continuity estimate (3.2.6) may be translated into continuous dependence for the corresponding functions C as in (3.2.2), although it is difficult to express this as a direct comparison because of the varying domain.

Finally, in view of our introduction of 'extensions' outside the physical range, we must actually show that this fixpoint of $\mathcal{M}$ does, indeed, provide the desired solution of Problem **(P)**. If $u_0(y) := C_0(y + R_0) - \Gamma_0$ happens to correspond to C_0 satisfying (3.2.1), then Lemma 3.2 gives $0 \leq C \leq 1$ (i.e., $0 \leq u + \Gamma \leq 1$) for all $(t, y) \in [0, \tau] \times [0, \infty)$ so the extensions become nugatory — see the beginning of section 2 — and we do have a solution of Problem **(P)**. Noting that, conversely, any solution of **(P)** gives a fixpoint of $\mathcal{M}$, we then have well-posedness of Problem **(P)** in the context of $\mathcal{H}_*$.

Such initial data are L^2-dense: if we are given any C_0 satisfying (1.15-iii), then we may find a sequence (C_0^ν) — satisfying $0 \leq C_0^\nu \leq 1$ and (3.2.1) for each ν — for which, correspondingly, $u_0^\nu \to u_0$ in $\mathcal{H} = L^2(0, \infty)$. By (3.2.6) this gives $u^\nu \to u$ for the solutions in the sense, say, of $L^2([0, \tau] \to H^1(0, \infty))$. By Lemma 3.2 we have $0 \leq u^\nu + \Gamma \leq 1$ for each ν and this inequality is preserved in the limit: $0 \leq [u + \Gamma] = C \leq 1$. Thus, we see that the fixpoint of $\mathcal{M}$ does, indeed, provide the desired solution of **(P)** so that Problem **(P)** is well-posed in this more general context as well. ■

3.3. Maximal time of existence

The results obtained in Theorem 3.1 give well-posedness locally in time — for some interval $[0, \tau]$ with $\tau = \tau_1 = \tau(R_0)$ obtained through (2.2.1). As is standard for ODEs, we note that the problem may be restarted at $t = \tau_1$ to get existence of a solution on $[\tau_1, \tau_2]$, where $\tau_2 = \tau_1 + \tau(R(\tau_1))$, with a simple verification that this provides, by concatenation, a (unique) solution on $[0, \tau_2]$ …and, of course, this construction can be iterated. A Zorn's Lemma argument (ordering solutions by their interval of existence and noting the uniqueness in Theorem 3.1) gives existence of a unique solution (R, C) of the problem with a maximal interval of existence $(0, \tau^*)$. We are now concerned with the possibility that $\tau^* < \infty$ and will defer to the forthcoming paper [10] discussion of the asymptotic behavior of $R(t)$ as $t \to \infty$ when $\tau^* = \infty$.

THEOREM 3.3: *Adjoin*[7] *to our hypotheses (1.14) a linear growth condition:*

$$H(r, \omega) \leq K(1 + r) \qquad \text{for } r > 0 \text{ and } 0 \leq \omega \leq 1. \tag{3.3.1}$$

Then $\tau^* < \infty$ is possible only if $R(t) \to 0$ as $t \nearrow \tau^*$.

If, in addition to (1.14) and (3.3.1), we have

$$H(r, \omega) \geq -K_0 r \qquad \text{for } 0 < r \leq r_0, \quad 0 \leq \omega \leq 1, \tag{3.3.2}$$

then we must have global existence: $\tau^* = \infty$.

Proof Since we know $0 \leq C \leq 1$ for solutions, the condition (3.3.1) gives a differential inequality $\dot{R} \leq K(1 + R)$ which, using the Gronwall Inequality, shows that $R(\cdot)$ remains bounded on bounded intervals: we cannot have blowup in finite time. If we were not to have $R(t) \to 0$ as $t \nearrow \tau^*$, there would be a sequence $t_n \nearrow \tau^*$ for which $R(t_n) \to R_* > 0$. We now observe that one could restart the problem at any t_n to get continued existence of the solution on $(t_n,\, t_n + \tau(R(t_n)))$, with $\tau(R)$ again given by (2.2.1), so one would have existence of a solution on $[0, \tau_n]$ with $\tau_n = t_n + \tau(R(t_n))$. Since $\tau(R(t_n)) \to \tau(R_*) > 0$, this gives $\tau_n \to \tau^* + \tau(R_*) > \tau^*$, which would contradict the maximality of τ^*.

It then follows from (3.3.2) that $\left(e^{K_0 t} R\right)^{\cdot} \geq 0$, which bounds $R(\cdot)$ away from 0 on finite time intervals so we cannot have $\tau^* < \infty$. ■

We observe that (3.3.1) (respectively, (3.3.2)) are each sufficient, but not necessary, to ensure the impossibility of having $R \to \infty$, (respectively, $R \to 0$) in finite time. It can also be shown, although we omit the analysis, that in the case $\tau^* < \infty$ (necessarily, then, with $R(\tau^*-) = 0$ if we have assumed (3.3.1)) we would have suitable convergence of $C(t, \cdot)$ for $r > 0$ as $t \nearrow \tau^*$ to

[7] This is automatic in the case (1.9) with $h(\cdot)$ as in (1.4) or (1.5) for example.

get admissible initial data for the reaction/diffusion equation (1.1) — now on $\mathbb{R}^d$ with the condition (1.12) at infinity but with no 'inner' boundary condition — to restart that reduced problem for the liquid phase alone at $t = \tau^*$ with global existence of a solution continuation.

4. Additional regularity of solutions

In this section we use a 'bootstrap' argument to prove some additional regularity for the solution (R, C) of **(P)**. For simplicity, we suppose that the kinetics of dissolution and growth are given by (1.9).

Before proceeding, we recall some technical properties (taken, e.g., from [11], [12]). For any convenient (fixed) choices of $\delta \geq 0$, $l > 0$, $T > 0$, $s \geq 0$ we write, for brevity, $\mathcal{Q}_\delta := (\delta, T) \times (0, \ell - \delta)$ and $\mathcal{W}_\delta^s$ for the Sobolev space $H^{2s,s}(\mathcal{Q}_\delta)$, as defined in [11].

LEMMA 4.1: *Suppose δ, ℓ, T, s are as above. Then*

1. *One has $H^{s'}(\delta, T) \subset H^s(\delta, T)$, $\mathcal{W}_\delta^{s'} \subset \mathcal{W}_\delta^s$ when $0 \leq s' \leq s$;*
   For any $\vartheta \in C^\infty(\mathbb{R})$ (i.e., with uniform bounds on each derivative), if $\omega \in H^s(0, T)$ or $w \in \mathcal{W}_\delta^s$, then $\vartheta \circ \omega \in H^s(0, T)$, $\vartheta \circ w \in \mathcal{W}_\delta^s$, respectively.
   If $w \in \mathcal{W}_\delta^s$ and satisfies $w \geq \eta$ on $\mathcal{Q}_\delta$ for $\eta > 0$ then $1/w \in \mathcal{W}_\delta^s$

2. *$H^s(\delta, T) \subset C^k([\delta, T])$ when $s > k + \frac{1}{2}$; in particular, $H^s(\delta, T) \subset L^\infty(\delta, T)$ when $s > \frac{1}{2}$;*
   If $s \geq 1$ and $s' < \lfloor s \rfloor + 1/2$, then $\mathcal{W}_\delta^s$ embeds in the Hölder space $C^{2s',s'}(\mathcal{Q}_\delta)$;
   $H^s(0, T) \cap L^\infty(0, T)$ is an algebra (i.e., a product $\omega_1\omega_2$ is there when both ω_j are);
   $\mathcal{W}_\delta^s \cap L^\infty(\mathcal{Q}_\delta)$ is an algebra.

3. *If $w \in \mathcal{W}_\delta^s$, then $w_y \in \mathcal{W}_\delta^{s-1/2}$ when $s > 1/2$;*
   If $w \in \mathcal{W}_\delta^s$, then $w(\cdot, 0) \in H^{s-1/4}(\delta, T)$ when $s > 1/4$ provided $s - 1/4$ is not an integer.

4. *If $[w_t - w_{yy}] =: f \in \mathcal{W}_\delta^s$ and $w_y(\cdot, 0) =: g \in H^{s+1/4}(\delta, T)$, then $w \in \mathcal{W}_{\delta'}^{s+1}$ for any $\delta' > \delta$ provided neither s nor $2s$ is of the form* [integer $+\frac{1}{2}$].

THEOREM 4.2: *Let (R, C) be the solution of* **(P)** *on $(0,T)$; assume that $T < \tau^*$ and that $R \geq \eta > 0$ on $[0,T]$. Suppose $[C_0(\cdot) - \Gamma_0] \in H^1(R_0, \infty)$ and the constitutive functions F in (1.6) and h in (1.9) are infinitely differentiable. Then R and C are infinitely differentiable*[8] *on $(0, \tau^*]$ and on $\mathcal{Q} := \{(t,r) : 0 < t \leq \tau^*,\ r \geq R(t)\}$, respectively.*

PROOF: The argument will be a bootstrapping induction on the degree of regularity. It is again convenient to convert **(P)** to its fixed-domain version, as in subsection 2.2, by introducing $y = r - R(t)$ and $u(t,y) = C(t,r) - \Gamma(t)$. As there, u satisfies (2.2.3) in $\hat{\mathcal{Q}} = [0,T] \times [0, \infty]$; we now restrict u to some bounded domain (as $\mathcal{Q}_0$ in Lemma 3.5) so

$$
(4.1) \qquad
\begin{aligned}
&i)\quad u_t - u_{yy} = f \quad \text{in } \mathcal{Q}_0 := (0,T) \times (0, \ell) \\
&\qquad\quad \text{with} \qquad f := \psi u_y + [F(u + \Gamma) - F(\Gamma)], \\
&ii)\quad u_y = (1 - \Gamma - u)\dot{R} \quad \text{at } y = 0
\end{aligned}
$$

In (4.1-i) we have $\psi := \dot{R} + (d-1)/(y+R)$ as in (2.2.2); in (4.1-ii) we have simplified from (2.2.3) since we now know that $0 \leq u + \Gamma = C \leq 1$. Since we will only use (4.1) for point 4 in Lemma 4.1, we have found it unnecessary to note any initial conditions or boundary conditions at $y = \ell$.

1) Inductive step: (for $\nu > 3/4$ with 4ν not an integer) we show

$$
(4.2) \qquad u \in \mathcal{W}^{\nu}_{\delta} \implies u \in \mathcal{W}^{\nu + 1/4}_{\delta'}
$$

for any $\delta' > \delta$.

Recall from (2.1.2) with (1.9) that the function R is now given by

$$
(4.3) \qquad \dot{R}(t) = \Lambda \left[u(t,0) + \Gamma(t) - h(1/R(t)) \right] \qquad \text{for } t \in (0,T]
$$

so Lemma 4.1 gives $u(\cdot, 0) \in H^{\nu - 1/4}(\delta, T)$ for $u \in \mathcal{W}^{\nu}_{\delta}$. Note also that, from (1.11) with F smooth, we know that $\Gamma(\cdot)$ is infinitely differentiable (with $0 \leq \Gamma \leq 1$) and, with our assumptions, we know that $0 < \eta \leq R(t) \leq \text{bound} =: \bar{\eta}$ so (cf., Lemma 4.1) $h(1/R)$ is as smooth as $R(\cdot)$. Standard ODE theory then shows from (4.3) that

$$
(4.4) \quad u(\cdot, 0) \in H^{\nu - 1/4}(\delta, T) \Longrightarrow R(\cdot) \in H^{\nu + 3/4}(\delta, T),\ \dot{R} \in H^{\nu - 1/4}(\delta, T).
$$

[8] We have imposed hypotheses giving infinitely differentiability — but note that the inductive argument can easily be interrupted to give lesser regularity under weaker hypotheses on $F(\cdot)$ and $h(\cdot)$.

We wish to apply Lemma 4.1 through (4.1). Set $g := (1 - \Gamma - u)\dot{R}$; since we always have $u(\cdot, 0) \in L^\infty(\delta, T)$ (as $\Gamma \in C^\infty$ and $0 \le u + \Gamma \le 1$), we note similarly that $\dot{R} \in L^\infty(\delta, T)$ whence

$$g \in H^{\nu - 1/4}(\delta, T). \tag{4.5}$$

We further note, from the definitions of $\mathcal{W}_\delta^\nu$ and of ψ, that

$$\psi, \psi_y \in \mathcal{W}_\delta^{\nu - 1/4} \cap L^\infty(\mathcal{Q}_\delta) \tag{4.6}$$

since $R \ge \eta$ keeps $(y + R) \ge \eta$ so we may apply Lemma 4.1 as earlier.

Suitably extending F in $C^\infty(\mathbb{R})$ from $[0, 1]$, we have $[F(u+\Gamma) - F(\Gamma)]$ also in $\mathcal{W}_\delta^\nu$ by Lemma 4.1. We cannot apply Lemma 4.1 directly to the product ψu_y since we do not know that u_y is bounded. However, we note that we do know ψ, ψ_y, u are each bounded and that $\psi u_y = (\psi u)_y - \psi_y u$ so using (4.5) gives (by Lemma 4.1) that $\psi_y u \in \mathcal{W}_\delta^{\nu-1/4}$ and that $\psi u \in \mathcal{W}_\delta^{\nu-1/4}$ whence, again by Lemma 4.1, one has $(\psi u)_y \in \mathcal{W}_\delta^{\nu-3/4}$. Combining gives $f \in \mathcal{W}_\delta^{\nu-3/4}$ in (4.1-i) and, using (4.5) for g in (4.1-ii), we may now apply Lemma 4.1 with $s := \nu - 3/4$ to get $u \in \mathcal{W}_{\delta'}^{s+1} = \mathcal{W}_{\delta'}^{\nu+1/4}$ for any $\delta' > \delta$ — provided s does not hit one of the inadmissible values. We have thus shown the inductive step.

2) Starting step: Proposition 2.2 and Theorem 3.1 give (2.2.5), i.e.,

$$u \in \boldsymbol{U} := L^\infty([0, \tau] \to \mathcal{H}) \cap L^2([0, \tau] \to \mathcal{V}),$$

where $\mathcal{H} = L^2(0, \infty)$ and $\mathcal{V} = \{v \in \mathcal{H} : v_y \in \mathcal{H}\}$. Moreover, if $u_0 \in H^1(0, \infty)$, we prove below in Lemma 4.2 that $u \in L^2([0, \tau] \to H^2(0, \infty))$, $u_t \in L^2(\hat{\mathcal{Q}})$ so u is in $\mathcal{W}_0^1$ for all $\ell > 0$. To avoid the inadmissible values of ν, we may use Lemma 4.1 to get $u \in \mathcal{W}_\delta^\nu$ with $\nu = \nu_0$ slightly less than 1 (say, $3/4 < \nu_0 < 1$) and $\delta = \delta_0 = 0$.

3) Finally, choosing any strictly increasing sequence $\{\delta_j\}$ with $\lim_j \delta_j < \varepsilon$ and setting $\nu_j = \nu_0 + j/4$, we have, inductively, that $u \in \mathcal{W}_{\delta_j}^{\nu_j} \subset \mathcal{W}_\varepsilon^{\nu_j}$ for all $j = 1, \ldots, \infty$. Using Lemma 4.1, it follows that u must be infinitely differentiable on $(0, \tau^*] \times [0, \infty)$, since $T < \tau^*$ and l were arbitrary, and that R is infinitely differentiable on $(0, \tau^*]$. Transforming back to working with (t, r), this immediately shows that C is infinitely differentiable on $\mathcal{Q}$. ■

LEMMA 4.2: *Let (R, C) be the solution of* **(P)** *on $(0, \tau)$; we assume that $\tau < \tau^*$ and that we know $R \geq \eta > 0$ on $[0, T]$. Suppose $[C_0(\cdot) - \Gamma_0] \in H^1(R_0, \infty)$. Then the unique solution u of (2.2.3) is in $L^2([0, \tau] \to H^2(0, \infty))$ with u_t in $L^2(\hat{\mathcal{Q}})$.*

PROOF: The function u satisfies (2.2.3) so, as in (4.1), we have

$$(4.7) \qquad \begin{aligned} i) &\quad u_t = u_{yy} + f \text{ in } \hat{\mathcal{Q}} \\ ii) &\quad u_y = g(t) \ \text{ at } y = 0 \\ iii) &\quad u(0, y) = u_0(y) \text{ for } y > 0, \end{aligned}$$

with $g(t) := G(R(t), u(t, 0) + \Gamma(t)) = (1 - \Gamma(t) - u(t, 0))\dot{R}(t)$.

Since $u \in L^2([0, \tau] \to \mathcal{V})$ and $R \in C^{0,1}([0, \tau])$, we have f and g in $L^2(\hat{\mathcal{Q}})$. First we multiply (4.7-i) by u_t and integrate on $\mathcal{Q}_t = (0, t) \times (0, \infty)$ for $t < \tau$ to get

$$(4.8) \int_0^t \|u_t\|^2 \, ds + \|u_y(t, \cdot)\|^2 \leq \int_0^t \|f\|^2 \, ds + \|u_{0,y}\|^2 - 2 \int_0^t g(s) u_t(s, 0) \, ds.$$

Multiplying (4.7-i) by $-u_{yy}$ and integrating on $\mathcal{Q}_t$, one obtains

$$(4.9) \int_0^t \|u_{yy}\|^2 \, ds + \|u_y(t, \cdot)\|^2 \leq \int_0^t \|f\|^2 \, ds + \|u_{0,y}\|^2 - 2 \int_0^t g(s) u_t(s, 0) \, ds.$$

Setting

$$\begin{aligned} \Theta(s, z) &:= \textstyle\int_0^z g(R(s), \omega + \Gamma(s)) \, d\omega \qquad (s \in [0, \tau], z \in [0, 1]) \\ I(t) &:= \textstyle\int_0^t g(s) u_t(s, 0) \, ds, \end{aligned}$$

we then have

$$(4.10) \qquad \begin{aligned} I(t) = \Theta(t, u(t, 0)) &- \Theta(0, u_0(0)) \\ &- \int_0^t \int_0^{u(s,0)} \Big[G_r(R(s), \omega + \Gamma(s)) \dot{R}(s) \\ &\qquad + G_z(R(s), \omega + \Gamma(s)) \dot{\Gamma}(s) \Big] \, d\omega \, ds. \end{aligned}$$

From (2.4) and (2.5), we know that there is a positive constant M such that

$$|\Theta(s, z)| \leq |z|, \qquad |G_r(r, z)| \leq M, \qquad |G_z(r, z)| \leq M$$

for $s \in [0,\tau], r \in [\eta, \bar{\eta}], z \in [0,1]$ and we deduce from (4.10) that

$$|I(t)| \leq M \left(|u(t,0)| + |u_0(0)| + \int_0^t |u(s,0)|\, ds \right)$$

for a convenient constant M. Using (2.2.10) then gives

$$\begin{aligned} |I(t)| \quad &\leq \sqrt{t}\, M \left(\int_0^t (\|u\|^2 + \|u_y\|^2)\, ds \right) + \frac{M}{2} \left(\|u_0\|^2 + \|u_{0,y}\|^2 \right) \\ &\quad + \frac{1}{2} \|u_y(t,\cdot)\|^2 + D\|u(t,\cdot)\|^2 + E \end{aligned} \tag{4.11}$$

where D and E are convenient positive constants. Combining (4.8), (4.9), (4.11), we have

$$\begin{aligned} &\tfrac{1}{2} \int_0^t \left(\|u_t\|^2 + \|u_{yy}\|^2 \right) ds \\ &\qquad \leq \int_0^t \|f\|^2\, ds + 2\sqrt{t}\, M \left(\int_0^t (\|u\|^2 + \|u_y\|^2)\, ds \right) \\ &\qquad\quad + (M+2) \left(\|u_0\|^2 + \|u_{0,y}\|^2 \right) + 2D\|u(t,\cdot)\|^2 + 2E \end{aligned} \tag{4.12}$$

whence u is in $L^2([0,\tau] \to H^2(0,\infty))$ with u_t in $L^2(\hat{\mathcal{Q}})$. ■

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