Homogenization in Random Media and Isotropy Considerations

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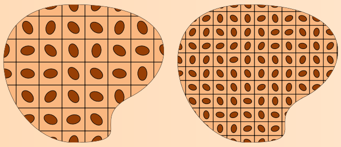
Diffusion in a Random Medium



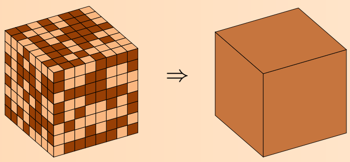
$$\begin{cases} -\operatorname{div}_{\boldsymbol{x}} \left(A(\boldsymbol{x},\omega)\nabla u(\boldsymbol{x},\omega)\right) = f(\boldsymbol{x}) & \text{on } \mathcal{D} \\ u(\boldsymbol{x},\omega) = 0 & \text{on } \partial\mathcal{D} \end{cases}$$

Some considerations:

- Random PDE $\Rightarrow$ difficult problem
- For fixed realization computation is a formidable task
- Need to have a discretization of the domain fine enough to resolve the complicated micro-structure
- The problem is either too computationally expensive or impossible to solve numerically

Homogenization

Physical intuition: instead of working with a medium with complicated structure work with a homogeneous medium with the same effective properties.



Consider the problem:

$$\begin{cases} -\operatorname{div}_{\boldsymbol{x}} \left(A(\boldsymbol{x},\omega)\nabla u(\boldsymbol{x},\omega)\right) = f(\boldsymbol{x}) & \text{on } \mathcal{D} \\ u(\boldsymbol{x},\omega) = 0 & \text{on } \partial\mathcal{D} \end{cases} \tag{1}$$

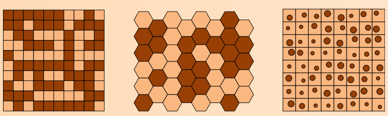
Given appropriate conditions on A, there is a homogenized problem [4, 2]:

$$\begin{cases} -\operatorname{div}(A^0\nabla u^0) = f & \text{on } \mathcal{D} \\ u^0 = 0 & \text{on } \partial\mathcal{D} \end{cases} \tag{2}$$

such that for almost all ω , the solution u^0 of (2) is a close approximation of the solution u of the problem (1).

Symmetries of random media

- Let Q be a rotation matrix
- A random structure is invariant under rotation by Q if rotation of a realization by Q gives an equally likely realization



Isotropy (diffusion)

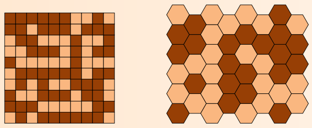
- A medium is isotropic if it behaves the same regardless of its orientation
- Consider the homogenized problem

$$\begin{cases} -\operatorname{div}(A^0\nabla u^0) = f & \text{on } \mathcal{D} \\ u^0 = 0 & \text{on } \partial\mathcal{D} \end{cases} \tag{3}$$

$$\text{Isotropy} \quad \Leftrightarrow \quad A^0 = \alpha I.$$

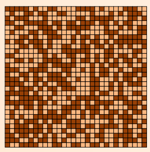
Question: Under what conditions the effective behavior of a composite medium is isotropic?
Answer in 2D:

- R_θ matrix of rotation by angle θ
- If $\theta \neq k\pi$, invariance under $R_\theta \Rightarrow$ isotropy



Numerical computations

Consider a random checkerboard:

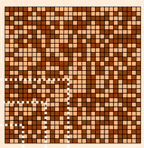


- Random checkerboard, fifty-fifty chance of white or gray
- gray conductivity $a_1 = 1$
- white conductivity $a_2 = 2$

We expect [2],

$$A^0 = \sqrt{a_1 a_2} I = \sqrt{2} I$$

Periodization



- Fix a realization, select a square $[0,\rho]^2$, and extend it periodically to $\mathbf{R}^2$
- Solve the unit cell problem from periodic homogenization
- Get the corresponding homogenized matrix $A^\rho(\omega)$

Then [1, 3],

As $\rho \to \infty$, $A^\rho \to A^0$ for almost all ω.

Sample computational results

$$A_1^\rho = \begin{bmatrix} 1.2397 & -0.0029 \\ -0.0029 & 1.2580 \end{bmatrix}$$

$$A_2^\rho = \begin{bmatrix} 1.3876 & 0.0013 \\ 0.0013 & 1.4074 \end{bmatrix}$$

$$A_3^\rho = \begin{bmatrix} 1.4099 & 0.0015 \\ 0.0015 & 1.4101 \end{bmatrix}$$

Linear elasticity

- $\mathcal{E}$: n-dimensional Euclidean space
- $\mathcal{D} \subset \mathcal{E}$

$$\begin{cases} -\operatorname{div} \mathsf{C}[\nabla \boldsymbol{u}] = \boldsymbol{f}, & \text{in } \mathcal{D}, \\ \boldsymbol{u} = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

- $\boldsymbol{f} \in \left(L^2(\mathcal{D})\right)^n$: body force
- C: elasticity tensor

$$\mathsf{C} : Sym(\mathcal{E}) \to Sym(\mathcal{E})$$

Isotropy (elasticity)

Definition: Symmetry group of an elasticity tensor C is a group $G_c \subseteq Orth(\mathcal{E})$ such that

$$Q\mathsf{C}[E]Q^T = \mathsf{C}[QEQ^T], \forall Q \in G_c. \tag{4}$$

C is Isotropic if

$$Q\mathsf{C}[E]Q^T = \mathsf{C}[QEQ^T], \quad \forall Q \in Orth(\mathcal{E}). \tag{5}$$

Question: Under what condition on G_c does (4) $\Rightarrow$ (5)?
Answer in 2D:

- $\mathcal{E}$: the 2-dimensional Euclidean space
- $R_\theta \in Orth(\mathcal{E})$: rotation by angle θ
- If $\theta \neq k\frac{\pi}{2}$, invariance under $R_\theta \Rightarrow$ isotropy

References

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[2] V. V. Jikov, S. M. Kozlov, and O. A. Olenik. *Homogenization of differential. operators and integral functionals*. Springer, 1994.

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[4] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. *Seria Colloquia Mathematica Societatis Janos Bolyai North-Holland, 27*, 1981.