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CONTROL OF A FLEXIBLE ROBOT ARM [1,2]

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Abstract

We model a flexible robot arm supported at a *prismatic joint* so the length of (the relevant part of) the rod is variable — indeed, this variation in the length of the arm is taken to be the control action. The principal results are: exact controllability for the consideration of purely longitudinal vibrations (wave equation) and the existence of optimal controls in the more practically interesting case of transverse vibrations (Euler–Bernoulli beam equation). In each case, of course, one is considering a free boundary problem.

1. Introduction

The original stimulation to consideration of a robot arm involving the motion of a prismatic joint as control came from Paul Wang's talk [5] at the IFIP WG7 Workshop on Control of PDEs, held here in Florida (Gainesville) in 1986. A previous version of some of this material [2] was presented at the

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Fourth International Conference on Free Boundary Problems, held in Irsee, Germany, the following year.

We consider the modeling of a robot arm consisting of a flexible beam with a *prismatic joint* — that is, a mounting of the arm in which one end is clamped but permits controlled in-out motion so the extension of the arm (i.e., the relevant portion of the beam) has variable length $\ell = \ell(t)$. In this case 'flexible' means only that the dynamics of the beam must be modeled by a partial differential equation: treatment as a rigid body is an inadequate approximation in that vibration (i.e., time-dependent deviation from the nominal rigid-body position) is a significant aspect of the problem. The space-time domain on which we consider the dynamics is thus

$$\mathcal{Q} = \mathcal{Q}_T := \{(t, s) : 0 < t < T;\ 0 < s < \ell(t)\} \tag{1.1}$$

where s denotes length (in *material coordinates*, measured from the tip of the arm). This is certainly a *moving boundary problem.* From a control-theoretic point of view, we note that the significant question is precisely the determination of the control $\ell(\cdot)$ so we will have a *free boundary problem.*

For our present purposes we take the beam to be (nominally) horizontal along the x-axis of the 'laboratory coordinate system' with the joint rigidly fixed at the origin. Included in our model is a load of mass m supported at the free end of the arm. We assume that the system is (except for the boundary condition at the joint) conservative in that dissipative mechanisms such as friction, internal damping, and, e.g., air resistance can be neglected. For the case of transverse vibrations we also assume that 'body forces' (e.g., gravity) are either not present or are negligible.

Numerous variations are obviously of practical interest. In practice one would expect that the joint may, itself, be mounted on a moving support (e.g., on another arm) and/or that there is coupling with a controlled rotation around a vertical axis. One might also wish to include the possibility that the mounting is *nominally* rigid yet subject to vibration giving 'noise' in the boundary condition there (compare [1]) or involves 'play' which one might then attempt to model along the lines of [4], etc. Many interesting and important problems remain open.

In the next section we analyze purely longitudinal vibration of such a beam. This is not, of course, the problem of greatest practical concern but we will be able to treat it with some degree of completeness — showing *exact* *controllability*, subject only to some rather mild conditions. The more significant problem of treating transverse vibration in this context is significantly more difficult and we will only sketch our approach in Section 3 and then briefly discuss the formulation of some related problems.

2. Longitudinal Vibrations

Here we restrict our consideration to the purely *longitudinal* vibrations of the arm. (This is, of course, not really physically plausible for an arm viewed as rigid enough to be supported at a prismatic joint and supporting a tip mass — one would have to think of something like a *Slinky*, lying along a frictionless surface. The free boundary problem we consider, however, is nevertheless of some interest.)

Suppose we let $x(t, s)$ be the actual position (in the laboratory coordinate system) of the material point s at time t. The formulation of the dynamics takes a much simpler form if we take the unknown to be $v := [s + x]$ so, e.g., $v = \ell$ at the joint where $x = 0$, $s = \ell$. We will assume a hyperelastic stress-strain law, making the usual assumption that the *elastic potential*

$\Psi = \Psi(s,r)$ is convex in r with minimum at $\Psi(s,0) = 0$. We set $\psi(s,r) := \Psi_r(s,r)$ and let $\rho(s)$ be the mass density. We then have total potential and kinetic energies given by

$$\begin{aligned} PE &= \tfrac{1}{2}\int_0^{\ell(t)} \Psi(s,v_s)\,ds \\ KE &= \tfrac{1}{2}m(v_t-1)^2\,|_{s=0} + \tfrac{1}{2}\int_0^{\ell(t)} \rho[v_t-1]^2\,ds \end{aligned}$$

Using Hamilton's Principle of Stationary Action and a bit of manipulation, one can then easily show that v must satisfy the one-dimensional wave equation:

$$\rho\, v_{tt} = \psi(\cdot, v_s)_s \qquad \textit{on } \mathcal{Q}_T \tag{2.1}$$

with the boundary condition:

$$m\, v_{tt} = \psi(v_s) \qquad \text{at the tip: } s = 0 \tag{2.2}$$

together with the original boundary condition:

$$v = \ell \qquad \text{at the joint: } s = \ell = \ell(t). \tag{2.3}$$

To these we adjoin specification of initial conditions:

$$\ell(0) = \ell^0 \text{ and } \quad v = v^0,\ v_t = w^0 \text{ on } [0,\ell^0] \text{ at } t = 0, \tag{2.4}$$

subject to (2.3): $v^0(\ell^0) = \ell^0$ as a *compatibility condition.*

Given a particular motion $\ell = \ell(\cdot)$ on $[0,T]$, this *moving boundary problem* is the 'direct problem'. We now seek $\ell(\cdot)$ to meet a terminal condition, comparable to (2.4), i.e., so that one attains a specified target state for a terminal time $t = T$:

$$\ell(T) = \ell^T \text{ and } \quad v = v^T,\ v_t = w^T \text{ on } [0,\ell^T] \text{ at } t = T, \tag{2.5}$$

where again we impose (2.3) as a compatibility condition. Our object is to prove *exact controllability*:

> *Given data (2.4), (2.5) satisfying the compatibility condition (2.3), there should be a time $T > 0$ and an admissible control function $\ell(\cdot)$ on $[0,T]$ such that (2.1), (2.4), (2.3), (2.2) give (2.5)*

under suitable (mild) hypotheses. (Note that the state is a triple $[\ell, v, v_t]$ and the terminal time T must, in general, depend on this — we will, in fact, require that $T > \ell^0 + \ell^T$.) By "admissible" we mean that $\ell(0) = \ell^0$, $\ell(T) = \ell^T$, and that

$$|\ell(t_1) - \ell(t_2)| \leq \kappa |t_1 - t_2| \qquad \text{with } \kappa < 1 \tag{2.6}$$

— which is equivalent, for smooth $\ell(\cdot)$, to having $|\dot{\ell}(t)| \leq \kappa < 1$. Physically, this corresponds to a requirement that the controlled motion at the joint must always be slower than the wave propagation speed along the beam. Among other things, this will ensure that the free boundary curve: $\Gamma := \{s = \ell(t)\}$ is 'time-like' so a characteristic $[s \pm t = const.]$ can intersect Γ only once (transversely). Of course, admissibility includes some presumption that a (weak) solution of the problem exists with sufficient regularity for appropriate interpretation but this actually follows from explicit the construction given.

For simplicity of exposition we present in detail only the linear case: $\Psi(s, r) = \frac{1}{2}\beta(s) r^2$ so $\psi = \beta r$ and further simplify by assuming the beam is homogeneous so, in suitable units, $\rho = \beta = 1$. Then (2.1) reduces to

$$v_{tt} = v_{ss} \qquad \text{on } \mathcal{Q}_T, \tag{2.7}$$

i.e., one has the standard wave equation — although, of course, not a standard problem for this. In this case we will have a fairly explicit (although somewhat messy) construction of the desired motion $\ell(\cdot)$ through the *method of characteristics* which corresponds to the simple decomposition

$$v(t, s) = f(s + t) + g(s - t) \tag{2.8}$$

into left- and right-moving waves and we note that differentiating (2.8) gives the identities:

$$\begin{aligned} f'(s + t) &= [v_s(t, s) + v_t(t, s)]/2, \\ g'(s - t) &= [v_s(t, s) - v_t(t, s)]/2. \end{aligned} \tag{2.9}$$

The representation (2.8) gives the general (weak) solution of (2.7) and, in terms of f, g, the boundary conditions (2.3), (2.2) now become:

$$f(\ell(t) + t) + g(\ell(t) - t) = \ell(t), \tag{2.10}$$

$$m[f''(s) + g''(-s)] = f'(s) + g'(-s). \tag{2.11}$$

Note that we can differentiate (2.10) with respect to t to obtain an ODE for ℓ:

$$\dot{\ell} = \frac{f'(\ell+t) - g'(\ell-t)}{1 - [f'(\ell+t) - g'(\ell-t)]} \quad . \tag{2.12}$$

From the inequality: $|u-v| + |u+v| \leq 2\max\{|u|,|v|\}$ it would follow that (2.12) is well-behaved — indeed, giving (2.6) — if we were always to have

$$|f'|, |g'| \leq \kappa/2 \tag{2.13}$$

where defined and we impose conditions on the data to ensure this.

The initial data v^0, w^0 in (2.4) determine f, g on $[0, \ell^0]$ using (2.9) with $t = 0$. In particular, we have g' determined on $\mathcal{D}_0^g := \{(t,s) : 0 \leq s-t \leq \ell^0\}$. We then can use (2.11) as an ODE for g' to obtain:

$$\begin{aligned} g'(s) \quad &= \quad \tfrac{1}{2}[v_s^0 + w^0](-s) + e^{s/m} w^0(0) \\ &\qquad + \tfrac{1}{m} \int_0^{-s} e^{(s+r)/m} [v_s^0 + w^0](r)\, dr \end{aligned} \tag{2.14}$$

for $-\ell^0 \leq s \leq 0$, which defines g' from (2.4) on $\mathcal{D}_1^g := \{(t,s) : -\ell^0 \leq s-t \leq 0\}$. Finally, we note that $v(\ell^0, 0)$ is determined by the data of (2.4).

Similarly, we note that the target data in (2.5) directly determine f' on $\mathcal{D}_0^f := \{(t,s) : T \leq s+t \leq \ell^T + T\}$, using (2.9) with $t = T$, and give, now using (2.11) as an ODE for f':

$$\begin{aligned} f'(T-\tau) \quad &= \quad \tfrac{1}{2}[v_s^T + w^T](\tau) + e^{-\tau/m} w^T(0) \\ &\qquad + \tfrac{1}{m} \int_0^{\tau} e^{-r/m} [v_s^T + w^T](r)\, dr \end{aligned} \tag{2.15}$$

for $0 \leq \tau \leq \ell^T$, which defines f' from (2.5) on $\mathcal{D}_1^f := \{(t,s) : T - \ell^T \leq s+t \leq T\}$. To avoid any possibility of conflict we assume that T is chosen so $T > \ell^0 + \ell^T$, giving $-\ell^0 < \ell^T - T$ and $\ell^0 < T + \ell^T$; later we will strengthen this requirement to (2.18). Since the equation is autonomous, we note that the computation, here, from (2.5) is independent (to within translation in t) of the particular choice of T. From (2.5) we also, then, determine $v(T - \ell^T, 0)$.

We now choose Cauchy data on the segment: $\mathcal{G} := \{(t,0) : \ell^0 \leq t \leq T - \ell^T\}$, taking $v(t,0)$ linear in t, making v_t constant:

$$v_t(t,0) = c := \frac{v(T-\ell^T, 0) - v(\ell^0, 0)}{T - [\ell^0 + \ell^T]} \tag{2.16}$$

on $\mathcal{G}$. This gives $v_s(t,0) = m\, v_{tt} = 0$ by (2.2) with $\psi(r) = r$. Using (2.9), we now have

$$(2.17) \qquad \begin{aligned} f' &= \ \ c/2 \quad \text{on } \mathcal{D}_2^f &&:= \{(t,s) : \ell^0 \leq s+t \leq T - \ell^T\}, \\ g' &= -c/2 \quad \text{on } \mathcal{D}_2^f &&:= \{(t,s) : \ell^T - T \leq s-t \leq -\ell^0\}. \end{aligned}$$

Note that if we require that

$$(2.18) \qquad T \geq [\ell^0 + \ell^T] + |v(T-\ell^T, 0) - v(\ell^0, 0)|/\kappa,$$

then (2.16), (2.17) give (2.13) on $\mathcal{D}_2^f, \mathcal{D}_2^g$. If we now impose the requirement that

$$(2.19) \qquad \begin{aligned} |v_s^0|, |w^0| \leq \kappa/4 \quad \text{on } [0, \ell^0], \\ |v_s^T|, |w^T| \leq \kappa/4 \quad \text{on } [0, \ell^T], \end{aligned}$$

then (2.13) holds on $\mathcal{D}_0^f, \mathcal{D}_0^g$ directly from (2.9) and it easily follows from (2.14), (2.15), that we will also have (2.13) on $\mathcal{D}_1^f, \mathcal{D}_1^g$.

THEOREM 1: *Suppose one is given the data of (2.4), (2.5) subject to compatibility with (2.3) and the condition (2.19) with $\kappa < 1$. (Assume w^0, w^T are continuous and v^0, $v^T \in C^1$.) Then there exists $T > 0$ and an admissible control function $\ell(\cdot) \in C^1[0,T]$ such that the solution v of (2.7), (2.4), (2.3), (2.2) also satisfies (2.5).*

PROOF: As above, solve (2.7), (2.4), (2.2) on the triangle $\Delta_0 = \{(t,s) : 0 \leq t, s;\ s+t \leq \ell^0\}$ and (2.7), (2.5), (2.2) on $\Delta_T = \{(t,s) :\ 0 \leq T-t, s;\ s-t \leq \ell^T - T\}$ — obtaining, in particular, $v(\ell^0, 0)$ and $v(T-\ell^T, 0)$. Then choose T subject to (2.18) and specify Cauchy data on $\mathcal{G}$ as above — i.e., (2.16) and $v_s = 0$. Thus, f is determined on $\mathcal{D}^f = \mathcal{D}_0^f \cup \mathcal{D}_1^f \cup \mathcal{D}_2^f$ and g on $\mathcal{D}^g = \mathcal{D}_0^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g$ with (2.13) satisfied on $\mathcal{S} = \mathcal{D}^f \cap \mathcal{D}^g$ as in the discussion above. Note that our assumptions give f', g' continuous on $\mathcal{S}$.

Starting with the initial condition: $\ell(0) = \ell^0$, solve (2.12) to obtain Γ. A solution exists since the right hand side of (2.12) is continuous in ℓ (as long as one remains in $\mathcal{S}$). Our assumptions do not give a Lipschitz condition; thus uniqueness — not really needed for the Theorem — would have to be obtained by a different argument: noting that (2.13) makes the right hand side of (2.10) contractive in ℓ. This solution curve satisfies (2.10) and (2.6). Note that the solution curve Γ cannot leave $\mathcal{S}$ before $t = T$ by, e.g., crossing the characteristic: $[s+t=\ell^T+T]$ — say, at a point (ℓ_*, t_*) — since this would give, by (2.10):

$$\begin{aligned}\ell^T+T-t_* &= \ell_* = f(\ell_*+t_*)+g(\ell_*-t_*)\\ &= f(\ell^T+T)+g(\ell^T+T-2t_*)\\ &= [\ell^T-g(\ell^T-T)]+g(\ell^T+T-2t_*),\end{aligned}$$

contradicting the fact that $|g'| \leq \kappa/2 < 1/2$; similarly, Γ cannot leave $\mathcal{S}$ by crossing $[s-t=\ell^T-T]$ since $|f'|<1/2$. It follows that $\ell(\cdot)$ is well-defined on $[0,T]$ with $\ell(T)=\ell^T$.

This specification of $T, \ell(\cdot)$ defines $\mathcal{Q}_T$. In view of (2.6), the geometry of Γ ensures that $\mathcal{Q}_T \subset \Delta_0 \cup \Delta_T \cup \mathcal{S}$ so f, g are both defined on all of $\mathcal{Q}_T$. Now (2.8) gives a (weak) solution v of (2.7) and the construction ensures that we also have (2.4), (2.5), (2.3), (2.2) as desired. □

Remarks: The regularity required for the data — continuity of v_s^0, w^0, v_s^T, w^T — can be weakened by a limit argument. In this case one retains (2.6) although $\dot{\ell}$ may no longer be continuous. There is also no really new difficulty in considering spatially variable coefficients: $\rho\, v_{tt} = (\beta\, v_s)_s$. The argument must be modified in detail but is essentially similar in spirit. E.g., the characteristics are no longer straight lines and the propagation along them may no longer be constant. We now must take κ to be a constant less than the *minimum* sound speed and impose a more stringent condition (2.19). □

It is worth noting that the construction in the proof above can be viewed from a slightly different perspective. After solving on Δ_0 and Δ_T one fills in Cauchy data on a 'gap' $\mathcal{G}$ so as to consider an *initial value problem* with the roles of t and s reversed — taking, say, v^0, v^T (perhaps suitably extended to larger s) as *boundary data*. The choice of data on $\mathcal{G}$ and the bounds (2.19) are such as to ensure that $|v_s| \leq \kappa < 1$ which then makes the right hand side of (2.10) contractive in ℓ so (2.10) determines $\ell(t)$ uniquely for each t. (One must also, of course, verify that this gives $\ell(\cdot)$ as an 'admissible control function'.)

Stated this way, it is clear that the argument does not really depend on the linearity of (2.7) or on the possibility of a decomposition (2.8). Without providing details, we note only that one can treat (2.1) from this point of view under quite mild conditions on $\psi(\cdot)$. If we set $\phi := \psi^{-1}$, $w = v_t$ and $u = \psi(v_s)$ so $v_s = \phi(u)$ we can rewrite (2.1) in terms as a system:

(2.20) $$u_s = \rho\, w_t \qquad w_s = \phi(u)_t$$

suggesting the role-reversed problem. The crux of the argument, now, is to show that 'reasonable bounds' on the initial and target data can be imposed to ensure that one will have (for a suitable construction of $\mathcal{G}$ and of the data there, satisfying $m\, w_t = u$) the necessary bound: $|v_s| = |\phi(u)| \leq \kappa$ to be able to apply the Contractive Mapping Theorem to (2.10). $\square$

3. <u>Transverse Vibration</u>

In this section we present an extremely brief introduction to the class of problems which might be considered. As above, we consider the robot arm as a nominally horizontal beam with a load m at the tip and with no dissipation or body forces (i.e., ignoring friction, gravity, ...). Here, however, we consider *transverse vibrations*. For simplicity we consider only vertical vibration, so the principal unknown is the vertical deflection $u = u(t, s)$. Thus, the 'laboratory coordinates' of a material point s will be $[x, y, z] = [\ell(t) - s, 0, u(t, s)]$.

For simplicity we will ignore shear and will work with a geometrically linearized theory. Note that the use of a linearized stress-strain relation is a consequence of the fundamental beam assumption: 'infinitesimal' cross-section, giving infinitesimal deformations. The significant assumption here is the simplifying but rather restrictive one that the arm always remains straight enough to ignore the so-called 'geometric nonlinearities' — essentially, that one can adequately approximate [radius of curvature]$^{-2}$ by u_{ss}^2, linear velocity by $[\dot{\ell}\,\mathbf{i} + u_t\mathbf{k}]$, angular velocity $[\arctan u_x]_t$ by $-u_{st}$, etc., e.g., (noting that a geometric analysis shows that the stress deformation for a segment ds of the beam is proportional to [radius of curvature]$^{-2}$) the strain energy (elastic potential) is $\approx \frac{1}{2}\beta\, u_{ss}^2 ds$ where β (sometimes written as EI) corresponds to stiffness.

We begin by observing that at the prismatic joint one has 'clamped' boundary conditions:

$$u = u_s = 0 \qquad \text{at the joint: } s = \ell(t). \tag{3.1}$$

In this linearized setting the potential energy PE and the corresponding kinetic energy KE are given by:

$$\begin{aligned} PE &= \tfrac{1}{2}\int_0^{\ell(t)} \beta\, u_{ss}^2\, ds \\ KE &= \tfrac{1}{2}\int_0^{\ell(t)} [\rho\,(\dot{\ell}^2 + u_t^2) + \alpha^2\, u_{st}^2]\, ds \\ &\qquad + \tfrac{1}{2}[m\,(\dot{\ell}^2 + u_t^2) + a\, u_{st}^2]|_{s=0} \end{aligned}$$

where α^2 is the cross-sectional moment. Again using Hamilton's Principle, to derive the dynamics, we obtain:

$$\rho\, u_t - (\alpha^2\, u_{stt})_s + (\beta\, u_{ss})_{ss} = 0 \qquad \text{on } \mathcal{Q}_T \tag{3.2}$$

(where $\mathcal{Q}_T$ is as above in (1.1)) and at the tip $s = 0$, the other boundary conditions:

$$m\, u_t - \alpha^2\, u_{stt} = (\beta\, u_{ss})_s, \qquad a\, u_{stt} + \beta\, u_{ss} = 0. \tag{3.3}$$

Note that we have just obtained the standard Euler-Bernoulli beam equation by the use of s as the independent variable. [This formulation is somewhat different from that of [5].]

The situation is somewhat different when $\alpha = 0$ so we restrict consideration to the case $\alpha > 0$. For simplicity we assume ρ, β, α are constants and note that we can now choose units of mass, time, length to make each of these 1. The hyperbolic nature of the equation is made evident by setting $w := u_{ss} - u$ so (3.2) now becomes:

$$w_{tt} = w_{ss} + (w + u) \tag{3.4}$$

which is 'almost' the telegrapher's equation.

It is physically reasonable to use the force at the prismatic joint as control — or, more-or-less equivalently, we take $\varphi := \ddot{\ell}$ as control. [Note that we now have specification of $\dot{\ell}$ as part of the state and we retroactively include these in (2.4) and (2.5).] At this point we introduce a cost functional and ask whether one attains optimal controls. The exact form of this functional is not very significant but we may, for definiteness, think of it as

$$\begin{aligned} \mathcal{J} \;&=\; \int_0^T \left[(\varphi - \bar{\varphi})^2 + \int_0^\ell |u_s - \bar{u}_s|^2\, ds \right] dt \\ &\quad + \mu \left[(\ell(T) - \lambda)^2 + \int_0^{\ell(T)} |u(T,s) - \omega(s)|^2\, ds \right] \end{aligned} \tag{3.5}$$

where $\bar{\ell}$, $\bar{u}$, μ, λ, ω are suitably given. We might, further, want to impose certain constraints — in particular, bounding $\ell(\cdot)$ away from 0 below by ℓ_* and by ℓ^* above as well as imposing some bound on $\dot{\ell}$ as in the previous section; with our choice of $\ddot{\ell}$ as control, these are well under control.

There is no difficulty is beginning the usual argument: it is clear that this functional is bounded below and we can assume a minimizing sequence $\{\varphi_\nu\}$, giving admissible motions $\{\ell^\nu\}$ (each satisfying the imposed constraints!) and corresponding solutions $\{u^\nu\}$ (once one verifies that, indeed, the direct problem is sufficiently well-posed to ensure that there are such solutions); the form of $\mathcal{J}$ ensures that $\{\varphi_\nu\}$ is L^2-bounded so one may extract a weakly convergent subsequence $\varphi_\nu \rightharpoonup \varphi_\infty$. One might anticipate little difficulty with the convexity, hence with the weak lower semicontinuity, of this functional from its form — until one observes that the functions $\{u^\nu\}$ will each be defined on a different set $\mathcal{Q}_T$.

As in [3], in a similar context, we proceed by reformulating the equation to put everything on a common domain. To this end we make the substitution $s = \sigma\ell$ so, always, $0 < \sigma < 1$. The equation now becomes rather messy, especially since it is convenient also to introduce a new 'time' variable τ such that $dt/d\tau = \ell$ ($\tau := \int^{\cdot}[1/\ell]$) and then set $\hat{w}(\tau,\sigma) := w(t,s)$. The computations are complicated but are manipulations at an 'Advanced Calculus' level and, working with the equation in, say, the form (3.4), we now obtain

$$\begin{aligned} \hat{w}_{\tau\tau} - 2\sigma\vartheta\hat{w}_{\sigma\tau} - [1 + \sigma\vartheta - 2\sigma^2\vartheta^2]\hat{w}_{\sigma\sigma} \\ = \vartheta\hat{w}_\tau - \sigma[\ell\varphi - 2\vartheta^2]\hat{w}_\sigma + (\hat{w} + \hat{u}) \end{aligned} \tag{3.6}$$

where $\vartheta := d\ell/dt = (1/\ell)d\ell/d\tau$. In this form we can at least look at the question of convergence of a sequence of solutions as functions defined for $0 < \sigma < 1$, etc.; note that we have $0 < \tau < \bar{\tau}_\nu \to \bar{\tau}_\infty$ (as $\varphi_\nu \rightharpoonup \varphi_\infty$ and we have imposed a constraint bounding ℓ away from 0) so we may look at convergence on a fixed domain $(0,1) \times (0, \bar{\tau}_\infty)$. To attack (3.6) we need hyperbolicity and in this form we see that the eigenvalues of the relevant quadratic form will have opposite signs (for each $\sigma \in (0,1)$) provided one has

$$-(\sqrt{5}+1) < 2\vartheta < (\sqrt{5}-1) \tag{3.7}$$

which, above, is a bit more stringent than the obvious condition $|\dot{\ell}| < 1$.

Without further detail, we assert at this point that one can, indeed, get the expected estimates to bound $\{u^\nu, \ldots, u^\nu_{sss}\}$ and obtain suitable convergence (for subsequences) as is needed for the terms in $\mathcal{J}$, thus ensuring that the limit is, indeed, the solution for the limit problem using ℓ_∞ — which then minimizes $\mathcal{J}$, providing the desired optimum. Again without further detail, even to state the final formula, we note that the derivation of the first order necessary conditions for optimality (including, of course, the effects of the constraints imposed, which actually were needed to work with the new form) can be obtained most easily for the problem after the reformulation to a common domain in terms of the variables $[\tau, \sigma]$.

As a rather different kind of problem, we note that Wang has considered stabilization of essentially this beam by a feedback at the prismatic joint. It would be interesting to see whether one might be able to work with a more general feedback involving an *observation operator* $\mathbf{C}$ which might be implementable by, say, visual observation of the tip position and placing an accelerometer on the beam at s_*, allowing for the (known) effect of the joint motion. This would give, as an example,

$$\ell(t) = \Phi(u(t,0), u_{tt}(t, s_*)) \tag{3.8}$$

for some given function Φ. If one were to attempt such an approach, it would be necessary to analyze the resulting free boundary problem to determine whether (a) it would be well-posed and (b) whether the desired steady state would, in fact, be a stable (local) attractor for the controlled dynamics.

References

[1] W.J. Bottega, "Dynamics and stability of support-excited beam-columns with end mass", *Dyn. and Stab. of Syst.* **1**, 1986, pp. 201–215.

[2] T.I. Seidman, "Free boundary problems arising in the control of a flexible robot arm", in *Free Boundary Problems: Theory and Applications* (Proc. 4^{th} Int'l. Conf on Free Boundary Problems; Irsee, 1987), (K.-H. Hoffmann and J. Sprekels, eds.), Birkhäuser, Basel, to appear.

[3] T.I. Seidman, "Some control-theoretic questions for a free boundary problem", in sl Control of Partial Differential Equations, (Lect. Notes Control Inf. Sci. **#114**, A. Bermúdez, ed.), pp. 265–276, Springer-Verlag, New York, 1989.

[4] M.W. Spong, "Modeling and control of elastic joint robots", *J. Dyn. Syst., Meas., Control* **109**, 1987, pp. 310–319.

[5] P.K.C. Wang, "Some nonlinear problems in the control of distributed systems", in *Control Problems for Systems Described by Partial Differential Equations and Applications* (Lect. Notes Control Inf. Sci. **#97**) (I. Lasiecka and R. Triggiani, eds.), pp. 369–388, Springer-Verlag, Berlin, 1987.