Some problems of distributed parameter control with unilateral constraints[1]

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ABSTRACT: A class of constrained LQ optimal control problems can be abstractly formulated as seeking a nearest point in a 'slice' $\mathcal{S} \cap \mathcal{C}$ where $\mathcal{S}$ is an affine subspace of a Hilbert space $\mathcal{H}$ and $\mathcal{C}$ is the (possibly unbounded) closed convex set corresponding to imposition of the constraint. Subject to a slackness condition, it is shown that the optimal control must then be a 'nearest point in $\mathcal{C}$' for *some control which would be optimal for some unconstrained problem of similar form.* In particular, this is exemplified by minimum-energy boundary control of the heat equation with a non-negativity constraint.

1 INTRODUCTION

We wish to consider linear control problems for distributed parameter systems in which the set $\mathcal{U}_{ad}$ of admissible controls involves nonlinear convex constraints but for which the constraints do not provide an *a priori* bound. Thus, we are looking at the same kind of LQ optimality as might be considered for a corresponding affine problem.

A prototypical problem is boundary control of the heat equation with a positivity constraint on the control. Assume $\Omega \subset \mathbb{R}^m$ is bounded with $\partial\Omega$ smooth and that we have specified $T > 0$ and $\omega_0, \omega_T \in L^2(\Omega)$. We seek a

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control $\varphi \geq 0$ such that

$$
\begin{aligned}
u_t &= \Delta u && \text{on} \quad \mathcal{Q} := (0,T) \times \Omega, \\
u &= \omega_0 && \text{at} \quad t = 0, \\
u &= \varphi && \text{on} \quad \Sigma := (0,T) \times \partial\Omega
\end{aligned}
\tag{1.1}
$$

gives $u = \omega_T$ at $t = T$. The control space is $\mathcal{U} = L^2(\Sigma)$ but the positivity constraint means that the set of *admissible controls* is here

$$
\mathcal{U}_{ad} = \{\varphi \in \mathcal{U} : \varphi \geq 0 \ \ ae\}. \tag{1.2}
$$

We now seek the *minimum-energy control* $\bar{\varphi}$, minimizing $\mathcal{J} = \|\varphi\|_{\mathcal{U}}^2$ among all admissible controls $\varphi \geq 0$ which verify the terminal condition. Setting

$$
\mathcal{S} = \{\varphi \in \mathcal{U} : (1.1) \ \text{ gives } u(T) = \omega_T\},
$$

we see that we are minimizing the quadratic functional $\mathcal{J}$ over $\mathcal{S} \cap \mathcal{U}_{ad}$.

Apparently changing the subject, for closed, convex subsets $\mathcal{C}$ of a Hilbert space $\mathcal{H}$, consider the 'nearest point projection' $\mathbf{P}(\cdot;\mathcal{C}) : \mathcal{H} \to \mathcal{C}$, defined by

$$
\mathbf{P}(\omega;\mathcal{C}) := \text{argmin}_\gamma\{\|\gamma - \omega\|_{\mathcal{H}} : \ \gamma \in \mathcal{C}\} \tag{1.3}
$$

so $\gamma_* = \mathbf{P}(\omega;\mathcal{C})$ is the (unique) nearest point in $\mathcal{C}$ to ω.

We will see that constrained LQ problems of the sort in which we are interested (as, e.g., the prototypical problem above of boundary control for (1.1) with a one-sided control constraint) can be reformulated as geometric *nearest point* problems in Hilbert space:

$$
\text{Find} \quad \mathbf{P}(\varphi_*;\mathcal{S} \cap \mathcal{C}) \tag{1.4}
$$

where $\mathcal{S}$ has the form: $[\bar{\omega} + \mathcal{N}]$ with $\mathcal{N}$ a closed subspace of $\mathcal{H}$. (The problem (1.4) can then be reduced to consider $\varphi_* = 0$ and $\mathcal{S} = \mathcal{N}$.)

Our principal geometric results for (1.4) with $\mathcal{S}$ of this form are that one has

$$
\{\mathbf{P}(0;\mathcal{N} \cap \mathcal{C})\} = \mathcal{N} \cap \mathbf{P}(\mathcal{N}^\perp;\mathcal{C}) \tag{1.5}
$$

if one has the *slackness condition*: $\mathcal{N} \cap \mathcal{C}^o \neq \emptyset$ while, more generally,

$$
\mathbf{P}(0;\mathcal{N} \cap \mathcal{C}) \in \mathcal{N} \cap \overline{\mathbf{P}(\mathcal{N}^\perp;\mathcal{C})} \tag{1.6}
$$

whenever the problem is *feasible*: $\mathcal{N} \cap \mathcal{C} \neq \emptyset$. Interpretation of this for the constrained LQ optimal control problems will show how such a geometric characterization can be of system-theoretic interest.

We observe that, e.g., Luenberger [4] has already given a geometric characterization — quite similar in spirit — for optimality for LQ control systems with convex constraints. Here, our emphasis is on exploitation of the specific structure of the constraint set as the intersection $\mathcal{S} \cap \mathcal{C}$. In control applications the convex set $\mathcal{C}$ is typically imposed independently of the dynamics and $\mathbf{P}(\cdot;\mathcal{C})$ is often easy to work with — e.g., for the positive cone of L^2, as in the example, this is given simply by taking the positive part pointwise. On the other hand, $\mathcal{S}$ is to be obtained by analysis of a purely linear problem: it is a translate of the nullspace of a given operator, hence expressible in terms of the range of the adjoint.[2] While the method of analysis here is by no means new, the structural context of the analysis seems of interest.

2 EXAMPLE

We consider boundary control for the heat equation:

$$\begin{aligned} u_t &= \Delta u \quad \text{on} \quad \mathcal{Q} := (0,T) \times \Omega, \\ u &= \varphi \quad \text{on} \quad \Sigma := (0,T) \times \partial\Omega \end{aligned} \tag{2.7}$$

where $\Omega \subset \mathbb{R}^m$ is bounded with $\partial\Omega$ smooth. Given ω_0, ω_T in $L^2(\Omega)$, we set

$$\mathcal{S} := \{\varphi \in L^2(\Sigma) : \ (2.7) \text{ and } [u(0,\cdot) = \omega_0] \text{ gives } u(T,\cdot) = \omega_T\},$$

i.e., $\mathcal{S}$ is the set of boundary controls steering[3] ω_0 to ω_T, and set $\mathcal{C} = \{\varphi \in L^2(\Sigma) : \varphi \geq 0 \ ae\}$. We *assume* that ω_1, ω_T, T are such that $\mathcal{C} \cap \mathcal{S}$ is non-empty, i.e., that there indeed are admissible controls, and seek to characterize

[2] For control problems with finite dimensional state space this is also finite dimensional, so closed. For distributed parameter problems the characterization of the closure, as with the result of [6] cited for the example, is often the deepest part of the analysis but, using the approach here, we at least have the advantage of considering that in a purely linear context.

[3] In general $\mathcal{S}$ is empty. Ignoring the positivity constraint, it is known [1] that $\omega_T = 0$ is always reachable for any ω_0; it is also known [7] that the reachability of ω_T is independent of $T > 0$ and of ω_0. Of course, the assumption that $\mathcal{C} \cap \mathcal{S} \neq \emptyset$ depends on the entire triple $[\omega_0, \omega_T, T]$.

the *optimal* control $\bar{\varphi}$ minimizing $\mathcal{J}(\varphi) := \|\varphi\|^2$:

(2.8) $\|\bar{\varphi}\|^2 = \min\{\|\varphi\|^2 : \varphi \geq 0;\ \ (2.7) \text{ with } u(0) = \omega_0 \text{ gives } u(T) = \omega_T\}.$

Note that, since $\mathcal{C} \cap \mathcal{S}$ is closed and convex, this minimum is (uniquely) attained.

Theorem 2.1 *Suppose there is a function w satisfying*

$$-w_t = \Delta w \text{ on } \mathcal{Q}, \quad w = 0 \text{ on } \Sigma \tag{2.9}$$

for which $\chi := \partial w/\partial n$ (normal derivative) is in $L^2(\Sigma)$ and such that $\bar{\varphi} := \chi_+ := \max\{\chi, 0\}$ is in $\mathcal{S}$. Then this $\bar{\varphi}$ is the optimal control, i.e., (2.8).

PROOF: Suppose φ were *another* control in $\mathcal{S} \cap \mathcal{C}$. We consider the solution v of

$$v_t = \Delta v \text{ in } \mathcal{Q}, \quad v = \varphi - \bar{\varphi} \text{ on } \Sigma, \quad v(0) = 0 \text{ on } \Omega \text{ at } t = 0.$$

Since $\varphi, \bar{\varphi} \in \mathcal{S}$ and $v(0) = \omega_0 - \omega_0$, we also have $v(T) = \omega_T - \omega_T = 0$ by linearity. Thus,

$$\begin{aligned} 0 &= \int_\Omega vw|_0^T = \int_\mathcal{Q} (vw)_t = \int_\mathcal{Q} [(\Delta v)w - w(\Delta w)] \\ &= \int_\Sigma [v_n w - v w_n] = -\int_\Sigma (\varphi - \bar{\varphi})\chi. \end{aligned}$$

Since $\bar{\varphi} = \chi_+$, we have $\bar{\varphi}\chi = \chi_+^2$; since $\varphi \geq 0$ and $\chi = \chi_+ + \chi_-$ with $\chi_- \leq 0$, we have $\varphi\chi \leq \varphi\chi_+$. Thus,[4]

$$\|\chi_+\|^2 = \int_\Sigma \bar{\varphi}\chi = \int_\Sigma \varphi\chi \leq \int_\Sigma \varphi\chi_+ \leq \|\varphi\|\|\chi_+\|$$

so $\|\varphi\| \geq \|\chi_+\| = \|\bar{\varphi}\|$. Hence $\bar{\varphi} = \chi_+$ is the unique minimizer of $\mathcal{J}$ over $\mathcal{S} \cap \mathcal{C}$. □

We will see that this is a special case of Theorem 3.4(i), below. At this point we remark only that the linearity of (2.7) ensures that differences of elements of $\mathcal{S}$ must be in

$$\mathcal{N} = \{\varphi \in \mathcal{U} : \ \ (2.7) \text{ with } u(0) = 0 \text{ gives } u(T) = 0\}$$

which is obviously a closed subspace of $\mathcal{U}$ whence $\mathcal{S}$ has the form: $[\mathcal{N} + \hat{\varphi}]$ for any particular element $\hat{\varphi}$.

[4] We can have equality only if φ, χ_+ are dependent — which would give $\varphi = \bar{\varphi}$.

3 GEOMETRIC ANALYSIS

Throughout this section we assume we are given a Hilbert space $\mathcal{H}$ with a specified closed, convex set $\mathcal{C} \subset \mathcal{H}$ and a specified closed subspace $\mathcal{N} \subset \mathcal{H}$. We will consistently denote elements of $\mathcal{N}$ by ν (or $\bar{\nu}$ or ...), elements of $\mathcal{N}^\perp$ by μ, elements of $\mathcal{C}$ by γ, and general elements of $\mathcal{H}$ by ω.[5] Our concern will be to characterize the minimum-norm element of the 'vertical slice (by $\mathcal{N}$) of $\mathcal{C}$', i.e., $\mathcal{N} \cap \mathcal{C}$. The problem is *feasible* if $\mathcal{N} \cap \mathcal{C} \neq \emptyset$ and we say we have a *slackness condition* if $\mathcal{N} \cap \mathcal{C}^o \neq \emptyset$ where $\mathcal{C}^o$ is the interior of $\mathcal{C}$ (relative to $\mathcal{H}$).

Theorem 3.1 *Suppose there is some $\hat{\mu} \in \mathcal{N}^\perp$ for which $\bar{\gamma} := \mathbf{P}(\hat{\mu}; \mathcal{C})$, the nearest point in $\mathcal{C}$, is in $\mathcal{N}$. Then $\bar{\gamma}$ is the unique minimizer of $\|\gamma\|$ over $\mathcal{N} \cap \mathcal{C}$.*

PROOF: The 'nearest point property' of $\bar{\mu} + \bar{\nu} = \bar{\gamma} := \mathbf{P}(\hat{\mu}; \mathcal{C})$ just means that for every $\gamma = \mu + \nu \in \mathcal{C}$ one has

$$\|\bar{\mu} - \hat{\mu}\|^2 + \|\bar{\nu}\|^2 = \|\bar{\gamma} - \hat{\mu}\|^2 \leq \|\gamma - \hat{\mu}\|^2 = \|\mu - \hat{\mu}\|^2 + \|\nu\|^2;$$

note that $\bar{\mu} = 0$, here, since $\bar{\gamma} \in \mathcal{N}$ by assumption. Considering only $\gamma = \nu \in \mathcal{N} \cap \mathcal{C}$, this gives $\|\bar{\nu}\| \leq \|\nu\|$. Hence one also has

$$\|\bar{\gamma}\|^2 = \|\bar{\nu}\|^2 \leq \|\nu\|^2 = \|\gamma\|^2 \quad \forall \gamma \in \mathcal{N} \cap \mathcal{C}$$

so $\bar{\gamma}$ is the (unique) minimizer of $\|\gamma\|$ over $\mathcal{N} \cap \mathcal{C}$. □

Theorem 3.2 *Suppose one has the slackness condition[6] $\mathcal{N} \cap \mathcal{C}^o \neq \emptyset$. Then there must be some $\hat{\mu} \in \mathcal{N}^\perp$ such that $\mathbf{P}(\hat{\mu}; \mathcal{C}) = \bar{\nu} \in \mathcal{N}$, whence (1.5):*
$\{\mathbf{P}(0; \mathcal{N} \cap \mathcal{C})\} = \{\bar{\nu}\} = \mathcal{N} \cap \mathbf{P}(\mathcal{N}^\perp; \mathcal{C})$.

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[5] Thus, if we write $\omega = \mu + \nu$, then it is to be understood that $\mu \in \mathcal{N}^\perp$, $\nu \in \mathcal{N}$ so, e.g., $\|\omega\|^2 = \|\mu\|^2 + \|\nu\|^2$ by orthogonality.

[6] Of course, this slackness condition requires *a fortiori* that $\mathcal{C}$ have nonempty interior. This is a rather restrictive requirement which, e.g., does not hold for our prototypical control problem for which $\mathcal{C} = \mathcal{U}_{ad}$ is the positive cone in $\mathcal{H} = L^2(\Sigma)$. On the other hand, one might expect to be able to replace the topological interior $\mathcal{C}^o$ by the *core* of $\mathcal{C}$.

PROOF: Our argument proceeds along quite familiar lines — see, e.g., [3], [4]. We do however, sketch the proof directly since this is not much longer than would be required to cite and state results in the literature from which the desired result would follow 'quickly'.

The case: $0 \in \mathcal{C}$ is trivial as we could then simply take $\hat{\mu} = 0 \in \mathcal{N}^\perp$ with $\mathbf{P}(\hat{\mu};\mathcal{C}) = 0 \in \mathcal{N}$; thus we assume $0 \notin \mathcal{C}$. Set $\bar{\nu} = \mathbf{P}(0;\mathcal{N} \cap \mathcal{C})$ and note that $\bar{\nu} \notin \mathcal{C}^o$ since $0 \notin \mathcal{C}$. A standard characterization of $\mathbf{P}(0;\mathcal{N} \cap \mathcal{C})$ is then:

$$\langle \bar{\nu}, \nu - \mathcal{N} \rangle > 0 \quad \text{for } \bar{\nu} \neq \nu \in \mathcal{N} \cap \mathcal{C}. \tag{3.10}$$

Let p be the *Minkowski functional* (cf., e.g., [2]) of the set $(\mathcal{C} - \tilde{\nu})$:

$$p(\xi) := \min\{s \geq 0 : \xi \in s(\mathcal{C} - \tilde{\nu})\} \quad \text{for } \xi \in \mathcal{H}; \tag{3.11}$$

this is well-defined (finite) and continuous on $\mathcal{H}$ since $\tilde{\nu} \in \mathcal{C}^o$ gives $0 \in (\mathcal{C} - \tilde{\nu})^o$. Now let λ_0 be the linear functional on $\mathcal{N}$ given by

$$\lambda_0(\xi) := \frac{-\langle \bar{\nu}, \xi \rangle}{\langle \bar{\nu}, \tilde{\nu} - \bar{\nu} \rangle} \quad \text{for } \xi \in \mathcal{N}.$$

Note that $\bar{\nu} \neq \tilde{\nu}$ as $\tilde{\nu} \in \mathcal{C}^o$ so the denominator is nonzero. Using (3.10) we obtain

$$\lambda_0(\bar{\nu} - \tilde{\nu}) = 1, \quad \lambda_0(\nu - \bar{\nu}) < 0 \text{ for } \bar{\nu} \neq \nu \in \mathcal{C}.$$

For any $\xi \in \mathcal{H}$ we have $\xi = p(\xi)\xi_0$ with $\xi_0 \in \partial(\mathcal{C} - \tilde{\nu})$. For $\xi \in \mathcal{N}$ this gives $\xi = p(\xi)(\nu - \tilde{\nu})$ with $\nu \in \mathcal{C}$. Then

$$\begin{aligned} \lambda_0(\xi) &= p(\xi)[\lambda_0(\bar{\nu} - \tilde{\nu}) + \lambda_0(\nu - \bar{\nu})] \\ &\leq p(\xi)\lambda_0(\bar{\nu} - \tilde{\nu}) = p(\xi) \quad (\xi \in \mathcal{N}). \end{aligned} \tag{3.12}$$

By the Hahn-Banach Theorem (cf., e.g., [2] 3.4) we can extend λ_0 to a linear functional λ on $\mathcal{H}$ with

$$\lambda(\xi) \leq p(\xi) \tag{3.13}$$

so λ gives a support functional to $\mathcal{C}$ at $\bar{\nu}$. (Note that $\bar{\nu} \in \partial\mathcal{C}$ so $p(\bar{\nu} - \tilde{\nu}) = 1 = \lambda_0(\bar{\nu} - \tilde{\nu})$.) Since λ extends λ_0 from $\mathcal{N}$ to $\mathcal{H}$, it must have the form:

$$\lambda(\xi) = \frac{\langle \hat{\mu} - \bar{\nu}, \xi \rangle}{\langle \bar{\nu}, \tilde{\nu} - \bar{\nu} \rangle}$$

for some $\hat{\mu} \in \mathcal{N}^\perp$. For any $\gamma \in \mathcal{C}$ we set $\xi := \gamma - \tilde{\nu} = \mu + (\nu - \tilde{\nu})$ and have $1 \geq p(\xi) \geq \lambda(\xi)$. Scaling by $\langle \bar{\nu}, \tilde{\nu} - \bar{\nu} \rangle$ then gives

$$\langle \hat{\mu} - \bar{\nu}, \mu + \nu - \tilde{\nu} \rangle = \langle -\bar{\nu}, \bar{\nu} - \tilde{\nu} \rangle \lambda(\xi) \leq \langle -\bar{\nu}, \bar{\nu} - \tilde{\nu} \rangle$$

Hence,

$$\langle \hat{\mu}, \mu \rangle \leq \langle \bar{\nu}, \nu - \bar{\nu} \rangle \qquad (\forall \gamma = \mu + \nu \in \mathcal{C}).$$

Then, for $\gamma \in \mathcal{C}$,

$$\begin{aligned} \|\hat{\mu} - \gamma\|^2 &= \langle \hat{\mu} - \mu\nu, \hat{\mu} - \mu - \nu \rangle = \|\hat{\mu} - \mu\|^2 + \|\nu\|^2 \\ &= \|\hat{\mu}\|^2 + \|\mu\|^2 - 2\langle \hat{\mu}, \mu \rangle + \|\nu\|^2 \\ &\geq \|\hat{\mu}\|^2 + \|\mu\|^2 - 2\langle \bar{\nu}, \nu - \bar{\nu} \rangle + \|\nu\|^2 \\ &= \|\hat{\mu}\|^2 + \|\mu\|^2 + \|\bar{\nu}\|^2 + \|\bar{\nu} - \nu\|^2 \end{aligned}$$

which shows that $\|\hat{\mu} - \gamma\|^2$ is minimized, for $\gamma \in \mathcal{C}$, by taking $\gamma = \bar{\nu}$ — i.e., $\mathbf{P}(\hat{\mu}; \mathcal{C}) = \bar{\nu} \in \mathcal{N}$. We have shown that $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C}) = \bar{\nu} = \mathbf{P}(\hat{\mu}; \mathcal{C}) \in \mathcal{N} \cap \mathbf{P}(\mathcal{N}^\perp; \mathcal{C})$ and Theorem 3.1 shows that $\mathcal{N} \cap \mathbf{P}(\mathcal{N}^\perp; \mathcal{C})$ must be the singleton $\{\bar{\nu}\}$ so we have (1.5). $\square$

Theorem 3.3 *Suppose the problem is feasible: $\mathcal{N} \cap \mathcal{C} \neq \emptyset$. Then we have (1.6): $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C}) \in \mathcal{N} \cap \overline{\mathbf{P}(\mathcal{N}^\perp; \mathcal{C})}$.*

PROOF: Let $\mathcal{B}_\varepsilon$ denote the closed ball of radius $\varepsilon > 0$ and set

$$\mathcal{C}_\varepsilon = \mathcal{C} + \mathcal{B}_\varepsilon = \{\omega : \|\omega - \gamma\| \leq \varepsilon \text{ for some } \gamma \in \mathcal{C}\}.$$

We note first the observation that (assuming $\|\hat{\omega} - \mathcal{C}\| > \varepsilon$):

$$\bar{\omega} =:= \hat{\gamma} - \varepsilon \frac{\hat{\omega} - \hat{\gamma}}{\|\hat{\omega} - \hat{\gamma}\|}, \qquad \hat{\gamma} = \bar{\omega} + \frac{\varepsilon(\hat{\omega} - \bar{\omega})}{\|\hat{\omega} - \bar{\omega}\| + 2\varepsilon} \tag{3.14}$$

where $\hat{\gamma} := \mathbf{P}(\hat{\omega}; \mathcal{C})$ gives $\bar{\omega} = \mathbf{P}(\hat{\omega}; \mathcal{C}_\varepsilon)$. [*Proof*: Consider any $\omega \in \mathcal{C}_\varepsilon$ with $\|\omega - \gamma\| \leq \varepsilon$ for some $\gamma \in \mathcal{C}$. Then

$$\|\hat{\omega} - \bar{\omega}\| = \|\hat{\omega} - \hat{\gamma}\| - \varepsilon \leq \|\hat{\omega} - \gamma\| - \|\gamma - \omega\| \leq \|\hat{\omega} - \omega\|$$

so $\bar{\omega} = \mathbf{P}(\hat{\omega}; \mathcal{C}_\varepsilon)$.] Now (with a sequence $0 < \varepsilon = \varepsilon_n \to 0$), note that the feasibility condition for $\mathcal{C}$ implies the slackness condition for each $\mathcal{C}_\varepsilon$ so we may apply Theorem 3.2 to assert the existence of a corresponding sequence

$$\hat{\mu} = \hat{\mu}_n \in \mathcal{N}^\perp \text{ with } \mathbf{P}(\hat{\mu}_n; \mathcal{C}_\varepsilon) = \bar{\nu}_n := \mathbf{P}(0; \mathcal{N} \cap \mathcal{C}_\varepsilon). \tag{3.15}$$

Setting $\hat{\gamma} = \hat{\gamma}_n := \mathbf{P}(\hat{\mu}_n; \mathcal{C})$, we see from (3.14) that $\|\hat{\gamma}_n - \bar{\nu}_n\| \to 0$. Since $\mathcal{C} \subset \mathcal{C}_\varepsilon = \mathcal{C} + \mathcal{B}_\varepsilon$, we have $\|\bar{\nu}\| - \varepsilon_n \leq \|\bar{\nu}_n\| \leq \|\bar{\nu}\|$ where $\bar{\nu} := \mathbf{P}(0; \mathcal{N} \cap \mathcal{C})$. Hence (extracting a subsequence if necessary), we may assume weak convergence: $\bar{\nu}_n \rightharpoonup \tilde{\nu}$ so also $\bar{\gamma}_n \rightharpoonup \tilde{\gamma}$. Since $\bar{\gamma}_n \in \mathcal{C}$ and $\mathcal{C}$ is closed and convex (hence, weakly closed), we have $\tilde{\nu} \in \mathcal{C}$ so $\|\tilde{\nu}\| \geq \|\bar{\nu}\|$. On the other hand, $\|\tilde{\nu}\| \leq \liminf \|\bar{\nu}_n\| = \|\bar{\nu}\|$ so one has *strong* convergence: $\bar{\nu}_n \to \bar{\nu}$ and so $\hat{\gamma}_n \to \bar{\nu}$. By construction we have each $\hat{\gamma}_n := \mathbf{P}(\hat{\mu}_n; \mathcal{C}) \in \mathbf{P}(\mathcal{N}^\perp; \mathcal{C})$ so $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C}) = \bar{\nu} = \lim \hat{\gamma}_n$ is in $\overline{\mathbf{P}(\mathcal{N}^\perp; \mathcal{C})}$ as desired. $\square$

4 DISCUSSION

So far, in the last section, we considered only the problem of characterizing $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C})$. Often, however, we need to consider $\mathbf{P}(\hat{\omega}; \mathcal{S} \cap \mathcal{C})$ where $\mathcal{S}$ has the form $\mathcal{S} = \mathcal{S}_{\bar{\omega}} = [\mathcal{N} + \bar{\omega}] := \{\omega = \nu + \bar{\omega}\}$. We proceed now to reduce this more general problem to the simpler one $[\bar{\omega} = 0 = \hat{\omega}]$ by a translation. To this end we introduce $\bar{\mu} = \mathbf{P}(\bar{\omega}; \mathcal{N}^\perp)$ and $\hat{\nu} = \mathbf{P}(\hat{\omega}; \mathcal{N})$; then set $\omega' := \bar{\mu} + \hat{\nu}$ and $\mathcal{C}' := [\mathcal{C} - \omega']$. With this notation we obtain the *translation formula*:

$$\mathbf{P}(\hat{\omega}; \mathcal{S}_{\bar{\omega}} \cap \mathcal{C}) = \mathbf{P}(0; \mathcal{N} \cap \mathcal{C}') + \omega'. \tag{4.16}$$

[*Proof*: Observe first that $\mathcal{S}_{\bar{\omega}} = \mathcal{S}_{\bar{\mu}}$ so the minimization defining $\mathbf{P}(\hat{\omega}; \mathcal{S}_{\bar{\omega}} \cap \mathcal{C})$ is over γ of the form: $\gamma = \bar{\mu} + \nu \in \mathcal{C}$ and we have

$$\|\hat{\omega} - \gamma\|^2 = \|\hat{\mu} - \bar{\mu}\|^2 + \|\hat{\nu} - \nu\|^2.$$

Note that $\|\hat{\mu} - \bar{\mu}\|^2$ is an additive constant for this minimization and that $(\bar{\mu} + \nu) \in \mathcal{C}$ if and only if $(\nu - \hat{\nu}) \in \mathcal{C}'$. This gives (4.16).] Using (4.16), we can interpret our results somewhat more generally.

Theorem 4.1 *Let $\mathcal{H}$ be a Hilbert space; let $\mathcal{N} \subset \mathcal{H}$ be a closed subspace and set $\mathcal{S} = [\mathcal{N} + \bar{\omega}]$ for some $\bar{\omega} \in \mathcal{H}$; let $\mathcal{C} \subset \mathcal{H}$ be a closed, convex set. Then:*
(i) $\mathbf{P}(\mu + \hat{\omega}; \mathcal{C}) \in \mathcal{S}$ *for some* $\mu \in \mathcal{N}^\perp \implies \mathbf{P}(\hat{\omega}; \mathcal{S} \cap \mathcal{C}) = \mathbf{P}(\mu + \hat{\omega}; \mathcal{C})$;[7]
(ii) $\mathcal{S} \cap \mathcal{C}^o \neq \emptyset \implies$ *there exists μ as above so* $\{\mathbf{P}(\hat{\omega}; \mathcal{S} \cap \mathcal{C})\} = \mathcal{S} \cap \mathbf{P}([\mathcal{N}^\perp + \hat{\omega}]; \mathcal{C})$;
(iii) $\mathcal{S} \cap \mathcal{C} \neq \emptyset \implies$ *there is $\{\mu_n\}$ in $\mathcal{N}^\perp$ with* $\mathbf{P}(\mu_n + \hat{\omega}; \mathcal{C}) \to \mathbf{P}(\hat{\omega}; \mathcal{S} \cap \mathcal{C})$.

[7] In particular, on taking $\mathcal{C} = \mathcal{H}$ one recovers the 'trivial' result that $\{\mathbf{P}(\hat{\omega}; \mathcal{S})\} = \mathcal{S} \cap [\mathcal{N}^\perp + \hat{\omega}]$.

PROOF: This is just the restatement of Theorems 3.1, 2, 3 on using the translation formula (4.16).□

We may now ask how close to necessary is the slackness condition $\mathcal{S}\cap\mathcal{C}^o \neq \emptyset$ in Theorem 4.1.

EXAMPLE (*i*): Let $\mathcal{H} = \mathbb{R}^2$ with $\mathcal{N} = \{x = 0\}$; consider $\mathcal{C} = \{(x-1)^2 + (y-1)^2 \leq 1\}$. (Note that $\mathcal{C}^o \neq \emptyset$.) For $\hat{\omega} = 0$ and $\mathcal{S} = \mathcal{N}$ (i.e., $\bar{\omega} = 0$), one has $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C}) = (0,1)$. One easily sees, however, that $(0,1) \notin \mathbf{P}(\mathcal{N}^\perp; \mathcal{C})$, i.e., there is no $\mu \in \mathcal{N}^\perp$ such that $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C}) = \mathbf{P}(\mu; \mathcal{C})$. Thus, *some* auxiliary condition must be imposed to obtain the conclusion of (ii) in Theorem 4.1.

EXAMPLE (*ii*): Again let $\mathcal{H} = \mathbb{R}^2$ with $\mathcal{N} = \{x = 0\}$, $\hat{\omega} = 0 = \bar{\omega}$; now consider $\mathcal{C} = [0,1] \times [1,2]$. Note that again we have $\mathcal{S} \cap \mathcal{C}^o = \emptyset$ although $\mathcal{C}^o \neq \emptyset$. Now we have $\mathbf{P}(0; \mathcal{N} \cap \mathcal{C}) = (0,1) \in \mathbf{P}(\mathcal{N}^\perp; \mathcal{C})$ since $\mathbf{P}((x,0); \mathcal{C}) = (0,1)$ for all $x \leq 0$. Thus the conclusion of Theorem 4.1(ii) holds here even *without* the slackness condition.

EXAMPLE (*iii*): Theorem 4.1(iii) asserts that one always has

$$\{\mathbf{P}(\hat{\omega}; \mathcal{S} \cap \mathcal{C})\} \subset \mathcal{S} \cap \overline{\mathbf{P}([\mathcal{N}^\perp + \hat{\omega}]; \mathcal{C})} \tag{4.17}$$

but we see that this need not be an equality: even if there is a sequence $\{\mu_n\}$ in $\mathcal{N}^\perp$ with $\mathbf{P}(\mu_n + \hat{\omega}; \mathcal{C}) \to \gamma \in \mathcal{S} \cap \mathcal{C}$, one need not have[8] $\gamma = \mathbf{P}(\hat{\omega}; \mathcal{S} \cap \mathcal{C})$. To see this, let $\mathcal{H} = \mathbb{R}^3$ with $\mathcal{S} = \mathcal{N} = \{x = y = 0\}$ (so $\mathcal{N}^\perp = \{z = 0\}$) and consider the cone

$$\mathcal{C} = \{z \geq 1 + \sqrt{(x - z + 1)^2 + y^2}\}$$

with $\hat{\omega} = 0$. It is clear from the geometry that $\mathbf{P}(\mathcal{N}^\perp; \mathcal{C}) = \partial\mathcal{C} \backslash \mathcal{L}$ where $\mathcal{L} := \{(0,0,z) : z > 1\}$. Thus, $\overline{\mathbf{P}(\mathcal{N}^\perp; \mathcal{C})} = \partial\mathcal{C}$ and $\overline{\mathbf{P}(\mathcal{N}^\perp; \mathcal{C})} \backslash \{\mathbf{P}(0; \mathcal{N} \cap \mathcal{C})\} = \mathcal{L} \neq \emptyset$.

At this point we return to control theory and consider a general *LQ optimal control problem.* Thus, one has dynamics/control given linearly:[9]

$$x(\cdot) = \mathbf{B}u + x_0 \tag{4.18}$$

[8] By Theorem 4.1(ii), one of the consequences of the slackness condition is that this cannot occur — although it is clear, here also, that the slackness condition is not strictly necessary. It would be of interest, certainly, to find weaker conditions giving equality in (4.17).

[9] For example, if one had $\dot{x} = Ax + Bu$, $x(0) = \xi$ with A generating a semigroup $S(\cdot)$, then $x_0(t) = S(t)\xi$ and $[\mathbf{B}u](t) = \int_0^t S(t-s)Bu(s)\,ds$.

where $x_0(\cdot)$ is the 'uncontrolled' solution and $\mathbf{B}$ is the operator giving the effect on the state x of the control $u = u(\cdot) \in \mathcal{U}$. One also has a (continuous) quadratic cost functional:

$$
\begin{aligned}
(4.19) \qquad \mathcal{J} \;&:=\; \int_0^T \begin{pmatrix} x-\bar{x} \\ u-\bar{u} \end{pmatrix}^* \begin{pmatrix} Q_1 & T \\ R & Q_2 \end{pmatrix} \begin{pmatrix} x-\bar{x} \\ u-\bar{u} \end{pmatrix} dt \\
&\qquad\qquad + [x(T)-\xi_T]^* Q_3 [x(T)-\xi_T] \\
&= \; \mathcal{Q}(u) + 2\mathcal{L}(u) + c
\end{aligned}
$$

where, using (4.18),

$$
\begin{aligned}
\mathcal{Q}(u) \;&:=\; \int_0^T \Big[[\mathbf{B}u]^* Q_1 [\mathbf{B}u] + 2[\mathbf{B}u]^* R[u] + [u]^* Q_2 [u] \Big]\, dt + [\mathbf{B}u]_T^* Q_3 [\mathbf{B}u]_T \,, \\
\mathcal{L}(u) \;&:=\; \int_0^T \Big[[x_0-\bar{x}]^* Q_1 [\mathbf{B}u] + [x_0-\bar{x}]^* R[u] - [\mathbf{B}u]^* R[\bar{u}] - [u]^* Q[\bar{u}] \Big]\, dt \\
&\qquad\qquad + [x_0(T)-\xi_T]^* Q_3 [\mathbf{B}u]_T \,.
\end{aligned}
$$

We assume the quadratic form $\mathcal{Q}$ is coercive so $[\mathcal{Q}(\cdot)]^{1/2} =: \|\cdot\|_{\mathcal{Q}}$ is a *norm*; denote the resulting Hilbert space by $\mathcal{H}$. The continuity of $\mathcal{J}$ means that this $\mathcal{H}$-norm is at least as strong as the original and that $\mathcal{L}$ is a continuous linear functional on $\mathcal{H}$ and so can be represented in the form: $\mathcal{L}(u) = -\langle \lambda, u \rangle_{\mathcal{H}}$ for some $\lambda \in \mathcal{H}$. To within an additive constant we thus have

$$
(4.20) \qquad \mathcal{J}(u) := \|u-\lambda\|_{\mathcal{Q}}^2 .
$$

Next, suppose we impose an *affine* constraint on u, requiring that

$$
(4.21) \qquad u \in \mathcal{S} := [\bar{u} + \mathcal{N}]
$$

for some $\bar{u} \in \mathcal{H}$ and some (closed) subspace $\mathcal{N} \subset \mathcal{H}$. For example, one might omit the Q_3 terminal penalty term in $\mathcal{J}$ and, instead, absolutely impose the terminal constraint: $x(T) = \xi_T$ with ξ_T given. (We can obviously formulate such a state constraint as: $\mathbf{B}_T u = \bar{\xi} := \xi_T - x_0(T)$ where $\mathbf{B}_T$ is just $\mathbf{B}$ followed by evaluation at $t = T$.) More generally, we might consider any constraint of the form: $\mathbf{M}u = \zeta$ for some closed linear operator $\mathbf{M} : \mathcal{H} \to \mathcal{Z}$. (Above we had $\mathbf{M} = \mathbf{B}_T$.) If we have *any* particular control $\bar{u} \in \mathcal{U}$ with $\mathbf{M}\bar{u} = \zeta$ then, by linearity, we must have

$$
\{u \in \mathcal{H} : \ \mathbf{M}u = \zeta\} = \{u \in \mathcal{H} : \ (u-\bar{u}) \in \mathcal{N}\} = [\bar{u} + \mathcal{N}]
$$

where $\mathcal{N} \subset \mathcal{H}$ is the (closed) nullspace of $\mathbf{M}$.

Clearly, the unique minimizer of $\mathcal{J}$ (as in (4.19)≈(4.20)) subject to (4.21) must just be $\mathbf{P}(\lambda; [\bar{u} + \mathcal{N}]) = \bar{u} + \mathbf{P}(\lambda - \bar{u}; \mathcal{N})$.

Finally, let us adjoin another constraint — now a (nonlinear) convex constraint of the abstract form: $u \in \mathcal{C}$ with $\mathcal{C}$ a closed convex subset of $\mathcal{H}$. For $\mathcal{H}$ a *function space* as in (4.19), a plausible such constraint would be to require positivity of u — i.e., to take $\mathcal{C}$ to be the cone $\mathcal{H}_+ := \{u \in \mathcal{H} : u \geq 0 \text{ ae}\}$.[10] One might, alternatively (or additionally), be imposing a state constraint, e.g., a requirement that $x(\cdot) \in \hat{\mathcal{C}}(\cdot)$ pointwise ae, with each set $\hat{\mathcal{C}}(\cdot)$ closed and convex so $\mathcal{C} = \{u \in \mathcal{H} : [\mathbf{B}u](\cdot) \in [x_0(\cdot) + \hat{\mathcal{C}}(\cdot)] \text{ ae}\}$ would also be (closed and) convex.

The problem of characterizing the minimizer (subject to *both* constraints (4.21) and $u \in \mathcal{C}$) of $\mathcal{J}$ now becomes, abstractly, the problem (1.4) of characterizing $\mathbf{P}(\lambda; \mathcal{C} \cap \mathcal{S})$ with $\mathcal{S} = [\bar{u} + \mathcal{N}]$).

At this point we return to the example of Section 2 and identify the relevant elements.[11] For the example we have simply $\mathcal{U} = \mathcal{H} = L^2(\Sigma)$ with $\lambda = 0$ in (4.20). The operator $\mathbf{B}$ is just the map: $\varphi \mapsto u$ defined by (1.1) with $\omega_0 = 0$ and $\mathcal{N}$ is the nullspace of $\mathbf{B}_T : \varphi \mapsto [\mathbf{B}\varphi](T)$, i.e., the set of boundary data φ for (1.1) giving terminal state 0 from initial state 0. Next, what corresponds to $\bar{u}$ in (4.21) will be any boundary data $\hat{\varphi}$ for (1.1) giving $u(T) = \omega_T$, so $\mathcal{S}$ is as in (1.2). Finally, $\mathcal{C}$ is just $\mathcal{H}_+$, i.e., $L^2_+(\Sigma) = \{\varphi \in L^2(\Sigma) : \varphi \geq 0 \text{ ae on } \Sigma\}$.

The deepest result involved is that for this problem we have [6]

$$\mathcal{N}^\perp = \{\chi \in \mathcal{H} : \chi = \partial w / \partial n \text{ with } (2.9)\}. \tag{4.22}$$

(Note that in (4.22) the pde (2.9) holds on the *open* set $\mathcal{Q}$ and there is no suggestion that any condition at all be imposed on w at $t = T$.) One then easily sees that $\mathcal{N}^\perp$ is just the set of all optimal [minimum norm] controls

[10]Another frequently seen convex constraint is the requirement that: $|u| \leq 1$ ae. In this case the set of admissible controls is *already* bounded so it is not customary to require that the cost functional $\mathcal{J}$ be coercive. In general, the analysis in which we are engaged would then be inapplicable since we would not, as above, obtain a norm $\|\cdot\|_\mathcal{Q}$ and a Hilbert space $\mathcal{H}$.

[11]Unfortunately, we have let ourselves fall into the usual notational confusion due to the conflicting traditions of the control and pde communities: in the present (control) section we have used x for the solution and u for the control while in Section 2 we followed the relevant tradition and used u for the solution of the pde (2.7) with φ as control.

for affine problems (obtaining the terminal condition $u(T) = \omega_T$ from (1.1) with $\omega_0 = 0$; no positivity constraint imposed on φ) as ω_T varies over the reachable set.

Thus, Theorem 2.1 asserts that whenever there is data χ which is an optimal control for *some* affine problem of this form (i.e., $\chi \in \mathcal{N}^\perp$) and for which $\bar{\varphi} = \chi_+$ is admissible for the given problem (with the positivity constraint) then this control $\bar{\varphi}$ is the optimal one. Since one has $\mathbf{P}(\chi; \mathcal{C}) = \chi_+$ here, it is clear, noting (4.22), that Theorem 2.1 is just a special case of Theorem 4.1(i). On the other hand, since $L^2_+(\Sigma)$ has empty interior in $\mathcal{H}$, the second part of Theorem 4.1 cannot possibly be applicable. Thus we cannot at present assert that the optimal control is *always* of this form: the characterization is sufficient for optimality but cannot be shown to be necessary using our present approach — although Theorem 4.1(iii) does show that the optimal positive control must always be a *limit* of positive parts of 'affinely optimal' controls.

The general validity of such a characterization is, of course, precisely the contribution of the Pontrjagin Maximum Principle in the finite-dimensional case. To prove the corresponding result here seems rather difficult — although we note that there remains important additional information which we have not used, provided in this control-theoretic context by the structure of the problem in depending optimally on the control independently on disjoint sets (generalizing the viewpoint of invariant embedding). For comparison, we note that the rather similar problem of obtaining a bang-bang principle for the time-optimal boundary control problem for the (one-dimensional) heat equation has been solved only subject to the imposition of a slackness condition; cf., [5].

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