Variational Problems in Weighted Sobolev Spaces with Applications to CFD

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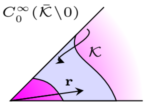
Weighted Sobolev spaces on a plane wedge

$u_n \in C_0^\infty(\bar{\mathcal{K}}\backslash 0)$

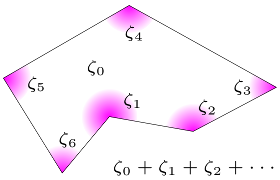


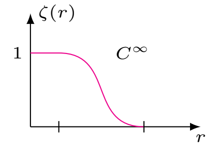
V_β^ℓ = closure of $C_0^\infty(\mathcal{K})\backslash\{0\}$ with respect to the norm

$$\|u\|_{V_\beta^\ell} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\le\ell} r^{2(\beta-\ell+|\alpha|)}|D_x^\alpha u|^2\,dx\Big)^{1/2}$$

$$\|u\|_{V_\beta^2} = \int_{\mathcal{K}} \left(r^{2\beta}|\nabla\nabla u|^2 + r^{2(\beta-1)}|\nabla u|^2 + r^{2(\beta-2)}|u|^2\right)dx$$

Weighted Sobolev spaces in domains with corners





$\zeta_0 + \zeta_1 + \zeta_2 + \cdots + \zeta_6 \equiv 1$

$V_{\vec\beta}^\ell(\Omega)$ = set of all functions on Ω such that $\zeta_0 u \in H^\ell(\Omega)$ and $\zeta_j u \in V_{\beta_j}^\ell(\mathcal{K}_j)$, $j = 1,\dots,d$

$$\|u\|_{\vec\beta}^\ell(\Omega) = \left(\|\zeta_0 u\|^2_{H^\ell(\Omega)} + \sum_{j=1}^d \|\zeta_j u\|^2_{V^\ell_{\beta_j}(\mathcal{K}_j)}\right)^{1/2}$$

$$\mathring{V}_{\vec\beta}^\ell(\Omega) = \{u \in V_{\vec\beta}^\ell(\Omega) : u|_{\partial\Omega} = 0\}$$

Variational formulation of the Poisson problem in $V_{\vec\beta}^2(\Omega)$

$\Omega \in \mathbb{R}^2$: bounded domain with corners $x^1,\dots,x^d$, with interior angles $\alpha_j \in (0,2\pi)$, $j = 1,\dots,d$. *Given* $f \in L_{2,\vec\beta}(\Omega)$, *find* $u \in \mathring{V}_{\vec\beta}^2(\Omega)$ *such that*

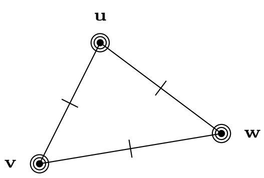
$$(\Delta u, \Delta v)_{L_{2,\vec\beta}(\Omega)} = -(f,\Delta v)_{L_{2,\vec\beta}(\Omega)} \quad \forall v \in \mathring{V}_{\vec\beta}^2(\Omega)$$

Theorem 1 *For any* $\vec\beta = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d$ *such that* $1-\pi/\alpha_j < \beta_j < 1+\pi/\alpha_j$, $j = 1,\dots,d$, *the weighted variational problem has a unique solution* $u \in \mathring{V}_{\vec\beta}^2(\Omega)$.

Theorem 2 *Let* $f \in L_{2,\vec\beta}(\Omega)$, $1-\pi/\alpha < \beta \le 1$. *Then the variational problem has a unique solution in* $\mathring{V}_{\vec\beta}^2(\Omega)$ *that coincides with the solution of the traditional* H^1 *variational problem.*

Finite element approximation

Argyris element:



Degrees of freedom:

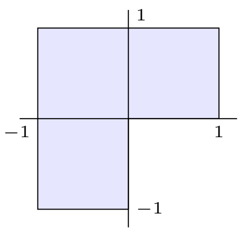
- Values at the three vertices.
- Values of the first derivatives at the three vertices.
- Values of the second derivatives at the three vertices.
- Values of the normal derivatives at the midpoint of each edge of the triangle.

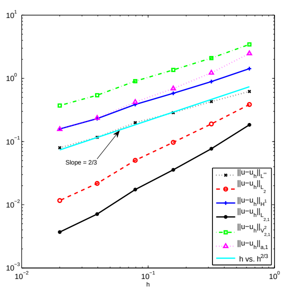
Theorem 3 *Let* Ω *be a domain with one reentrant corner with interior angle* α *and* $1 \le \beta < 1+\pi/\alpha$. *Then* $\lim_{h\to 0}\|u-u_h\|_{V^2_{\vec\beta}(\Omega)} = 0$, *where* $\vec\beta = (\beta,\dots,0)$.

Test against exact solution

Exact solution with singularity: $u = 2(1=x^2)(1-y^2)r^{2/3}\sin\frac{2}{3}\theta$

$-\Delta u = f$ in Ω, $\quad u = 0$ on $\partial\Omega$





Navier-Stokes equations

Fluid flow in a domain $\Omega \subset \mathbb{R}^2$ governed by:

$$\begin{aligned} \mathbf{u}_t + (\nabla\mathbf{u})\mathbf{u} + \nabla p &= \nu\Delta\mathbf{u} + \mathbf{f} && \text{in } \Omega\times(0,T)\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega\times(0,T)\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega\times(0,T)\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega\times\{0\}\end{aligned}$$

$\mathbf{u}$: fluid velocity $\quad\nu$: kinematic viscosity

p: pressure $\quad\mathbf{f}$: external force per unit volume

Numerical scheme of Liu, Liu & Pego

1. Given an approximation $\mathbf{u}^n \in [H^2(\Omega)\cap H^1_0(\Omega)]^2$ to the velocity at n^{th} time step, determine $\nabla p^n \in [L_2(\Omega)]^2$ from a weak form pressure Poisson equation

$$\langle\nabla p^n,\nabla\phi\rangle = \langle\mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n + \nu\Delta\mathbf{u}^n - \nu\nabla(\operatorname{div}\mathbf{u}^n),\nabla\phi\rangle \quad \forall\phi \in H^1(\Omega).$$

2. Next, determine $\mathbf{u}^{n+1} \in [H^2(\Omega)\cap H^1_0(\Omega)]^2$ from the Helmholtz problem

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{k} - \nu\Delta\mathbf{u}^{n+1} = \mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n - \nabla p^n, \qquad \mathbf{u}^{n+1}|_{\partial\Omega} = \mathbf{0}.$$

J.-G. Liu, J. Liu & R. Pego, Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate, Comm. Pure & Appl. Math., 60 (2007)

Fully-discrete C^1/C^0 FEM without compatibility conditions on smooth domains

- $X_h \subset [H^2(\Omega)\cap H^1_0(\Omega)]^2$ finite dimensional subspace containing approximate velocity field
- $Y_h \subset H^1(\Omega)$ finite-dimensional subspace containing approximate pressure

Given $\mathbf{u}_h^n$ at the n^{th} step, compute $p_n^n \in Y_h$ and $\mathbf{u}_h^{n+1} \in X_h$ from:

$$\langle\nabla p_h^n,\nabla\phi_h\rangle = \langle\mathbf{f}^n - (\nabla\mathbf{u}_h^n)\mathbf{u}_h^n + \nu\Delta\mathbf{u}_h^n - \nu\nabla(\operatorname{div}\mathbf{u}_h^n),\nabla\phi_h\rangle \quad \forall\phi_h \in Y_h$$

$$\Big\langle -\Delta\mathbf{u}^{n+1} + \frac{1}{\nu k}\mathbf{u}^{n+1},\Delta\mathbf{\Psi}\Big\rangle = \Big\langle\frac{1}{\nu k}\mathbf{u}^n + \frac{1}{\nu}(\mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n + \nabla p^n),\Delta\mathbf{\Psi}\Big\rangle,$$

$$\forall\mathbf{\Psi} \in [H^2(\Omega)\cap H^1_0(\Omega)]^2$$

Numerical scheme in weighted Sobolev spaces on polygonal domains

Given and approximation $\mathbf{u}^n \in \mathring{V}_{\vec\beta}^2(\Omega)$ to the velocity at the n^{th} time step, determine $p^n \in V^1_{\vec\beta}(\Omega)$ from

$$(\nabla p^n,\nabla\phi)_{L_{2,\vec\beta}(\Omega)} = (\mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n + \nu\Delta\mathbf{u}^n - \nu\nabla(\operatorname{div}\mathbf{u}^n),\nabla\phi)_{L_{2,\vec\beta}(\Omega)} \quad \forall\phi \in V^1_{\vec\beta}(\Omega),$$

then determine $\mathbf{u}^{n+1} \in \mathring{V}_{\vec\beta}^2(\Omega)$ from:

$$\Big(-\Delta\mathbf{u}^{n+1} + \frac{1}{\nu k}\mathbf{u}^{n+1},\Delta\mathbf{\Psi} + \frac{1}{\nu k}\mathbf{\Psi}\Big)_{L_{2,\vec\beta}(\Omega)}$$

$$= \Big(\frac{1}{\nu k}\mathbf{u}^n + \frac{1}{\nu}(\mathbf{f}^n - (\nabla\mathbf{u}^n)\mathbf{u}^n + \nabla p^n),\Delta\mathbf{\Psi} + \frac{1}{\nu k}\mathbf{\Psi}\Big)_{L_{2,\vec\beta}(\Omega)} \forall\mathbf{\Psi} \in \mathring{V}_{\vec\beta}^2(\Omega)$$

Backstep flow