October 26, 1987

SOME CONTROL-THEORETIC QUESTIONS FOR A FREE BOUNDARY PROBLEM[1]

Thomas I. Seidman
Department of Mathematics and Statistics,
University of Maryland Baltimore County,
Baltimore, MD 21228, USA.
(BITNET:seidman@umbc; Internet:seidman@umbc1.umd.edu).

ABSTRACT: We consider a number of control-theoretic questions related to a free boundary problem modeling growth and dissolution of a crystal grain surrounded by a dilute solution of the same substance.

KEY WORDS: *Free boundary problem, nonlinear, parabolic, partial differential equation, boundary control, optimal control.*

1 Introduction

Consider a solution of some substance surrounding a 'crystal grain' of the pure material. We assume radial symmetry[2] in $\mathbb{R}^d$ and consider the concentration C of the solution for $t > 0$ and $R(t) < r = |x| < 1$ where $L = 1$ is the fixed outer radius and $R(t) > 0$ is the radius of the grain at time t.

The underlying model involves diffusion/reaction in the solution with the concentration C satisfying

$$\dot{C} = \Delta C + F(C) = r^{1-d}(r^{d-1}C_r)_r + F(C) \qquad \text{in } \mathcal{Q}. \tag{1.1}$$

[1]This is to appear in the Proceedings of the IFIP conference on Optimal Control of Systems Governed by Partial Differential Equations, held at the University of Santiago de Compostela in July, 1987. This research was partially supported under grants #AFOSR-87-0190 and #AFOSR-87-0350.

[2]Physically, of course, we would have $d = 2$ or 3. It would certainly be of interest to consider less symmetrical settings but these would seem to be far more complicated — e.g., the free boundary would then be a *time-varying surface* and the function H appearing in the analogue of (1.4), governing the 'flow' of the surface along its normal, would become a function of the concentration C at the boundary and the local radius of curvature R.

Here F is some nonlinear reaction rate, for simplicity, taken nonincreasing in C and $\mathcal{Q} := \{(t,r) \in \mathbb{R}_+ \times \mathbb{R}^2 : R(t) < r < 1\}$. Conservation of mass gives the boundary condition:

$$-C_r = -\dot{R}[1-C] \qquad \text{at } r = R(t), \tag{1.2}$$

while at the outer boundary we impose the Dirichlet condition:

$$C = \gamma \qquad \text{at } r = 1. \tag{1.3}$$

We also, of course, have initial conditions.

In comparison with [3], note that (with no loss of generality) we are here scaling t, x, C so that the (constant) diffusion coefficient is $D \equiv 1$, the 'outer radius' is $L = 1$, the 'pure' concentration in the crystal is $\Gamma = 1$.

The rate of growth or dissolution of the grain, i.e., the motion of the free boundary surface $\{r = R(t)\}$, is not specified *a priori* but is governed by an equation of the form:

$$\dot{R} = H(R, \omega) \qquad [\omega(t) := C(t, R(t))]. \tag{1.4}$$

Thus we have a *free boundary problem*, here called the *direct problem*, defined by coupling (1.1) and (1.4) with the associated conditions.

A principal *control-theoretic* concern will be to view the data γ appearing in (1.3) as a possible *boundary control* and then to consider such questions as the existence of an optimal control for standard optimality criteria. An interesting alternative is to take the functions $F(\cdot)$ and $H(\cdot)$ appearing in (1.1) and (1.4) to depend also on the temperature ϑ and assume the availability of a 'heater' with radially symmetric effect β and variable intensity — so $\vartheta(t,r) = \alpha(t)\beta(r)$. Such a setting, in which the intensity $\alpha(\cdot)$ would be taken as under our control (within some specified range) is physically plausible, perhaps even more so than the boundary control setting. Apart from optimality, we also formulate (without much in the way of results already obtained) some questions of feedback stabilization and identification.

Most of the technical work of this paper is in the next section, discussing the direct problem from the perspective of the control-theoretic applications. These can then be presented comparatively briefly in the final section.

2 The direct problem

The paper [3] presents a detailed argument for the existence and uniqueness of solutions to the direct problem. Our present considerations are somewhat different since we are not as concerned with uniqueness but are extremely concerned to know that we obtain limiting solutions by a weak limit of controls. Further, the treatment in [3] does not consider as general boundary data γ as we would like. We therefore sketch the relevant theory for the direct problem.

Our strategy is first to obtain existence for a modified problem and then to show, using a maximum principle argument, that the modification is nugatory — the solution stays within the range for which the two problems coincide — so one has, indeed, a solution of the original problem. Initially we restrict our consideration to a fixed time interval $[0, T]$ with T depending only on R_0.

We begin by assuming:

(H-1): $F : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing; $F(0) \geq 0 \geq F(1)$.

(H-2): $H : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is continuous with $H(\cdot, 0) \leq 0$.

(H-3): $0 < R_0 < 1$ and $C_0 \in L^\infty(R_0, 1)$ with $0 \leq C_0 \leq 1$.

(H-4): γ is in $\mathcal{G}_T := \{\gamma \in L^2(0, T) : 0 \leq \gamma \leq 1\}$.

as our basic set of hypotheses.

Let us restate the problem: we wish to find a pair $[R, C]$ of functions such that R satisfies on the interval $[0, T]$ the ordinary differential equation

$$\dot{R} = H(R, \omega) \qquad \text{with } R(0) = R_0, \tag{2.5}$$

$$\omega(t) := C(t, R(t)) \tag{2.6}$$

and C satisfies on $\mathcal{Q} = \mathcal{Q}_T$ the parabolic moving boundary problem

$$\text{(1.1) with } C = C_0 \text{ at } t = 0 \text{ and (1.2), (1.3)} \tag{2.7}$$

For the *modified* problem, define functions $\hat{H}$, $\hat{\varphi} : (0, 1) \times \mathbb{R} \to \mathbb{R}$ by:

$$\hat{H}(R, \omega) \;:=\; \begin{cases} H(R, 0) & \text{if } \omega \leq 0 \\ H(R, 1) & \text{if } \omega \geq 1 \\ H(R, \omega) & \text{for } 0 \leq \omega \leq 1; \end{cases} \tag{2.8}$$

$$\hat{\varphi}(R, \omega) \;:=\; \begin{cases} -\hat{H}(R, \omega)[1 - \omega] & \text{if } \omega \leq 1 \\ -\hat{H}(R, \omega)[1 - \omega]/2 & \text{if } \omega \geq 1 \end{cases} \tag{2.9}$$

and then replace $-H[1 - \omega] =: \varphi$ by $\hat{\varphi}$ in (1.2), obtaining

$$C_t = r^{1-d}(r^{d-1}C_r)_r + F(C) \qquad \text{with } C = C_0 \text{ at } t = 0 \tag{2.10}$$

$$\text{and } \begin{cases} \text{(i)} & -C_r = \hat{\varphi}(R, \omega) & \text{at } r = R(t) \\ \text{(ii)} & \quad C = \gamma & \text{at } r = 1, \end{cases}$$

and H by $\hat{H}$ in (2.5), obtaining

$$\dot{R} = \hat{H}(R, \omega) \qquad \text{with } R(0) = R_0. \tag{2.11}$$

We refer to this modified problem [(2.10), (2.11), (2.6)] as $(\mathbf{MP}) = (\mathbf{MP})_T$ and to the original problem as $(\mathbf{OP})$.

Suppose, now, we fix $0 < R_- < R_0 < R_+ < 1$ and find upper and lower bounds h_-, h_+ for H on the compact set $[R_-, R_+] \times [0,1]$ so $h_- \le \hat{H}(R,\omega) \le h_+$ on $[R_-, R_+] \times \mathbb{R}$. Choose $T > 0$ so

$$R_- \le R_0 + h_- T \le R_0 + h_+ T \le R_+. \tag{2.12}$$

This is then the T for consideration of $(\mathbf{MP})_T$.

We will use a 'fixpoint approach' to existence. Given a function $\omega(\cdot)$, we solve (2.11) to obtain $R(\cdot)$ and set

$$h(\cdot) := \hat{H}(R,\omega) = \dot{R} \quad \text{ and } \rho(\cdot) := 1/[1 - R(\cdot)]. \tag{2.13}$$

One easily sees that for every possible solution $R(\cdot)$ of (2.11) we always have the bounds $R_- \le R \le R_+$ on $[0,T]$ by the definition of T and $h = \hat{H}$ must satisfy $h_- \le h \le h_+$. Also, we note that $\rho(\cdot)$ is in a fixed compact subset of $C[0,T]$ with $1 < 1/[1-R_+] \le \rho \le 1/[1-R_-]$.

To treat the moving boundary problem, we make the change of variables $r \mapsto y := \rho[1-r] = \frac{1-r}{1-R}$ and set $u(t,y) = C(t,r)$. The parabolic problem (2.7) then becomes

$$u_t = \rho^2 u_{yy} - \psi u_y + F(u) \text{ on } \tilde{\mathcal{Q}}_T \qquad \text{with } u = u_0 \text{ at } t = 0 \tag{2.14}$$

$$\text{and } \begin{cases} \text{(i)} & \rho^2 u_y = \tilde{\varphi} \quad \text{at } y = 1 \\ \text{(ii)} & \quad u = \gamma \quad \text{at } y = 0. \end{cases}$$

Here $\tilde{\mathcal{Q}}_T := (0,T) \times (0,1)$ and, with R, ρ, h as above, we have set

$$\begin{aligned} \psi(t,y) = \psi(t,y;\omega) &:= \rho\left[\frac{(d-1)\rho(t)}{\rho(t)-1} + y h(t)\right]; \qquad &(2.15)\\ \tilde{\varphi}(t) = \tilde{\varphi}(t;\omega) &:= \rho(t)\hat{\varphi}(R(t),\omega(t)). \qquad &(2.16)\end{aligned}$$

Note that ψ, $\tilde{\varphi}$, and $\psi_y = \rho h$ are uniformly bounded, independently of ω.

Provided we show that the solution u of (2.14) will have enough regularity at $y = 1$ to justify taking the trace (as a suitable function on $[0,T]$), we can now define a map $\mathbf{T} = \mathbf{T}_\gamma : \omega \mapsto \omega_{new}$ by setting

$$[\mathbf{T}_\gamma(\omega)] = \omega_{new} := u(\cdot, 1) \tag{2.17}$$

with u obtained as above. We will next verify, for $\mathbf{T}$, the hypotheses of the Schauder Fixpoint Theorem.

Introduce the solution $z = z(\cdot;\gamma,\rho)$ of the linear partial differential equation

$$z_t = \rho^2 z_{yy} \text{ on } \tilde{\mathcal{Q}}_T \tag{2.18}$$

with $z = u_0$ at $t = 0$, $\rho^2 z_y = 0$ at $y = 1$, and $z = \gamma$ at $y = 0$. In view of **(H-3), (H-4)**, we note that one also has $0 \le z \le 1$ by the maximum principle. Now set $v := u - z$ so

$$v_t = \rho^2 v_{yy} - \psi v_y + F(v+z) - \psi z_y \quad \text{on } \tilde{\mathcal{Q}}_T \tag{2.19}$$

with $v = 0$ at $t = 0$, $\rho^2 v_y = \tilde{\varphi}$ at $y = 1$, and $v = 0$ at $y = 0$ (without γ appearing explicitly). Multiplying this by v and integrating over $\tilde{\mathcal{Q}}_t$ gives the identity

$$\begin{aligned}\tfrac{1}{2}\|v(t)\|^2 \;&+\; \int_0^t \rho^2\|v_y\|^2 \\ &= \int_0^t [\tilde{\varphi} - \psi z]v|_{y=1} + \int_0^t\int_0^1 (z-v)\psi v_y + \int_0^t\int_0^1 [F(z) + z\psi_y]v \\ &\qquad + \int_0^t\int_0^1 [F(v+z) - F(z)][(v+z) - z].\end{aligned}$$

As usual,[3] this provides uniform bounds (independent of ρ,γ) on the solution v in $L^\infty((0,T)\to L^2(0,1)) \cap L^2((0,T)\to H^1(0,1))$ and so, using the equation, on v_t in $L^2((0,T)\to H^{-1}(0,1))$. By the Aubin Compactness Theorem [1], this shows that v is in a compact, convex subset of $\mathcal{V} := L^2((0,T)\to H^s(0,1))$ (with, e.g., $s < 1$) independent of ρ,γ. A similar argument shows that v is in a compact subset of $\mathcal{U} := C([0,T]\to L^2(0,1))$. By trace theory (taking $s > /2$) $v(\cdot,1)$ is in a fixed compact subset of $L^2(0,T)$.

Returning to (2.18), we consider the further change of variables

$$\tau = \vartheta(t) = \vartheta(t;\rho) := \int_0^t \rho^2, \qquad t = \Theta(\tau) := \int_0^\tau \rho^{-2}, \tag{2.20}$$

so $\Theta = \vartheta^{-1}$ for given $\rho(\cdot)$. Setting $\zeta(\tau,y) = \zeta(\tau,y;\gamma,\rho) = z(t,y;\gamma,\rho)$, we then have

$$\zeta_t = \zeta_{yy} \quad \text{on } \tilde{\mathcal{Q}}_T \tag{2.21}$$

with $\zeta = 0$ at $t = 0$, $\zeta_y = 0$ at $y = 1$, $\zeta = \hat{\gamma}$ at $y = 0$, and

$$\hat{\gamma}(\tau) = \hat{\gamma}(\tau;\rho) = [\mathbf{L}_\rho\gamma](\tau) := \gamma(\Theta(\tau;\rho)) = \gamma(t). \tag{2.22}$$

[3] Note that $|v(t,1)| \le C\|v(t,\cdot)\|^2_{H^1}$ and the last integral is non-positive by **(H-1)**. Recall that $\rho > 1$ and we have uniform bounds for $\rho, \tilde{\varphi}, \psi, z, \psi_y$. One can then estimate the terms on the right so as to apply the Gronwall Inequality.

Note that the linear maps $\mathbf{L}_\rho$ are continuous from L^2 to L^2 (uniformly in ρ for the functions ρ arising here — already noted to be in a compact subset of $C[0,T]$). Note also that $[\rho \mapsto \mathbf{L}_\rho^*]$ is uniformly continuous from (the relevant compact subset of) $C[0,T]$ to the strong (pointwise) operator topology for $\mathbf{L}_\rho^*$; from this it follows that the map: $[\gamma,\rho] \mapsto \hat{\gamma}$ is continuous (topologizing γ and $\hat{\gamma}$ in L^2_{weak} and ρ in $C[0,T]$). It follows from linearity, closedness, and well-known interior regularity results that the solution map: $\hat{\gamma} \mapsto \zeta$ for (2.21) is continuous and compact to $\mathcal{V}_\varepsilon := L^2(\to H^s(\varepsilon,1))$ $(s<1)$ for each $\varepsilon > 0$ from L^2_{weak}. Inverting $\mathbf{L}_\rho$ to obtain z again, this means that the map: $[\gamma,\rho] \mapsto z(\cdot;\gamma,\rho)$ defined by (2.18) is also continuous and compact to $L^2(\tilde{\mathcal{Q}}_T)$ — indeed, to $\mathcal{U}$, using the bound $0 \le z \le 1$ and obtaining continuity for each $z : [0,T] \to L^2(0,1)$ by a density argument — and also to each $\mathcal{V}_\varepsilon$. It follows that the map: $[\gamma,\rho] \mapsto z(\cdot,1;\gamma,\rho)$ to $L^2(0,T)$ is also continuous and compact.

Combining this with our compactness result for $v(\cdot,1)$, we see that

$$\mathbf{T}_\gamma(\omega) = z(\cdot,1) + v(\cdot,1)$$

lies in a *compact* subset $\mathcal{W}$ of $L^2(0,T)$. Re-interpreting $\mathbf{T}_\gamma$ as its restriction to $\mathcal{W}$, we have shown that $\mathbf{T}_\gamma$ is a self-map of a compact set (which we may assume is also convex). Note that $\mathcal{W}$ is here independent of $\gamma \in \mathcal{G}_T$.

Returning to (2.19), by a standard estimate[4] the map: $[\omega,z] \mapsto v$ defined by (2.19), (2.15), (2.16), etc., is continuous to $\mathcal{V}$ (taking $\omega \in \mathcal{W}$ and $z \in L^2(\mathcal{Q}_T)$ with $0 \le z \le 1$). Combining this with our earlier continuity result for z (and using the continuity of the map: $\omega \mapsto \rho : \mathcal{W} \to C[0,T]$), this gives the continuity of $\mathbf{T}_\gamma$. The Schauder Fixpoint Theorem is now applicable to give existence (but, of course, not uniqueness) of a solution pair $[R,u]$ for (2.11), (2.14) with $\omega = u(\cdot,1)$.

This solves $(\mathbf{MP})_T$ (for fixed $\gamma \in \mathcal{G}_T$) since we may obviously invert the '$r \mapsto y$–substitution' to obtain a solution C — with the same regularity as u — for (2.10). Next we wish to observe a property of continuous dependence on $\gamma \in L^2_{weak}(0,T)$. The nature of our analysis makes it convenient to work with u rather than with C but, in any reasonable sense, the notions of convergence will correspond.

Suppose $\gamma_k \rightharpoonup g_*$ (weak convergence in $L^2(0,T)$, subject to **(H-4)**) and let $[R_k,u_k]$ be corresponding solutions of $(\mathbf{MP})_T$. We wish to know, at least for a subsequence, that $[R_k,u_k] \to [R_*,u_*]$ in some suitable sense, where $[R_*,u_*]$ is a solution corresponding to γ_*. We use, subscripted by k, the notation of

[4] For solutions v_1, v_2 corresponding to two pairs of functions $[\omega_1,z_1]$, $[\omega_2,z_2]$ set $v = v_1 - v_2$. Much as in the estimate above (giving the *a priori* bound), we multiply the resulting equation by v, integrate over $\tilde{\mathcal{Q}}_t$, estimate, and apply the Gronwall Inequality.

the analysis above (at the fixpoint of $\mathbf{T}_{\gamma_k}$, of course). Thus, e.g., $\{\omega_k\}$ is in $\mathcal{W}$ (which we have already noted is independent of γ and compact). Extracting a subsequence if necessary, we may thus assume $\omega_k \to \omega_*$ in the sense of $L^2(0,T)$ for *some* function $\omega_* \in \mathcal{W}$ and have $[R_k, \rho_k] \to [R_*, \rho_*]$ in $C[0,T]$ and $h_k \to h_*$ from (2.11), (2.13). Note that this gives suitable convergence of $\psi_k, (\psi_k)_y$ from (2.15) and $\tilde{\varphi}_k$ from (2.16) to the corresponding $\psi_*, (\psi_*)_y$ and $\tilde{\varphi}_*$. It also gives $\hat{\gamma}_k \rightharpoonup \hat{\gamma}_*$ in L^2 and so $z_k = z(\cdot;\gamma_k,\rho_k) \to z_* = z(\cdot;\gamma_*,\rho_*)$ in each $\mathcal{V}_\varepsilon$, as noted above. It follows that $z_k(\cdot,1) \to z_*(\cdot,1)$ in $L^2(0,T)$ and also that $z_k \to z_*$ in $\mathcal{U}$. We now have, again as noted above, $v_k \to v_*$ in $\mathcal{V}$ so $v_k(\cdot,1) \to v_*(\cdot,1)$ in $L^2(0,T)$. Thus, $u_k := z_k + v_k \to z_* + v_* =: u_*$ in, say, the senses of $\mathcal{U}$ and of $\mathcal{V}_\varepsilon$ for each $\varepsilon > 0$ so u_* satisfies (2.14). Finally, we have convergence of the traces: $u_k(\cdot,1) \to u_*(\cdot,1) =: \mathbf{T}_{\gamma_*}(\omega_*)$. Since $\omega_k = \mathbf{T}_{\gamma_k}(\omega_k) := u_k(\cdot,1)$ and $\omega_k \to \omega_*$ by construction, we have $\mathbf{T}_{\gamma_*}(\omega_*) = \omega_*$ so the limit $[R_*, u_*]$ is indeed a solution of the coupled system $(\mathbf{MP})_T$ using γ_*.

Finally, we use a maximum principle argument to show that each solution $[R,C]$ of (2.10), (2.11) is actually already a solution of (2.7), (2.5). Clearly, this will be the case if $0 \le C \le 1$ ae on $\tilde{Q}_T$ so $\hat{H}, \hat{\varphi}$ coincide with the original functions H, φ. To this end, multiply (2.10) by w, taken as either $w_0 := \min\{C, 0\}$ or $w_1 := \max\{C-1, 0\}$. In either case we have

$w = 0$ at $r = 1$, $t = 0$, $wF(C) \le 0$, $wC_t = (\frac{1}{2}w^2)_t$, $w_r C_r = {w_r}^2$

and

$$\frac{d}{dt}\left(\tfrac{1}{2}\int_{R(t)}^1 r^{d-1} w^2\right) = -\tfrac{1}{2}R^{d-1}w^2\dot{R} + \int_{R(t)}^1 r^{d-1} w C_t.$$

A standard computation then gives

$$\tfrac{1}{2}\int_{R(t)}^1 r^{d-1}w^2 + \int_0^t \int_{R(t)}^1 r^{d-1}{w_r}^2 \le \int_0^t R^{d-1} w[\hat{\varphi} - \tfrac{1}{2}wR]|_{r=R(\cdot)}. \tag{2.23}$$

CASE 1: $\boxed{w = w_1}$ When $w_1|_{r=R} \ne 0$ we have $\omega = C > 1$ so (2.9) gives $\hat{\varphi} := \frac{1}{2}(C-1)\dot{R} = \frac{1}{2}w_1\dot{R}$. Thus, the integrand of the right hand side of (2.23) vanishes identically and one concludes that $w_1 \equiv 0$, i.e., $C \le 1$.

CASE 2: $\boxed{w = w_0}$ When $w_0|_{r=R} \ne 0$ we have $w = \omega < 0$ and $\hat{\varphi} = (\omega - 1)\dot{R}$ so $[\hat{\varphi} - \frac{1}{2}\dot{R}] = [\frac{1}{2}w]\dot{R}$ and also $\dot{R} = \hat{H}(R,0) \le 0$. Thus the integrand of the right hand side of (2.23) must be non-positive ae and one concludes, as above, that $w_0 \equiv 0$, i.e., $C \ge 0$. □

Summarizing the arguments above, we see that we have shown:

Direct Theorem 1: *Consider the system* (2.7), (2.5), (2.6) *with initial data satisfying* **(H-3)** *and with T given as in* (2.12); *assume* **(H-1,2)**. *Then for each*

$\gamma \in \mathcal{G}_T$ there is always at least one solution $[R, C]_\gamma$ on the interval $[0, T]$. This solution lies in a fixed compact set, independent of $\gamma \in \mathcal{G}_T$, in $H^1(0,T) \times \mathcal{U}$. Further, if H is Lipschitzian and one additionally has $\gamma \in H^1(0,T)$, then[5] $u \in \mathcal{V}$ and R are unique. □

Direct Theorem 2: *Consider the system as in Direct Theorem 1 with fixed initial data as in* **(H-3)**. *Then the graph $\{[\gamma, R, u]\}$ is compact in $\mathcal{G}_T \times H^1(0,T) \times [\mathcal{U} \cap \mathcal{V}_\varepsilon]$ where $\mathcal{G}_T$ is taken with the L^2_{weak}-topology.* □

So far we have determined the value of T by (2.12). Before proceeding further, a word is in order about the *global existence* of solutions: we note that it is always possible to restart at $t = T$, taking $R(T), C(T)$ as new initial data but the new T obtained for the restarted problem may be smaller and proceeding recursively need not yield an extended solution defined for all t — in general, the interval of definition can be expected to vary with the solution.

Direct Theorem 3: *Let $[R_k, u_k]$ be solutions — defined on time intervals $[0, \tau_k)$ — corresponding to γ_k $(0 \le \gamma_k \le 1)$ with fixed initial data satisfying* **(H-3)**. *Suppose $\gamma_k \rightharpoonup \gamma_*$ in $\mathcal{G}_{\tau_*}$ with $\tau_* := \underline{\lim}\tau_k$. Then there is a solution $[R_*, u_*]$ — defined on $[0, \tau_*)$ — corresponding to γ_* and such that, possibly for a subsequence, one has suitable convergence $[R_k, u_k] \to [R_*, u_*]$ on compact subintervals of $[0, \tau_*)$.*

Proof In view of Direct Theorem 2, we see (possibly extracting a subsequence) that $R_k \to R_*$ uniformly on the fixed initial interval $[0, T]$ and that $u_k \to u_*$ suitably where $[R_*, u_*]$ is a solution corresponding to γ_*. In particular, $R_k(T) \to R_*(T)$ so $T_k \to T_*$ for the 'guaranteed existence' times. This gives $[R_*, u_*]$ defined up to $\tau = T + \lim T_k$ and $[R_k, u_k] \to [R_*, u_*]$. We apply this argument recursively (using a diagonal process to get the final subsequence) to obtain the result. □

We will also want to know similar results in preparation for the alternate mode of control under consideration.

Direct Theorem 4: *Let $F = F(\cdot, \vartheta, C)$ be bounded, continuous in ϑ, C, nonincreasing in C, with $F(\cdots, 0) \ge 0 \ge F(\cdots, 1)$. Let $H = H(\vartheta, R, \omega)$ be continuous and bounded for R in $[R_-, R_+] \subset (0,1)$ and $0 \le \omega \le 1$. Fix $\gamma = \gamma_0 = const.$ $(0 \le \gamma_0 \le 1)$ in (2.7.ii), $0 < R < 1$ in (2.5), and $C_0 = u_0$ in (2.7) with $0 \le u_0 \le 1$. Let $\beta(\cdot)$ be a fixed L^2 function of $r \in [R_-, R_+]$. Then the map:*

$$\alpha \longmapsto \vartheta = \alpha(t)\beta(r) \longmapsto [R, u] \quad (\leftrightarrow [R, C]) \tag{2.24}$$

[5] See [3] for the uniqueness argument. The hypothesis that $\gamma \in H^1(0,T)$ is certainly stronger than needed but makes it easy to show the map: $[\gamma, \rho] \mapsto z$ is Lipschitzian to $\mathcal{V}$. One certainly needs $\gamma \in H^{1/2}(0,T)$to have $z \in \mathcal{V}$, rather than $\mathcal{V}_\varepsilon$, as here; compare [4].

is continuous from $L^2_{weak}(0,T)$ *to* $H^1(0,T)\times[\mathcal{U}\cap\mathcal{V}]$. *If one has solutions* $[R_k, C_k]$ *— defined on maximal intervals* $[0,\tau_k)$ *— corresponding to* $\alpha_k \rightharpoonup \alpha_*$, *then one has suitable convergence to a limit solution — defined on* $[0,\tau_*)$ *with* $\tau_* = \lim \tau_k$ *— corresponding to* α_*.

PROOF These hypotheses give, for each fixed α, the hypotheses of Direct Theorem 1 so the map: $\alpha \mapsto [R,u] \leftrightarrow [R,C]$ is well-defined. Noting the uniqueness now available with constant γ, the proof of continuity is much along the same lines as the argument for Direct Theorem 2 showing closedness of the graph $\{[\gamma,R,u]\}$. Finally, the extension to maximally defined solutions is much as for Direct Theorem 3. □

3 Some control-theoretic questions

In this section we will be considering a number of control-theoretic problems for our system. We assume throughout that F,H satisfy **(H-1,2)** and that we always have $0\le\gamma\le 1$. Note that most of the technical preparation has already been done in the previous section so our presentation here is quite brief.

First, consider the *minimum time problem*, taking γ as the control: we are given a *target set* $\mathcal{T}$ and seek a boundary control γ_* for which $[R_*, C_*]|_{t=\tau} \in \mathcal{T}$ with τ as small as possible. Actually, it is more convenient to think of $\mathcal{T}$ as a subset of $(0,1)\times L^2(0,1)$ and ask that $[R_*(\tau), u_*(\tau)] \in \mathcal{T}$.

Control Theorem 1: *Consider the system as in Direct Theorems 1,2,3. Let* $\mathcal{T}$ *be closed in* $(0,1)\times L^2(0,1)$ *and suppose the set*

$$\mathcal{S} := \{[\tau,\gamma,R,u] : \gamma\in\mathcal{G}_\tau,\ \text{(2.14) on } \tilde{\mathcal{Q}}_\tau,\ \text{(2.5), (2.6) on } [0,\tau],\ [R(\tau),u(\tau)]\in\mathcal{T}\}$$

is nonempty. Then τ *attains its minimum: there exists some* γ_* *and a corresponding solution* $[R_*, u_*]$ *of* (2.14), (2.5) *for which* $[R_*(\tau), u_*(\tau)] \in \mathcal{T}$ *with* $\tau = \tau_{min}$.

PROOF Let $\{[\tau_k,\gamma_k,R_k,u_k]\}\subset\mathcal{S}$ be a minimizing sequence for τ. Since, by assumption, we have $0\le\gamma_k\le 1$ on $[0,\tau_{min}+\varepsilon]$ (say, extending each γ_k as 0 past τ_k), we may extract a subsequence weakly convergent in $L^2(0,\tau_{min}+\varepsilon)$ to some γ_*. Now apply Direct Theorem 3. Note that the closure assumption for $\mathcal{T}$ ensures that $R_k(\tau_k)$ must be bounded away from $0,1$ so the intervals in question are not *maximal* intervals of definition but can be prolonged by some uniform T. Hence the application of Direct Theorem 3 gives uniform convergence in $\mathbb{R}\times L^2(0,1)$ of $[R_k,u_k]$ on, say, $[0,\tau_{min}+\varepsilon]$. Thus we have $[R_k(\tau_k),u_k(\tau_k)]\to[R_*(\tau_{min}),u_*(\tau_{min})]$, whence $[R_*(\tau_{min}),u_*(\tau_{min})]\in\mathcal{T}$. □

One might reasonably conjecture a ‘bang-bang principle’: that the time-optimal control satisfies $\gamma_* \equiv 0, 1$ ae. The difficulty of *proving* this is indicated by the fact that the corresponding assertion is open for boundary control of the one-dimensional heat equation. We also note that essentially the same argument as above, but now using Direct Theorem 4, gives existence of a time-optimal control when we use temperature as our control mode (with some constraint giving coercivity).

Now consider an *optimal control problem* with an integral cost functional. As a typical example of the kind of result available here, we take the control to be α as in (2.24) and seek to minimize

$$\mathcal{J} := \int_0^T \int_{R(t)}^1 |C - C^*| + \int_0^T |\alpha - \alpha^*|^2 \tag{3.25}$$

for some given $[T, C^*, \alpha^*]$.

Control Theorem 2: *Consider the system as in Direct Theorem 4 with fixed initial data and γ_0. Let $T > 0$ and assume the problem is feasible: there is some $\alpha \in L^2(0,T)$ for which a solution exists on the interval $[0,T]$. Let C^* and α^* be given in $L^1(\tilde{\mathcal{Q}}_T)$ and $L^2(0,T)$, respectively. Then $\mathcal{J}$ attains its minimum: there is a control α_* for which there is a solution $[R_*, C_*]$ on $[0,T)$ and such that $\mathcal{J}$ attains its infimum.*

PROOF Take a minimizing sequence $\{\alpha_k\}$ for $\mathcal{J}$ and, noting the coercivity of $\mathcal{J}$ so this sequence is bounded in $L^2(0,T)$, extract a weakly convergent subsequence. Then apply Direct Theorem 4 to obtain α_* and $[R_*, u_*]$ on[6] $[0,T)$. Consider each C_k extended as 0 for $0 \le r < R_k(t)$ and similarly for C_*. On each $\tilde{\mathcal{Q}}_\tau$ ($\tau < T$), we have L^2 convergence $C_k \to C_*$ so, extracting a subsequence, we may assume $C_k \to C_*$ pointwise ae on $\tilde{\mathcal{Q}}_T$. Since $0 \le C_k \le 1$ ae, the Dominated Convergence Theorem ensures suitable convergence for the first term of (3.25). The second term is $\|\alpha\|^2$ and so is weakly lower semicontinuous. Hence, $\mathcal{J}(\alpha_*) \le \lim \mathcal{J}(\alpha_k) := \inf \mathcal{J}$. □

A similar result holds when one uses boundary control (except that coercivity is ‘built in’ by the assumption: $0 \le \gamma \le 1$ which we have needed.) Assuming differentiability of F, H with respect to ϑ, it is tedious but not difficult to compute (formally!) the derivative of the solution with respect to the control for either of

[6]It is conceivable that we have $R_* \to 0$ or $R_* \to 1$ as $t \to T$ but only in such a way might $[R_*, C_*]$ fail to be a solution on the closed interval $[0,T]$: one cannot have infinite oscillation of R_* since H has been assumed uniformly bounded for compact subintervals of $(0,1)$. If one *does* have $R_* \to 0$, the equation (1.1) continues to hold in $(0,1)$ at $t = T$.

the possibilities. This should be justifiable and lead to regularity rsults for the optimal controls (e.g., along the lines of Barbu's work) but this is conjectural.

Finally, we note some problems for which, as yet, no results are available. The particular model originally treated in [2] took H in (1.4) to have the special form:

$$H(R, \omega) = K[\omega - G(R)] \tag{3.26}$$

with K a positive constant and $G(\cdot)$ an empirically determined positive (smooth) function on $(0, L]$. Assuming this form, one could seek to identify the function $G(\cdot)$ (over an appropriate range) from observations of $R(\cdot)$. With exact observation (and assuming K and the data $C_0, \gamma(\cdot)$ are known) we could solve the uncoupled equation (2.7) for u to obtain $\omega = u(\cdot, 1)$ and then use the relation: $G(R) = \omega - \dot{R}/K$. The problem is to obtain a useable result if, say, K, γ are unknown — or to find a control $\gamma(\cdot)$ which makes the determination of $G(\cdot)$ most reliable over the largest range.

Another class of problems is related to stabilizing the solution near a steady state. Some consideration of stability is provided in [2] for (3.26) and $\gamma = const.$ but we envision, here, the construction of a stabilizing feedback in a more general context, say, by linearizing the problem around the steady state. It would be of particular interest to seek a feedback of the form:

$$\gamma := \{c_1 \text{ if } R > R_0; \quad c_2 \text{ if } R < R_0; \quad \text{in } [c_1, c_2] \text{ for } R = R_0\}$$

where $0 \leq c_1 < c < c_2 < 1$. Here, the steady state solution considered has $\gamma \equiv c$ and $R \equiv R_0$. Still more interesting would be a feedback of thermostat type: γ switches from c_1 to c_2 when R moves 'left' of some prescribed $R_- < R_0$ and switches from c_2 to c_1 when R moves right of some $R_+ > R_0$; compare [5].

References

[1] J.P. Aubin, Un théorème de compacité, CRAS de Paris **265**, pp. 5042-5045 (1963).

[2] F. Conrad and M. Cournil, Free boundary problems in dissolution-growth processes, preprint (1987).

[3] F. Conrad, D. Hilhorst, and T.I. Seidman, Well-posedness of the free boundary problem for a dissolution-growth process, to appear.

[4] J.L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. II, Springer-Verlag, Berlin (1972).

[5] T.I. Seidman, Switching systems, I: general theory, to appear.