Existence of optimal controls for a free boundary problem

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For Larry Markus on the occasion of his 70^{th} *birthday.*

ABSTRACT: We consider a model for the radially symmetric evolution (by accretion or dissolution) of the interface between a 'pure' solid grain and a liquid solution of the same substance with diffusion both of material and heat. After analyzing the direct problem, we show existence of optimal controls.

1. Introduction

We take up a 'promise' from [4] to consider the controlled evolution (growth/dissolution) of a grain in a diffusive medium with temperature coupling, using a heat source as control.

Some of the technical difficulties addressed in [4], [3], [5] were due to the necessary consideration of possibly rough boundary data for the concentration as control which now become unnecessary; compare [1]. On the other hand, the setting now becomes a system of partial differential equations whose coupling involves new technical difficulties. We also address an implicit consideration of controlability to verify that there may be *any* admissible control for the optimization since the existence theory for the direct problem need not ensure that the grain 'survives' throughout the relevant time interval. We defer any characterization of optimal controls in terms of first order necessary conditions for optimality but anticipate that they would be rather comparable to those obtained in [3].

We continue to consider the problem only in the simplified context of radial symmetry. We are considering the problem in a fixed region $\Omega = \{|x| < L\}$ where we have scaled so $L = 1$; the nature of the 'outer' boundary condition at $r = L$ is somewhat arbitrary and the problem could alternatively have been considered in an unbounded region $L = \infty$ as in [2].

The physical setting for the problem begins with consideration of diffusion of some substance in a liquid whose concentration C satisfies

$$\dot{C} = \Delta C = r^{1-d}\left(r^{d-1}C_r\right)_r - F(C,\Theta) \qquad \text{on } \boldsymbol{Q} \tag{1.1}$$

where, with $\mathcal{I} := (0,T)$,

$$\boldsymbol{Q} := \{(t,x) \in \mathcal{I}\times\Omega : R(t) < r := |x|\} \quad \Omega := \{x \in \mathbb{R}^d : |x| < 1\}. \tag{1.2}$$

Here $R(t) > 0$ is the radius of a solid grain[1] of the substance and at this solid/liquid interface conservation gives

$$C_r = \dot{R}[1 - C] \qquad \text{at } r = R(t)+ \tag{1.3}$$

where we have normalized the 'pure' concentration in the grain to 1, just as in (1.1) we chose units to make the diffusion coefficient 1. Note that the concentration C must physically take values in $[0,1]$. Note also that (1.3) only applies when $R(t) > 0$, as otherwise the grain has (permanently) 'disappeared' and we would shift to using (1.1) on all of Ω.

This problem is a *free boundary problem* because the time-varying grain radius $R(\cdot)$, defining the boundary of $\boldsymbol{Q}$, is to be determined as part of the problem. We have an ordinary differential equation

$$\dot{R} = h(t) \qquad \text{on } \mathcal{I} \tag{1.4}$$

— with the constraint that

$$R(t_*) = 0 \qquad \Rightarrow \qquad R(t) \equiv 0 \text{ for } t_* \le t \le T \tag{1.5}$$

since, once the grain is completely dissolved, we have no provision for its resurrection — and otherwise (when $R > 0$) we assume for growth/dissolution a constitutive law of the form

$$h := K(\vartheta)[\omega - G(\kappa,\vartheta)] =: H(\kappa,\omega,\vartheta) \tag{1.6}$$

where $\kappa := 1/R$ is the *curvature* of the interface, $\omega(t) := C(t,r)\Big|_{r=R(t)+}$ is the local concentration (in the liquid next to the grain boundary), and $\vartheta(t) := \Theta(t,R(t))$ is the temperature there. Here K is a 'reaction rate'

[1] Physically, $d = 2$ or 3 with $d = 2$ corresponding, e.g., to a thin surface film. The problem as described will actually be a physically somewhat implausible 'mixture' since we take $d = 2$ for descriptive convenience and adopt a heat source model somewhat appropriate to a planar problem whereas we continue to speak of a 'grain', which is more appropriate to $d = 3$.

coefficient and G is the 'critical concentration', determining whether one has accretion (growth of the grain if $C > G$ at the interface) or dissolution of grain material into the liquid (if $C < G$). Of course, $0 < K$ and $0 < G < 1$. The curvature dependence in G comes from a surface component in the Gibbs Free Energy (surface tension) so, plausibly, G increases with (positive) curvature — i.e., decreases with increasing $R(t)$. We model the surface reaction of adsorption as endothermic, making this 'easier' for higher temperatures, so plausibly G decreases (and K increases) with increasing ϑ. In the same spirit, we consider an endothermic reaction in the liquid so $F(C, \Theta)$ would plausibly be an increasing function of each of its arguments

For the temperature we have

$$(1.7)\quad \dot{\Theta} = \nabla \cdot D\nabla\Theta - \alpha_0 F(C, \Theta) + S \qquad \text{for } 0 \le r < 1 \text{ with } r \ne R(t)$$

where we permit different diffusion coefficients within the grain ($D = D_1$) and in the liquid ($D = D_0$). The term $\alpha_0 F$ represents the heat absorbed by the reaction included in (1.1). It is precisely the exogenous source term S in (1.7) which will involve the (scalar) control function $u(\cdot)$: we model[2] this as

$$(1.8)\quad S = u(t)\beta \qquad \text{with } \beta = \beta(t, r) := \begin{cases} \beta_1(r) & \text{for } 0 \le r < R(t) \\ \beta_0(r) & \text{for } R(t) < r < 1 \end{cases}.$$

Since (1.7) holds separately in the two phases, we must balance the heat transport mechanisms across $r = R$; we simplify this by assuming the heat content is the same in both phases so material transport is temperature–neutral. If we assume that the heat absorbed by the surface reaction is simply proportional to the reaction, we adjoin to (1.7) the *jump condition*

$$(1.9)\qquad D\Theta_r\big|_{r=R(t)-}^{r=R(t)+} =: [D\nabla\Theta] = \alpha\dot{R}$$

with the constant α positive, corresponding to the physical assumption that adsorption is endothermic. Formally, we may write (1.7)+(1.9) as

$$(1.10)\qquad \Theta_t = r^{-1}(rD\Theta_r)_r - \alpha_0 F + u\beta - \alpha h \delta_R$$

where $h = \dot{R}$ as in (1.4) and $\delta_R = \delta(r - R(t))$ — a δ-function with respect to the variable r (using the measure $r\,dr$ to correspond to integration over Ω).

The 'direct problem' consists of the coupled system of differential equations (1.4), (1.1)+(1.3), (1.10), together with boundary conditions at $r = 1$ and initial conditions. The 'control problem' then comes from the attempt to specify $u(\cdot)$ optimally in (1.8) so as to minimize a *cost functional* $\mathcal{J}$. Our objectives in this paper are to verify solvability for the direct problem for 'general' $u(\cdot)$ and then to demonstrate existence of an optimal control.

[2]This might, for example, correspond to the effect of a heatlamp with adjustable intensity $u(t)$ whose radiation has a radial spatial pattern $\beta_*(r)$ with possibly different absorption coefficients $\tilde{\beta}_j$ for the two phases so $\beta = \beta_*(r)\tilde{\beta}_j$ with $j = 0, 1$ for the liquid *vs.* solid phases. Alternatively, one could consider $\tilde{\beta} = \tilde{\beta}(C)$, so $\tilde{\beta}_1 = \tilde{\beta}(1)$ and $\tilde{\beta}_0 \approx \tilde{\beta}(C)$ for $C << 1$, just as one might take $D = D(C)$ in (1.7).

2. Formulation

We begin by stating the relevant hypotheses for the constitutive functions F in (1.1), (1.7) and $H = K[C-G]$ in (1.6):

(H-1) $K = K(\vartheta)$ is positive, increasing, and continuous on $\mathbb{R}$ with at most linear growth as $\vartheta \to \infty$

(H-2) $G = G(\kappa, \vartheta)$ is continuous with values in $(0,1)$

(H-3) $G(1, \gamma_1) > \gamma_0$

(H-4) $\vartheta G(\kappa, \vartheta)$ is bounded as $\vartheta \to \infty$, uniformly on bounded κ-intervals

(H-5) we extend H outside the physical range[3] by setting

$$H(\kappa, C, \vartheta) := \begin{cases} H(\kappa, 0, \vartheta) & \text{if } C < 0, \qquad H(\kappa, 1, \vartheta) \quad \text{if } C > 1 \\ H(1, C, \vartheta) & \text{if } \kappa < 1. \end{cases}$$

(H-6) $F = F(C, \Theta)$ is uniformly Lipschitzian and is increasing in each argument, extended so

$$F(C,\Theta) = F(0,\Theta) = 0 \text{ if } C < 0, \qquad F(C,\Theta) = F(1,\Theta) \text{ if } C > 1.$$

As in [1], etc., we extend (1.3) outside the physical range as

$$C_r = \begin{cases} \dot{R}[1-C] & \text{if } C \le 1 \\ \dot{R}[1-C]/2 & \text{if } C \ge 1 \end{cases} \qquad \text{at } r = R(t)+ \tag{2.1}$$

and then make a change of variables to the fixed domain $(t, y) \in \mathcal{Q} := \mathcal{I} \times (0,1)$ with

$$y = \rho(1-r), \qquad r := |x|, \qquad \rho := \frac{1}{1-R(t)}. \tag{2.2}$$

In this setting — abusing notation slightly more by letting C denote not only $C(t,x)$ and $C(t,r)$, as earlier, but also $C(t,y)$ — (1.1), with boundary conditions and initial conditions, then becomes

$$C_t = \rho^2 C_{yy} - \psi C_y - F(C,\Theta) \qquad \text{on } \mathcal{Q}$$

$$\left.\begin{aligned} C(\cdot,0) &= \gamma_0 \\ -\rho C_y(\cdot,1) &= \begin{cases} h(\cdot)[1-C] & \text{if } C \le 1 \\ h(\cdot)[1-C]/2 & \text{if } C \ge 1 \end{cases} \end{aligned}\right\} \quad \text{on } \mathcal{I}; \quad C = C(\cdot,1) \tag{2.3}$$

with

$$\psi(t,y) := \rho(t)\left[\frac{(d-1)\rho(t)}{\rho(t)-y} + y h(t)\right]. \tag{2.4}$$

[3] I.e., $0 \le C < 1$ and $0 < R < 1$ so $\kappa \ge 1$ — noting that $R = 0$ would mean there is no grain at all.

The function $h(\cdot)$ is, for the moment, a specified function of t as in (1.4) — only later to be determined as in (1.6). The weak formulation of (2.3) is standard, taking test functions in

$$
\begin{aligned}
\boldsymbol{\mathcal{U}} &:= L^2(\mathcal{I} \to \mathcal{U}) \cap H^1(\mathcal{I} \to \mathcal{U}^*) \text{ with} \\
\mathcal{U} &:= \{w \in H^1(0,1) : w(0) = 0\};
\end{aligned}
\tag{2.5}
$$

requiring that $[C(t) - \gamma_0] \in \boldsymbol{\mathcal{U}}$ imposes the Dirichlet boundary condition at $y = 0$.

The equation (1.7) for Θ is already given on a fixed domain $\hat{\mathcal{Q}} := \mathcal{I} \times \Omega$ and, noting that this is purely radial, we again abuse notation slightly to obtain the weak formulation of (1.10):

$$
\int_0^1 [\eta\dot{\Theta} + \eta_r \Theta_r]\, r\, dr = \int_0^1 \eta[\alpha_0 F + S]\, r\, dr - \alpha h(t)\eta(t, R(t))
\tag{2.6}
$$

for test functions $\eta = \eta(t,r) \in \boldsymbol{\mathcal{V}} := L^2(\mathcal{I} \to \mathcal{V}) \cap H^1(\mathcal{I} \to \mathcal{V}^*)$ where $\mathcal{V}$ is the space of functions on $[0,1)$ corresponding to consideration of the purely radial functions in $H_0^1(\Omega)$. The Dirichlet boundary condition

$$
\Theta = \gamma_1 \qquad \text{at } r = 1-
\tag{2.7}
$$

is imposed by requiring that $[\Theta(t,\cdot) - \gamma_1] \in \mathcal{V}$.

Finally, we will impose the initial conditions

$$
\left.\begin{aligned}
C(0,\cdot) &= C_0(\cdot) \\
\Theta(0,\cdot) &= \Theta_0(\cdot)
\end{aligned}\right\} \text{on } \Omega \qquad\qquad R(0) = R_0,
\tag{2.8}
$$

subject to the requirements that C_0 and Θ_0 are in $L^2(\Omega)$ and that $0 < C_0 < 1$, $0 < R_0 < 1$.

3. Existence

The existence and compactness arguments for the direct problem — i.e., given a fixed 'control function' u in (1.8) for which we assume

$$
\int_0^t u(s)\, ds = U(t) < \infty \qquad \text{with } U \text{ increasing } (u \geq 0)
\tag{3.1}
$$

— are similar enough to the arguments of [1], etc., that it suffices merely to sketch them, with particular attention to the modification due to the coupling with (1.10). As there, we initially obtain a semi-local existence result: on an interval $\mathcal{I}_\tau := [0, \tau]$. We note one difference: that *a priori* we no longer can expect bounded $h(\cdot)$ as in [1], etc., but instead will only have $h(\cdot) \in L^2(\mathcal{I}_\tau)$. We will be using a fixpoint argument for existence and a further difference is related to that between the approaches of [1] and of [4]: while we could have used a Contraction Mapping argument and gotten uniqueness at the same time as in [1], we actually use a Schauder Theorem argument since the estimates are slightly simpler and, in any case, we do not need uniqueness but do need the compactness obtained in order to consider optimal controls later, as in [4]. Thus, this section is the technical heart of the paper.

We wish to construct a map $\boldsymbol{\mathcal{F}} : [C, h] \mapsto [\hat{C}, \hat{h}]$ whose fixpoint will provide the desired solution of the system, at least on some interval $\mathcal{I}_\tau$. We start with C on $\mathcal{Q}$ such that $0 \le C \le 1$ and $h \in L^2$ with an *a priori* bound M_* to be specified later

$$h \in \mathcal{X}_* := \{h \in L^2(\mathcal{I}) : \|h\| \le M_*\} \tag{3.2}$$

and proceed as follows:

Step 1 Solve (1.4) for $R(\cdot)$, stopping[4] at t_* if there is a t_* giving $R(t_*) = 0, 1$; else $t_* = T$.

Step 2 Use R, h to define ρ, ψ by (2.2), (2.4) for $0 \le t < t_*$ and solve (2.6), (2.7), (2.8) to obtain Θ.

Step 3 Use (2.3) to obtain $\tilde{C}$. Let $\hat{C}$ be the truncation of $\tilde{C}$ to take values in $[0, 1]$.

Step 4 Obtain the traces at the interface:

$$\omega := \hat{C}(\cdot, R(\cdot)), \qquad \vartheta := \Theta(\cdot, R(\cdot)) \tag{3.3}$$

and define $\hat{h} \in \mathcal{X}_*$ by setting

$$\hat{h} := \begin{cases} \tilde{h} & \text{if } \|\tilde{h}\| \le M_* \\ M_*\tilde{h}/\|\tilde{h}\| & \text{else} \end{cases} \quad \text{with } \tilde{h} := H(R, \omega, \vartheta). \tag{3.4}$$

We accept, here, the solvability of the equations in Steps 2,3; note that $t_* \ge \min\{T, [\min\{R_0, 1 - R_0\}/M_*]^2\} =: \tau$ in Step 1. A standard estimate, taking $[C - \gamma_0]$ as test function in the weak formulation of (2.3) and using Gronwall (compare [1], etc.), gives a bound on $C - \gamma_0$ in $\boldsymbol{\mathcal{U}}_\tau$ so the trace ω is well defined in, say, $L^2(\mathcal{I}_\tau)$. Indeed, the $\boldsymbol{\mathcal{U}}$-bound gives C in a compact subset of, say, $L^2(\mathcal{I}_\tau \to H^s(0, 1)$ for any $s < 1$ by the Aubin Compactness Theorem and taking $s > \frac{1}{2}$ gives ω in a compact subset of $L^2(\mathcal{I}_\tau)$ by trace theory. We will also, somewhat similarly, obtain a $\boldsymbol{\mathcal{V}}$-bound for $\Theta - \gamma_1$ and the corresponding argument then gives ϑ in a compact subset of $L^2(\mathcal{I}_\tau)$. The hypotheses (H-1,2,5) ensure that $\hat{h}$ depends continuously on ω, ϑ and is also in a compact subset of $L^2(\mathcal{I}_\tau)$. It is not difficult to verify the continuity of $\boldsymbol{\mathcal{F}}$ on $\mathcal{X}_*$ and the Schauder Theorem then gives existence of a fixpoint, determining C, Θ. The precise extension we made in (2.1) just permits a weak Maximum Principle argument[5] (using C_- and $(C - 1)_+$ as test functions) to show that $0 \le C \le 1$; cf., [1], etc.

[4] We will not be interested in situations in which the grain disappears ($R = 0$) and will show that $R = 1$ cannot occur for the solution.

[5] Since the direction of motion of the interface is not fixed, this argument is necessarily a bit more delicate than, e.g., for the one-phase Stefan problem.

We return to the estimation of Θ to show, in particular, that we may choose M_* so as to have (3.4) give (1.6) at the solution. Using $\eta = 2[\Theta - \gamma_1]$ in (2.6), we first note that

$$|\vartheta - \gamma_1|^2 \le \delta \|\Theta_r\|^2 + C_\delta \|\Theta - \gamma_1\|^2$$

for any $\delta > 0$ so, with $\delta > 0$ a lower bound for $D(r)$, we use (H-6), etc., to get

$$\|\Theta - \gamma_1\|^2 + \delta \int_0^t \|\Theta_r\|^2 \, dt' \le M \left[M_0 + C_\delta M_*^2 + \int_0^t [1 + u(t)] \|\Theta - \gamma_1\|^2 \, dt' \right]$$

and Gronwall gives a $\boldsymbol{\mathcal{V}}$-bound, here depending on M_* but adequate to justify our previous argument. However, at the solution, where we have $\omega \in [0,1]$ and (3.4), we note that the last term in (2.6) gives $-(\vartheta - \gamma_1)\tilde{\alpha}\vartheta$ with $\tilde{\alpha} = \alpha$ when $\|\vartheta\| \le M_*$ and $\tilde{\alpha} = \alpha M_* / \|\vartheta\| < \alpha$ else; using (H-1,2,4), one has

$$\begin{aligned} -(\vartheta - \gamma_1)\tilde{\alpha}\vartheta \ &\le\ \begin{cases} \alpha[\gamma_1 + \vartheta G(\kappa, \vartheta)] K(\vartheta) & \text{if } \vartheta \ge 0 \\ \alpha[\gamma_1 - \vartheta] K(\gamma_1) & \text{if } \vartheta \le 0 \end{cases} \\ &\le\ M_1 + M_2 |\vartheta - \gamma_1|^2 \end{aligned}$$

where it is important to note that the constants M_1, M_2 are independent of M_* and of $u(\cdot)$. We now have (with a new M)

$$\|\Theta - \gamma_1\|^2 + \delta \int_0^t \|\Theta_r\|^2 \, dt' \le M \left[M_0 + \int_0^t [u(t) + C_\delta] \|\Theta - \gamma_1\|^2 \, dt' \right]$$

and Gronwall now gives a $\boldsymbol{\mathcal{V}}$-bound on $[\Theta - \gamma_1]$ which depends only on the data (and on an integral bound for the control u) and so an $L^2(\mathcal{I}_\tau)$-bound for the trace ϑ. The linear growth of $K(\cdot)$ then gives a corresponding bound for $\tilde{h}$ in (3.4) and this gives (1.6) if we have chosen M_* at least equal to that.

We note that (H-3) ensures — together with the Dirichlet conditions for C, Θ at $r = 1-$ — that $H(R, C(\cdot, R), \Theta(\cdot, R))$ would become negative if we were to have $R \to 1$. Hence, (1.4)+(1.6) ensures that R can never reach 1 for a solution of the direct problem. One consequence of this is that we need never contend with a possible degeneracy of having $\rho \to 0$ in (2.3).

This procedure gives such a solution on the interval $\mathcal{I}_\tau$ with τ as noted above. Obviously, if we do not actually have $R(\tau) = 0$ then we can restart the problem and continue the solution to a maximal interval of existence, again denoted by $\mathcal{I}_\tau$ with either $\tau = T$ or $R(\tau) = 0$; one sees as in [1] that a solution cannot become non-continuable for any other reason so, now with $\mathcal{I}_\tau$ the maximal interval of existence, we could only have $\tau < T$ if the grain were to dissolve completely: $R(\tau) = 0$.

4. Optimal Controls

We will consider as admissible controls only those satisfying (3.1) which provide a solution with $R > 0$ on all of $\mathcal{I}$. We will simply assume the existence of such controls but do observe heuristically that, if we have no upper constraint on $u(\cdot)$, it should always be possible to pump enough heat into the system to make ϑ large so, using (H-4), $G(\kappa,\vartheta) \leq \omega$ and $\dot{R} \geq 0$ — noting that we cannot be prevented from this by having $\dot{R} < 0$ with $\omega = 0$ since (1.3) with $\dot{R} < 0$ implies $C_r\Big|_{r=R} < 0$ which would be impossible for $\omega = 0$, which shows that the possible effect of increasing Θ to increase F and so decrease C must be overbalanced at the interface, itself, by the surface reaction.

Somewhat arbitrarily, we may consider $\mathcal{J}$ as having the form

$$\mathcal{J} := \int_0^T |\kappa(t) - \kappa_*(t)|^2 \, dt + \lambda U(T) \tag{4.1}$$

with U as in (3.1); here $\kappa_*(\cdot)$ corresponds to some specified desirable 'trajectory' for the free boundary. One significance of using this particular form for $\mathcal{J}$ is that we will have $\mathcal{J} = \infty$ if the grain is not there for the full interval $\mathcal{I} = [0,T]$ — so the solution gives $R(\tau) = 0$ for some $\tau < T$ and so, by our interpretation of (1.4), gives $\kappa(t) \equiv \infty$ on $[\tau, T]$. Indeed, if we were to set

$$\mathcal{J}_\tau := \int_0^\tau [|\kappa(t) - \kappa_*(t)|^2 + \lambda u(t)] \, dt,$$

then $R(\tau) = 0$ would give

$$\begin{array}{c} R(\tau - s) = -\int_{\tau-s}^{\tau} h(\sigma)\, d\sigma \leq \sqrt{s}\|h\| \\ \int_{\tau-s}^{\tau} \kappa^{-2}\, d\sigma \geq \|h\|^{-2} \int_{\tau-s}^{\tau} \frac{1}{\sigma}\, d\sigma \end{array}$$

and this would make $\mathcal{J}_\tau = \infty$. Thus, any control u for which $\mathcal{J}$ or $\mathcal{J}_\tau$ would be finite necessarily implies a solution existing on the full interval $\mathcal{I}$ with $R(t)$ bounded away from $0, 1$ and, in fact, a bound on the control will imply a uniform such bound for $R(\cdot)$.

We will permit the inclusion of positive measures as controls so controls are taken in the dual space $[\mathcal{C}(\mathcal{I})]^*$; another consequence of the use of (4.1) is that any minimizing sequence $\{u_k\}$ for $\mathcal{J}$ is then necessarily bounded and so, possibly extracting a subsequence, we have weak-$*$ convergence $u_k \rightharpoonup \bar{u}$ (whence strong convergence $u_k \to \bar{u}$ in, say, $H^{-1}(\mathcal{I})$). By the discussion in the previous section, the corresponding $\{\vartheta_k, \omega_k, h_k\}$ lie in a compact subset of $L^2(\mathcal{I})$ and the functions $R_k(\cdot)$ are uniformly bounded away from $0, 1$. Again taking a subsequence, we may assume that that $\vartheta_k \to \bar{\vartheta}$, etc. By Krasnosel'skĭi's Theorem on the continuity of Nemitsky operators, one has (1.6) in the limit for $\bar{h}$, etc., and, using smooth test functions (which are dense) one easily verifies that $\bar{u}$ is an admissible control with uniform convergence $R_k \to \bar{R}$. Since the last term in $\mathcal{J}$ is actually continuous in u with respect to weak-$*$ convergence (noting the positivity of the controls), this shows that $\bar{u}$ is an *optimal control*, as desired.

An essentially similar argument — indeed, technically slightly simpler — would give the same result for, e.g., a cost functional of the form

$$\mathcal{J} := \int_0^T |\kappa(t) - \kappa_*(t)|^2 \, dt + \lambda \int_0^T |u(t)|^2 \, dt \tag{4.2}$$

and it is for this that one might expect a derivation of the first order necessary conditions along the lines of [3]. On the other hand, if we were to try to use, say,

$$\mathcal{J} := \int_0^T |R(t) - R_*(t)|^2 \, dt + \lambda U(T), \tag{4.3}$$

then we do not have a similar argument that $\bar{R}(\cdot)$ never vanishes, which was needed to have $\bar{u}$ admissible. It is likely, in that case, that the existence of an optimal control would depend on the particular initial conditions. Finally, one can also obtain, by now-standard arguments, existence of time-optimal controls: minimizing the time T at which one would have $R(t) = R_* < R_0$ — where the requirement that $0 < R_* < R_0$ ensures that there can be no intermediate difficulties with disappearance of the grain.

Note that we could obtain the existence results for optimal controls as rapidly as we have only because the relevant compactness results for $\{h_k\}$, etc., were obtained in the course of the existence arguments in the previous section.

References

[1] F. Conrad, D. Hilhorst, and T.I. Seidman, *Well-posedness of a moving boundary problem arising in a dissolution–growth process,* Nonlinear Anal-TMA **15**, pp. 445–465 (1990).

[2] D. Hilhorst, F. Issard-Roch, and T.I. Seidman, *On a free boundary problem on an unbounded domain arising in a dissolution growth process*, in *Proc. 5^{th} International Conf. on Free Boundary Problems*, (Montreal, 1990), to appear [Longman, London, 1993].

[3] P. Neittaanmäki and T.I. Seidman, *Optimal solutions for a free boundary problem for crystal growth,* in *Control and Estimation of Distributed Parameter Systems* (Vorau, 1988) (F. Kappel, K. Kunisch, W. Schappacher, eds.), pp. 323–334, Birkhäuser Verlag, Basel (1989).

[4] T.I. Seidman, *Some control-theoretic questions for a free boundary problem,* in *Control of Partial Differential Equations* (LNCIS #114; A. Bermúdez, ed.), Springer-Verlag, New York, pp. 265–276, (1989).

[5] T.I. Seidman, *Optimal control and well-posedness for a free boundary problem*, in *Proc. 5^{th} International Conf. on Free Boundary Problems*, (Montreal, 1990), to appear [Longman, London, 1993].