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An Approximation of the Budyko-Widiasih Model with the Jormungand Albedo Function

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The Budyko-Widiasih Model

Energy Balance Models (EBM) are conceptual models that are developed to analyze the long-term behavior of specific system variables. The Budyko-Widiasih model describes the Earth's ice-albedo feedback system and incorporates the dynamics of the ice line, the outermost edge of the polar glaciers. It is defined by the Budyko Model and Widiasih's ice line condition.
The Budyko Model is defined as follows:

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T).$$

T is the annual average surface temperature as a function latitude. The latitude is described by the variable y which is defined as the sine of the compliment of the polar angle. The Budyko model assumes that the temperature is distributed symmetrically across the equator and thus it suffices to consider $y \in [0,1]$ where $y=0$ is the equator and $y=1$ is the North Pole. Q is the global average incoming solar radiation (insolation) while s describes the distribution of the insolation as a function of the latitude. The albedo α describes the percent of the Sun's energy reflected by the Earth. Thus the energy that enters the Earth's system at a given latitude is $Qs(y)(1-\alpha(y,\eta))$.
The process through which the Earth radiates energy is complicated by the greenhouse effect. The Budyko model approximates the outgoing radiation using the equation $A+BT$ where A and B are determined by satellite data. The last part of the Budyko model describes the transport of energy between latitudes, also known as the meridonal heat transport. This process is complex and involves winds and ocean currents but its effects can be approximated by a simple linear relaxation to the mean. Thus we define $\overline{T}$ as the mean Earth temperature:

$$\overline{T} = \int_0^1 T(y)dy.$$

Therefore the equation $C(\overline{T}-T)$ describes the meridional heat transport. The constant C is determined from data.
We will denote the ice line as $\eta \in [0,1]$. Widiasih's ice line condition models the change in the ice line as a function of the temperature:

$$\frac{d\eta}{dt} = \epsilon(T(\eta,\eta) - T_c).$$

$T(\eta,\eta)$ is defined as the average of the temperatures just above and just below the ice line, or more rigorously

$$T(\eta,\eta) = \frac{1}{2}(\lim_{y\to\eta^-} T^*(y,\eta) + \lim_{y\to\eta^+} T^*(y,\eta))$$

The constant T_c denotes the critical temperature for melting the glaciers.

Our goal is to understand the equilibrium behavior of the ice line.

The Jormungand Albedo Function

The albedo α is the percent of solar radiation reflected off the Earth's surface. In our model, α is described by the Jormungand Albedo Function. The Jormungand Albedo Function assumes the Earth's surface is covered with water. Sea ice is only covered with snow if there is sufficient precipitation. An argument can be made to show that the parts of the glaciers that extend below the latitude $\rho = .35$ would not be covered in snow. Denote the albedo of snow covered ice as α_s, non-snow covered ice as α_i, and the albedo of water as α_w. We define the albedo function as:

$\eta < \rho$	$\eta > \rho$
$\alpha(y,\eta) = \begin{cases} \alpha_s & y > \eta \\ \alpha_i(\eta) & \rho < y < \eta \\ \alpha_w & y < \rho \end{cases}$	$\alpha(y,\eta) = \begin{cases} \alpha_s & y > \eta \\ \alpha_w & y < \eta \end{cases}$

where $\alpha_i(\eta) = \frac{\alpha_s - \alpha_i}{\rho}\eta + \alpha_i$ and $\alpha_w < \alpha_i < \alpha_s$.

Simulation

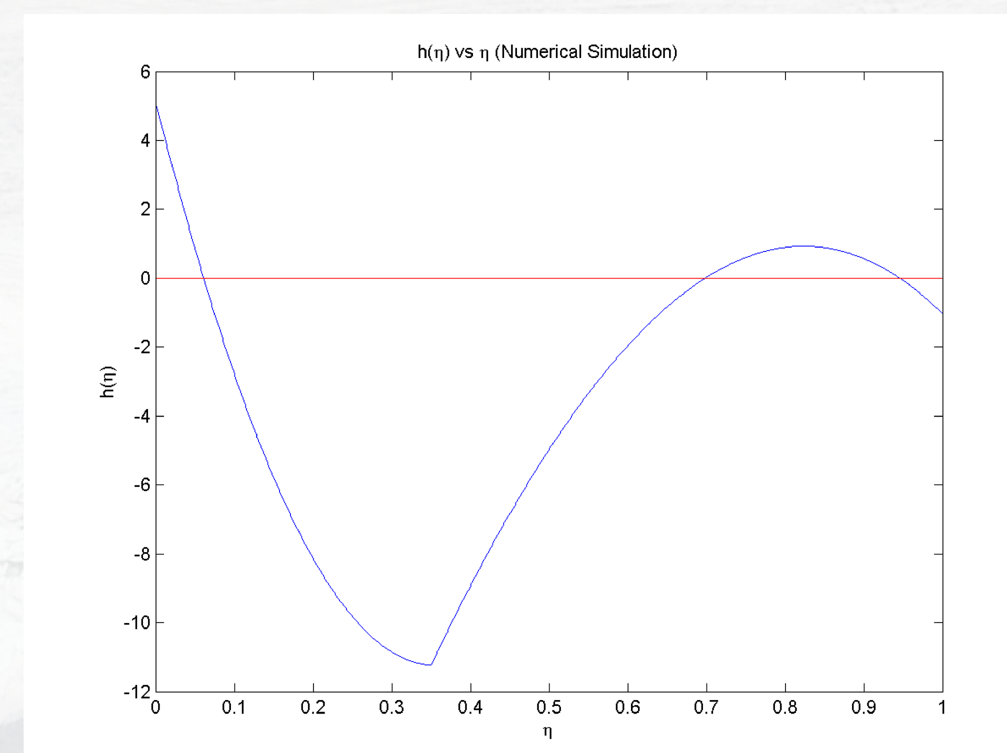
We wish to understand the equilibrium solution of the ice line. We can solve an attracting one-dimensional curve

$$h(\eta) = T^*(\eta,\eta) - T_c = \frac{Q}{B+C}(s(\eta)(1-\alpha(\eta,\eta)) + \frac{C}{B}(1-\overline{\alpha}(\eta))) - \frac{A}{B} - T_c$$

where

$$\overline{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy.$$

Below we show the resulting $h(\eta)$ found computationally using MATLAB.



This curve can be shown to satisfy the equation

$$\frac{d\eta}{dt} = \epsilon h(\eta).$$

Thus the equilibrium ice line latitudes are found by setting $h(\eta) = 0$. These results show the existence of three equilibrium solutions. $h(\eta) > 0$ leads to the melting of ice which decreases the ice line. $h(\eta) < 0$ leads to the freezing of water which increases the ice line. From these facts we see that the solution $\eta \approx .06$ is a stable equilibrium, $\eta \approx .65$ is an unstable equilibrium, and $\eta \approx .95$ is a stable equilibrium.

Solving the Model

Previous research has solved for the case where $\eta > \rho$. Thus we solve for the equilibrium where $\eta < \rho$. Define the temperature piecewise as

$$T(y) = \begin{cases} U(y) & y < \eta \\ V(y) & \eta < y < \rho \\ W(y) & y \geq \rho \\ \frac{1}{2}(U(\eta)+V(\eta)) & y = \eta \end{cases}$$

Approximate the functions U, V, and W as quadratic functions and substitute to write the model as

$$\begin{aligned} \frac{d\eta}{dt} &= \epsilon(T_b - T_c), \\ \frac{da}{dt} &= \frac{1}{R}(Q(1-\frac{1}{2}(\alpha_s + \frac{1}{2}(\alpha_w + \alpha_i(\eta)))) - A - (B+C)a + C\bar{T}(\eta)), \\ \frac{dz}{dt} &= \frac{1}{R}(Q(\alpha_i(\eta) - \alpha_w) - (B+C)z), \end{aligned}$$

where a and z are functions of η, U, V, and W.

Geometric Singular Perturbation Theory Approximation

Since $0 < \epsilon \ll 1$, the functions can be approximated away from the equilibrium solutions by the fast subsystem:

$$\begin{aligned} \frac{d\eta}{dt} &= 0, \\ \frac{da}{dt} &= \frac{1}{R}(Q(1-\frac{1}{2}(\alpha_s + \frac{1}{2}(\alpha_w + \alpha_i(\eta)))) - A - (B+C)a + C\bar{T}(\eta)), \\ \frac{dz}{dt} &= \frac{1}{R}(Q(\alpha_i(\eta) - \alpha_w) - (B+C)z). \end{aligned}$$

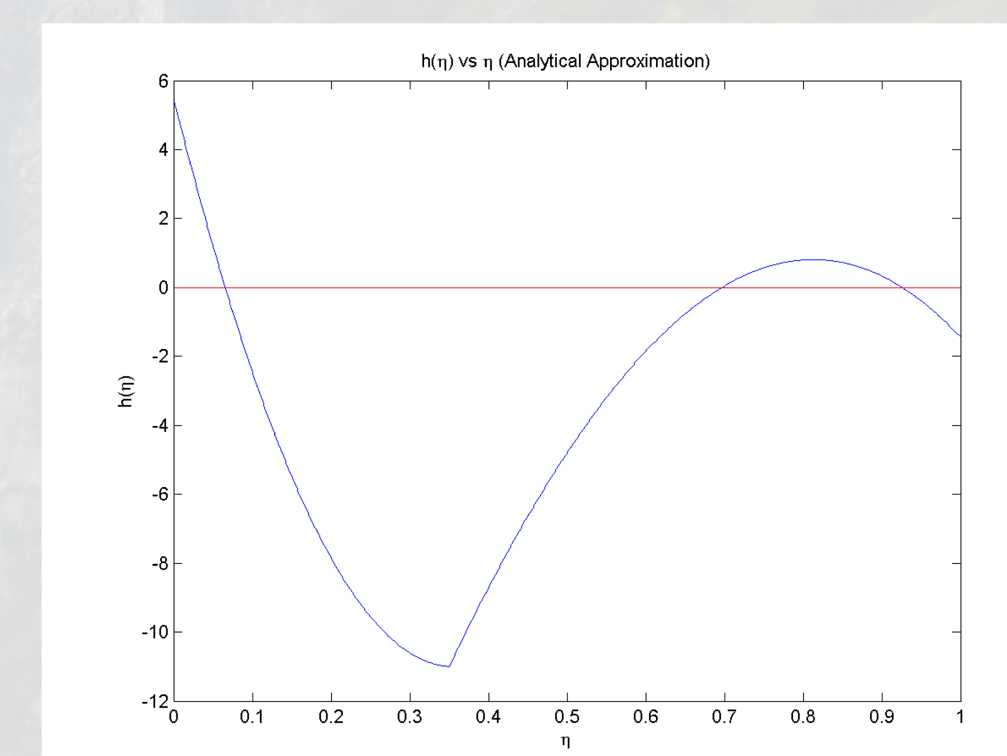
Thus, since η is a constant, we can solve for the equilibrium values of z and a:

$$\begin{aligned} z(\eta) &= \frac{Q(\alpha_i(\eta) - \alpha_w)}{B+C}, \\ a(\eta) &= \frac{1}{B}(Q(1-\frac{1}{2}(\alpha_s + \frac{1}{2}(\alpha_w + \alpha_i(\eta))) - A + C\mu(\eta)). \end{aligned}$$

We can thus solve for the attracting curve

$$h^0(\eta) = a + \frac{1}{4}(e-z) + \frac{1}{2}(u_2 + d - s_2 z)p_2(\eta) - T_c,$$

Results from geometric singular perturbation theory show that h^0 is an $\mathcal{O}(\epsilon)$ approximation to h. Equilibrium ice lines are found when $h(\eta) = 0$. Below is a graph of the h function we receive from this approximation. The same analysis on this graph shows the lowest equilibrium, the Jormungand state, is a stable equilibrium.



References

- James Walsh, Richard McGehee. *Climate Modeling, Dynamically Speaking.* http://www.oberlin.edu/math/faculty/walsh/walshpubl.html
- Richard McGehee, Esther R. Widiasih *A simplification of Budyko's ice-albedo feedback model* http://www.math.umn.edu/~mcgehee/publications/McGeheeWidiasih2012/index.html
- Christopher K.R.T. Jones. *Geometric Singular Perturbation Theory* Dynamical Systems, Lecture Notes in Mathematics Volume 1609, 1995, pp 44-118.