Variational Problems in Weighted Sobolev Spaces with Applications to CFD

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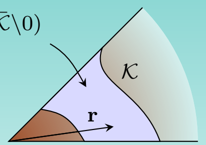
Weighted Sobolev spaces on a wedge

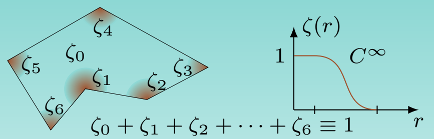


$V_\beta^\ell(\mathcal{K})$ = closure of $C_0^\infty(\mathcal{K})\backslash\{0\}$ with respect to the norm

$$\|u\|_{V_\beta^\ell} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\le\ell} r^{2(\beta-\ell+|\alpha|)}|D_x^\alpha u|^2\,dx\Big)^{1/2}$$

$$\|u\|_{V_\beta^2} = \int_{\mathcal{K}} \Big(r^{2\beta}|\nabla\nabla u|^2 + r^{2(\beta-1)}|\nabla u|^2 + r^{2(\beta-2)}|u|^2\Big)\,dx$$

Weighted Sobolev spaces in domains with corners



$$\zeta_0 + \zeta_1 + \zeta_2 + \cdots + \zeta_6 \equiv 1$$

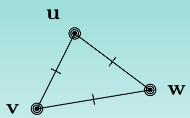
$V_{\vec\beta}^\ell(\Omega)$ = set of all functions on $\Omega \in \mathbb{R}^2$ such that $\zeta_0 u \in H^\ell(\Omega)$ and $\zeta_j u \in V_{\beta_j}^\ell(\mathcal{K}_j)$, $j = 1,\dots,d$

$$\|u\|_{\vec\beta}^\ell(\Omega) = \Big(\|\zeta_0 u\|_{H^\ell(\Omega)}^2 + \sum_{j=1}^d \|\zeta_j u\|_{V_{\beta_j}^\ell(\mathcal{K}_j)}^2\Big)^{1/2}$$

$$\mathring{V}_{\vec\beta}^\ell(\Omega) = \{u \in V_{\vec\beta}^\ell(\Omega) : u|_{\partial\Omega} = 0\}$$

C^1 finite elements

The Argyris element:



21 degrees of freedom:

- Values of 0^{th}, 1^{st} and 2^{nd} derivatives the three vertices.
- Values of the normal derivatives at midpoints of the edge

Theorem 1 *Let $1 \le \delta_i < (\beta + \alpha_i - 1)^{-1}$, $i = 1,\dots,d$. The on an appropriately graded mesh we obtain optimal convergence rate:*

$$\|u - u_h\|_{\vec\beta}^2(\Omega) \le Ch^{\min\{k-2,q\}}, \quad q = \min\delta_i(\beta + \alpha_i - 1)$$

The Poisson problem in $\mathring{V}_{\vec\beta}^2(\Omega)$

$\Omega \in \mathbb{R}^2$, bounded domain with corners $x^1,\dots,x^d$, with interior angles $\alpha_j \in (0, 2\pi)$, $j = 1,\dots,d$. *Given $f \in L_{2,\vec\beta}(\Omega)$, find $u \in \mathring{V}_{\vec\beta}^2(\Omega)$ such that*

$$(\Delta u, \Delta v)_{L_{2,\vec\beta}(\Omega)} = -(f, \Delta v)_{L_{2,\vec\beta}(\Omega)} \quad \forall v \in \mathring{V}_{\vec\beta}^2(\Omega)$$

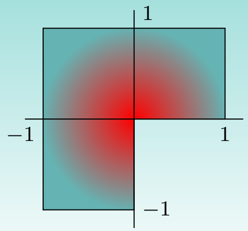
Theorem 2 *For any $\vec\beta = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d$ such that $1 - \pi/\alpha_j < \beta_j < 1 + \pi/\alpha_j$, $j = 1,\dots,d$, the weighted variational problem has a unique solution $u \in \mathring{V}_{\vec\beta}^2(\Omega)$.*

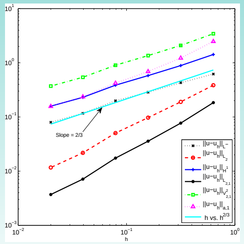
Theorem 3 *Let $f \in L_{2,\vec\beta}(\Omega)$, $1 - \pi/\alpha < \beta \le 1$. Then the variational problem has a unique solution in $\mathring{V}_{\vec\beta}^2(\Omega)$ that coincides with the solution of the traditional H^1 variational problem.*

Test against exact solution

Exact solution with singularity: $u = 2(1 = x^2)(1 - y^2)r^{2/3}\sin\frac{2}{3}\theta$

$-\Delta u = f$ in Ω, $\quad u = 0$ on $\partial\Omega$





Literature

[1] Ana Maria Soane and Rouben Rostamian, *Free Boundary Problems in Fluid Mechanics*, The Legacy of Ladyzhenskaya and Oleinik, available at http://topo.math.auburn.edu/pub/20lgas-proceedings/.

[2] Ana Maria Soane and Rouben Rostamian, *Variational problems in weighted Sobolev spaces on non-smooth domains*, Quarterly of Applied Mathematics (2010), in press.

[3] Ana Maria Soane, Manil Suri, and Rouben Rostamian, *The optimal convergence rate of a C^1 finite element method for non-smooth domains*, Journal of Computational and Applied Mathematics **233** (2010), no. 10, 2711–2723.

[4] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, American Mathematical Society, Providence, R.I., 1997.

[5] Jian-Guo Liu, Jie Liu, and Robert L. Pego, *Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate*, Comm. Pure Appl. Math. **60** (2007), no. 10, 1443–1487.

The Navier-Stokes equations

Fluid flow in a domain $\Omega \subset \mathbb{R}^2$ governed by:

$$\begin{aligned} \mathbf{u}_t + (\nabla\mathbf{u})\mathbf{u} + \nabla p &= \nu\Delta\mathbf{u} + \mathbf{f} && \text{in } \Omega\times(0,T)\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega\times(0,T)\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega\times(0,T)\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega\times\{0\} \end{aligned}$$

u: fluid velocity — ν: kinematic viscosity

p: pressure — **f**: external force per unit volume

Navier-Stokes in weighted spaces

Algorithm: Given and approximation $\mathbf{u}^n \in \mathring{V}_{\vec\beta}^2(\Omega)$ to the velocity at the n^{th} time step, determine $p^n \in V_{\vec\beta}^1(\Omega)$ from

$$(\nabla p^n, \nabla\phi)_{L_{2,\vec\beta}(\Omega)} = \big(\mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n + \nu\Delta\mathbf{u}^n - \nu\nabla(\operatorname{div}\mathbf{u}^n), \nabla\phi\big)_{L_{2,\vec\beta}(\Omega)} \quad \forall\phi \in V_{\vec\beta}^1(\Omega),$$

then determine $\mathbf{u}^{n+1} \in \mathring{V}_{\vec\beta}^2(\Omega)$ from:

$$\Big(-\Delta\mathbf{u}^{n+1} + \frac{1}{\nu k}\mathbf{u}^{n+1}, \Delta\mathbf{\Psi} + \frac{1}{\nu k}\mathbf{\Psi}\Big)_{L_{2,\vec\beta}(\Omega)} = \Big(\frac{1}{\nu k}\mathbf{u}^n + \frac{1}{\nu}\big(\mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n + \nabla p^n\big), \Delta\mathbf{\Psi} + \frac{1}{\nu k}\mathbf{\Psi}\Big)_{L_{2,\vec\beta}(\Omega)} \quad \forall\mathbf{\Psi} \in \mathring{V}_{\vec\beta}^2(\Omega)$$

Computational results: Backstep flow