THE DIFFERENTIAL GEOMETRY OF TUBE PLOTS AND COMPUTER GRAPHICS

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ABSTRACT. We describe a robust method for constructing a tube-like surface surrounding a 3D curve. Our method is designed to eliminate undesirable twists and wrinkles in the tube's skin at points where the curve experiences high torsion. In our construction the tube's twist is bounded by the curve's curvature and is independent of the curve's torsion. We also give an algorithm for numerical implementation of our approach. The algorithm has been incorporated in DYNAGRAPH, a graphics software for rendering high-quality three dimensional curves and surfaces.

1. INTRODUCTION

Consider a curve $\mathbf{x}(s)$ in $\mathbf{R}^3$, parameterized by its arc length s. Let $\mathbf{T}(s)$ be its tangent vector, i.e., $\mathbf{T}(s) = \mathbf{x}'(s) = d\mathbf{x}(s)/ds$. The arc length parameterization of the curve makes $\mathbf{T}(s)$ a unit vector; i.e., $\|\mathbf{T}(s)\| = 1$, therefore its derivative $\mathbf{T}'$ is orthogonal to $\mathbf{T}$. The *principal normal vector* $\mathbf{N}$ is defined as $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$. The *binormal vector* $\mathbf{B}$ is defined as the cross product $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The *Frenet-Serret equations*, cf. [3], express the rate of change of the moving orthonormal triad $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along the curve:

$$
\begin{cases} \mathbf{T}' = \kappa \mathbf{N}, \\ \mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \mathbf{B}' = -\tau \mathbf{N}. \end{cases} \tag{1}
$$

The coefficients κ and τ are the curve's *curvature* and *torsion*.

With the Frenet-Serret system in hand, we can construct a "tube" of radius r about the curve by defining a surface with parameters s and t:

$$
\mathbf{tube}(s,t) = \mathbf{x}(s) + \mathbf{N}(s) r \cos t + \mathbf{B}(s) r \sin t. \tag{2}
$$

Figure 1 shows two views of such a tube built about the trefoil knot $\mathbf{x}(s) = \big(x_1(s), x_2(s), x_3(s)\big)$, $0 \le s < 2\pi$, with $r = 4$ where[1]

$$
\begin{aligned}
x_1(s) &= -10\cos s - 2\cos 5s + 15 \sin 2s, \\
x_2(s) &= -15\cos 2s + 10\sin s - 2\sin 5s, \\
x_3(s) &= 10\cos 3s.
\end{aligned}
$$

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[1] As presented, this curve is not parametrized by its arc length. In principle it can be re-parametrized to make it a unit-speed curve, but since this is not our focus, we will side step the issue here.

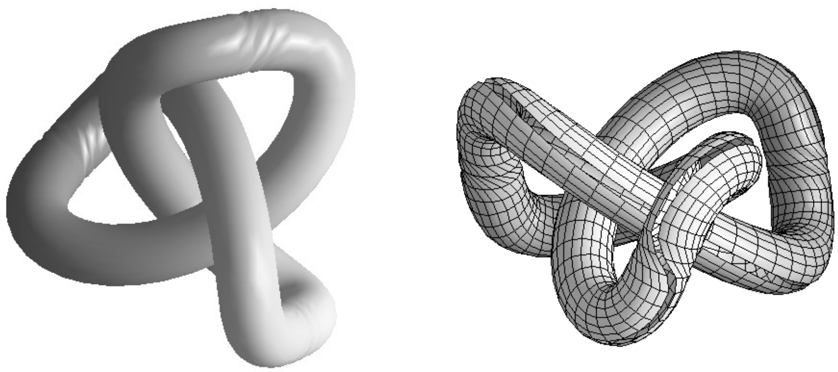

Figure 1. Two views of the trefoil knot drawn using the Frenet-Serret system, showing wrinkling of the tube surface and twisting of the coordinate grid. The longitudinal slice in the figure on the right is made to make the twisting more clearly visible.

Note the pronounced wrinkling of the tube's "skin" in at least two places in the figure on the left. The knot has a three-way symmetry and there is a third location with severe wrinkles but which is not visible in this view.

The wrinkles in the skin are due to the severe twisting of the tube in those locations. The figure on the right shows the coordinate grid and a longitudinal slice to make the twisting more visible. All three location of severe twist are visible in this view.

Gray [1] has noted that wild gyrations of the Frenet-Serret system can be expected at points where the curvature κ is small and the torsion τ is large. An inspection of the Frenet-Serret equations (1) shows that at such points $\mathbf{T}'$ is small and $\mathbf{N}'$ and $\mathbf{B}'$ are large, in effect indicating that the Frenet-Serret system is spinning about its $\mathbf{T}$ axis.

The reason that the twisting occurs in seemingly innocuous locations in the knot of Figure 1 is that a curve's torsion is a function of the curve's *third derivative* as evidenced by the explicit formulas

$$\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3}, \qquad \tau = \frac{\mathbf{x}' \times \mathbf{x}'' \cdot \mathbf{x}'''}{\|\mathbf{x}' \times \mathbf{x}''\|^2}.$$

The effect of the third derivative on the shape of a space curve is not easy to discern by visual inspection. It's only after we install a tube around the curve, utilizing the Frenet-Serret system, that the large third derivative manifests itself in the form of localized twisting of the tube.

Figure 2 shows the graphs of the curvature κ and torsion τ along this trefoil knot. The three locations where the curvature is small and torsion is most negative correspond to the wrinkled patches in Figure 1.

2. An alternative moving triad

To avoid the undesirable twisting and wrinkling in tubular surfaces produced by computer graphics, we introduce an alternative, more tamely behaved moving triad

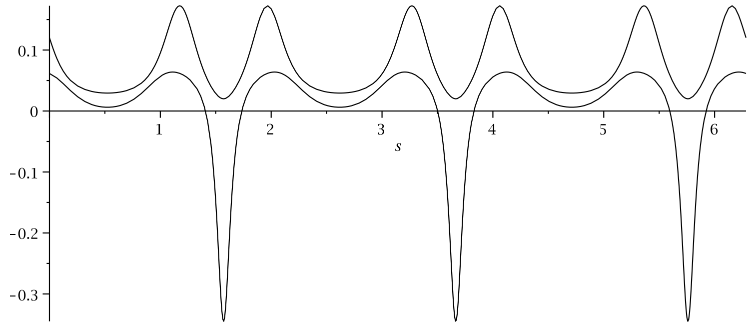

FIGURE 2. The graphs of curvature (top) and torsion (bottom) of the trefoil knot.

for curves. For this, we let $\mathbf{x}(s)$ be a regular curve in $\mathbf{R}^3$ parameterized by its arc length, and $\mathbf{T}(s) = \mathbf{x}'(s)$ be its unit tangent vector—the same as before. We will define unit vector fields $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ such that $\{\mathbf{T}, \mathbf{P}, \mathbf{Q}\}$ is orthonormal at each point along the curve. We will chose $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ in such a way as to minimize its gyrations as the triad moves along the curve.

Unlike the Frenet-Serret triad which is defined locally on the curve, our triad is defined in terms of the solution of a differential equation, hence it depends not only on the curve's local properties but also on the location and value of the differential equation's initial conditions.

The construction of the new triad is based on the set of solutions of the system of differential equations

$$\frac{d}{ds}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & \tau(s) \\ -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{3}$$

where a and b are functions of s and $\tau(s)$ is the torsion of the curve $\mathbf{x}(s)$. Let $\phi_{s_0,s}$ denote the semigroup of solutions of this differential equation, that is, $\big(a(s), b(s)\big) = \phi_{s_0,s}\big(a(s_0), b(s_0)\big)$ is the solution of the differential equation corresponding to the initial data $\big(a(s_0), b(s_0)\big)$ at $s = s_0$. If the curve $\mathbf{x}(s)$ is sufficiently regular, then the standard theory of linear systems, see e.g. [2], guarantees the existence, uniqueness and maximal continuation of its solutions.

Definition 1. Let $\mathcal{U}(s)$ denote the *normal plane* of the curve $\mathbf{x}(s)$ at any s. That is, $\mathcal{U}(s)$ is perpendicular to $\mathbf{T}(s)$ and contains the principal normal and binormal vectors $\mathbf{N}(s)$ and $\mathbf{B}(s)$. We equip each $\mathcal{U}(s)$ with the inner product inherited from $\mathbf{R}^3$. We define the mapping $\Phi_{s_0,s} : \mathcal{U}(s_0) \to \mathcal{U}(s)$ with $\Phi_{s_0,s} : \mathbf{V}_0 \in \mathcal{U}(s_0) \mapsto a(s)\mathbf{N}(s) + b(s)\mathbf{B}(s)$ where $\big(a(s), b(s)\big) = \phi_{s_0,s}\big(\mathbf{V}_0 \cdot \mathbf{N}(s_0), \mathbf{V}_0 \cdot \mathbf{B}(s_0)\big)$.

Lemma 1. *Let $\Phi_{s_0,s}$ be as in Definition 1. Then for any $\mathbf{V}_0 \in \mathcal{U}(s_0)$ the vector-valued function $\mathbf{V}(s)$ defined by $\mathbf{V}(s) = \Phi_{s_0,s}(\mathbf{V}_0)$ satisfies the differential equation*

$$\mathbf{V}' = -(\mathbf{V} \cdot \mathbf{T}')\mathbf{T},$$

where $\mathbf{T}$ is the curve's unit tangent vector.

Proof. According to the definition of $\Phi_{s_0,s}$, the components $\mathbf{V}(s) \cdot \mathbf{N}(s)$ and $\mathbf{V}(s) \cdot \mathbf{B}(s)$ of $\mathbf{V}(s)$ satisfy the differential equation (3), therefore $(\mathbf{V} \cdot \mathbf{N})' = \tau \mathbf{V} \cdot \mathbf{B}$ and

$(\mathbf{V}\cdot\mathbf{B})' = -\tau\mathbf{V}\cdot\mathbf{N}$. We now compute $\mathbf{V}'(s)$ while making use of the Frenet-Serret equations:

$$\begin{aligned}
\mathbf{V}'(s) &= \big[(\mathbf{V}\cdot\mathbf{N})\mathbf{N} + (\mathbf{V}\cdot\mathbf{B})\mathbf{B}\big]' \\
&= (\mathbf{V}\cdot\mathbf{N})'\mathbf{N} + (\mathbf{V}\cdot\mathbf{N})\mathbf{N}' + (\mathbf{V}\cdot\mathbf{B})'\mathbf{B} + (\mathbf{V}\cdot\mathbf{B})\mathbf{B}' \\
&= \tau(\mathbf{V}\cdot\mathbf{B})\mathbf{N} + (\mathbf{V}\cdot\mathbf{N})(-\kappa\mathbf{T} + \tau\mathbf{B}) - \tau(\mathbf{V}\cdot\mathbf{N})\mathbf{B} + (\mathbf{V}\cdot\mathbf{B})(-\tau\mathbf{N}) \\
&= -\kappa(\mathbf{V}\cdot\mathbf{N})\mathbf{T} \\
&= -(\mathbf{V}\cdot\mathbf{T}')\mathbf{T},
\end{aligned}$$

as asserted. □

Theorem 1. *The mapping $\Phi_{s_0,s} : \mathcal{U}(s_0) \to \mathcal{U}(s)$ is an isometry.*

Proof. Take any two vectors $\mathbf{P}_0$ and $\mathbf{Q}_0$ in $\mathcal{U}(s_0)$ and let $\mathbf{P}(s) = \Phi_{s_0,s}(\mathbf{P}_0)$ and $\mathbf{Q}(s) = \Phi_{s_0,s}(\mathbf{Q}_0)$. Then applying the result of Lemma 1 we get

$$\begin{aligned}
(\mathbf{P}(s)\cdot\mathbf{Q}(s))' &= \mathbf{P}'\cdot\mathbf{Q} + \mathbf{P}\cdot\mathbf{Q}' \\
&= -(\mathbf{P}\cdot\mathbf{T}')\mathbf{T}\cdot\mathbf{Q} - \mathbf{P}(\mathbf{Q}\cdot\mathbf{T}')\mathbf{T} \\
&= 0,
\end{aligned}$$

because $\mathbf{P}\cdot\mathbf{T} = 0$ and $\mathbf{Q}\cdot\mathbf{T} = 0$. Therefore $\Phi_{s_0,s}$ preserves the inner product hence is an isometry. □

Definition 2. Let $\mathbf{x}(s)$ be a regular curve in $\mathbf{R}^3$ parameterized by its arc length, and let $\mathbf{T}(s)$ denote its unit tangent. Arbitrarily fix a parameter value s_0 and let $\mathbf{T}_0 = \mathbf{T}(s_0)$. Choose any pair of vectors $\mathbf{P}_0$ and $\mathbf{Q}_0$ such that $\{\mathbf{T}_0, \mathbf{P}_0, \mathbf{Q}_0\}$ forms and orthonormal set. Then let $\mathbf{P}(s) = \Phi_{s_0,s}(\mathbf{P}_0)$ and $\mathbf{Q}(s) = \Phi_{s_0,s}(\mathbf{Q}_0)$. According Theorem 1, the triad $\big\{\mathbf{T}(s), \mathbf{P}(s), \mathbf{Q}(s)\big\}$ is orthonormal for all s. We call $\big\{\mathbf{T}(s), \mathbf{P}(s), \mathbf{Q}(s)\big\}$ *a tube triad* for the curve.

Remark 1. If the initial triad $\{\mathbf{T}_0, \mathbf{P}_0, \mathbf{Q}_0\}$ is chosen such that it is positively oriented, i.e., $\det|\mathbf{T}_0, \mathbf{P}_0, \mathbf{Q}_0| = 1$, then the tube triad will be positively oriented for all s. This is an immediate consequence of the continuity of solutions of the differential equations (3).

The rate of change of the tube triad is expressed in equations akin to Frenet-Serret's:

Theorem 2. *Let $\big\{\mathbf{T}(s), \mathbf{P}(s), \mathbf{Q}(s)\big\}$ be a positively oriented tube triad. Then its rate of change is expressed by:*

$$\left\{\begin{aligned}
\mathbf{T}' &= \alpha\mathbf{P} + \beta\mathbf{Q} \\
\mathbf{P}' &= -\alpha\mathbf{T}, \\
\mathbf{Q}' &= -\beta\mathbf{T},
\end{aligned}\right. \tag{4}$$

where $\alpha(s)$ and $\beta(s)$ are scalar functions defined along the curve.

Proof. The second and third equations in (4) are consequences of Lemma 1. To obtain the first equation we note that the positive orientation of the tube triad implies $\mathbf{T} = \mathbf{P}\times\mathbf{Q}$ and $\mathbf{Q} = \mathbf{T}\times\mathbf{P}$ and $\mathbf{P} = \mathbf{Q}\times\mathbf{T}$. Therefore we have

$$\mathbf{T}' = \mathbf{P}'\times\mathbf{Q} \,+\, \mathbf{P}\times\mathbf{Q}' = -\alpha\mathbf{T}\times\mathbf{Q} \,+\, \mathbf{P}\times(-\beta\mathbf{T}) = \alpha\mathbf{P} + \beta\mathbf{Q}$$

as asserted. □

Remark 2. Rather than computing both $\mathbf{P}$ and $\mathbf{Q}$ components of the tube triad with $\mathbf{P}(s) = \Phi_{s_0,s}(\mathbf{P}_0)$ and $\mathbf{Q}(s) = \Phi_{s_0,s}(\mathbf{Q}_0)$ as in Definition 2, it is more practical to compute one, say $\mathbf{P}(s)$, then define $\mathbf{Q}(s)$ as $\mathbf{Q}(s) = \mathbf{T}(s) \times \mathbf{P}(s)$, which also implies the permutations $\mathbf{P} = \mathbf{Q} \times \mathbf{T}$ and $\mathbf{T} = \mathbf{P} \times \mathbf{Q}$. We can verify directly that $\mathbf{Q}$, defined this way, satisfies the differential equation (3):

$$\begin{aligned}
\mathbf{Q}' &= (\mathbf{T} \times \mathbf{P})' = \mathbf{T}' \times \mathbf{P} + \mathbf{T} \times \mathbf{P}' \\
&= \mathbf{T}' \times \mathbf{P} - \mathbf{T} \times (\mathbf{P} \cdot \mathbf{T}')\mathbf{T} \\
&= \mathbf{T}' \times \mathbf{P} \\
&= \mathbf{T}' \times (\mathbf{Q} \times \mathbf{T}) \\
&= (\mathbf{T}' \cdot \mathbf{T})\mathbf{Q} - (\mathbf{T}' \cdot \mathbf{Q})\mathbf{T} \\
&= -(\mathbf{Q} \cdot \mathbf{T}')\mathbf{T}
\end{aligned}$$

therefore $\mathbf{Q}(s) = \Phi_{s_0,s}(\mathbf{Q}_0)$, as required. In this derivation we have made use of the fact that $\mathbf{T}'.\mathbf{T} = 0$. We have also used the general vector algebra identities $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$.

The following theorem establishes bounds on the rate of change of the tube triad:

Theorem 3. *Let $\{\mathbf{T}, \mathbf{P}, \mathbf{Q}\}$ be the tube triad as in Definition 2 and let κ be the curvature of the curve. We have the following pointwise bounds on the rate of change of the triad:*

$$\|\mathbf{T}'(s)\| = \kappa(s), \quad \|\mathbf{P}'(s)\| \le \kappa(s), \quad \|\mathbf{Q}'(s)\| \le \kappa(s). \tag{5}$$

Proof. The first of the estimates (5) is merely the definition of curvature. To verify the second, we refer to equation (3):

$$\|\mathbf{P}'\| = |\mathbf{P} \cdot \mathbf{T}'|\,\|\mathbf{T}\| \le \|\mathbf{P}\|\,\|\mathbf{T}'\| = \|\mathbf{T}'\| = \kappa.$$

The third equation is verified in the same way. □

Unlike the Frenet-Serret system, the rate of change of the tube triad $\{\mathbf{T}, \mathbf{P}, \mathbf{Q}\}$ is bounded by the curvature but is *independent of curve's torsion*. A tube based on the tube triad, i.e.,

$$\mathbf{tube}(s,t) = \mathbf{x}(s) + \mathbf{P}(s) r \cos t + \mathbf{Q}(s) r \sin t. \tag{6}$$

will have fewer twists and wrinkles in its skin compared to one based on the Frenet-Serret formulas as in (2). Figure 3 shows the previous trefoil knot plotted using the new system. Compare this with the images in Figure 1 and note the absence of wrinkles and twists.

Example: For illustration, we compute the $\mathbf{P}$ and $\mathbf{Q}$ vectors of the tube triad for the unit speed helix:

$$\mathbf{x} = \left[a \cos \frac{s}{\sqrt{a^2+b^2}},\ a \sin \frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right] \tag{7}$$

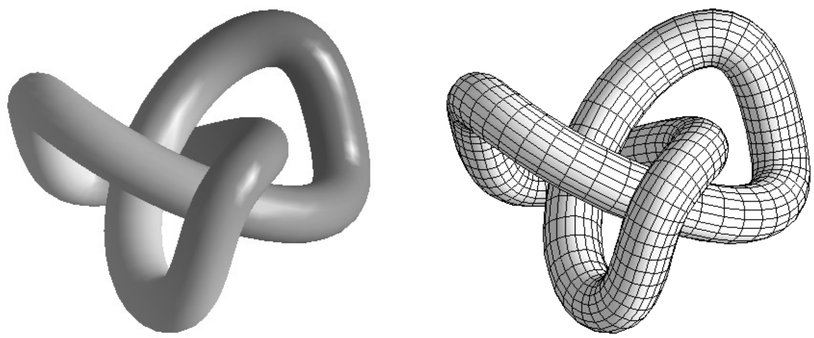

FIGURE 3. Tube plots of trefoil knot drawn using the modified system. Compare with the images in Figure 1 and note the smooth rendition and the absence of twists and wrinkles in this case.

with $a \neq 0$. The corresponding Frenet-Serret triad is:

$$\begin{aligned}
\mathbf{T}(s) &= \Big[-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}\Big],\\
\mathbf{N}(s) &= \Big[-\cos\frac{s}{\sqrt{a^2+b^2}},\ -\sin\frac{s}{\sqrt{a^2+b^2}},\ 0\Big],\\
\mathbf{B}(s) &= \Big[\frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}},\ -\frac{b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}},\ \frac{a}{\sqrt{a^2+b^2}}\Big].
\end{aligned}$$

The curvature and torsion are constants, independent of s, and are given by

$$\kappa = \frac{a}{a^2+b^2}, \qquad \tau = \frac{b}{a^2+b^2}.$$

Since torsion is constant, the system (3) reduces to a differential equation with constant coefficients which can be solved readily once a set of initial conditions is supplied. Suppose that we take $\mathbf{P}(0) = \mathbf{B}(0)$. Note that $\mathbf{P}(0)$ is of unit length and orthogonal to $\mathbf{T}(0)$, as required. Then $a(0) = \mathbf{P}(0)\cdot\mathbf{N}(0) = 0$ and $b(0) = \mathbf{P}(0)\cdot\mathbf{B}(0) = 1$. Solving (3) with these initial conditions we obtain

$$a(s) = \mathbf{P}\cdot\mathbf{N} = \sin\tau s \quad \text{and} \quad b(s) = \mathbf{P}\cdot\mathbf{B} = \cos\tau s,$$

whence, according to Definition 1:

$$\mathbf{P}(s) = \mathbf{N}(s)\sin\tau s + \mathbf{B}(s)\cos\tau s,$$

where $\mathbf{N}(s)$, $\mathbf{B}(s)$, and τ are given in terms of s by the explicit formulas above. The vector $\mathbf{Q}$ can also be computed in the same manner, or simply through $\mathbf{Q}(s) = \mathbf{T}(s)\times\mathbf{P}(s)$. Since $\mathbf{T}\cdot\mathbf{N} = \mathbf{B}$ and $\mathbf{T}\cdot\mathbf{B} = -\mathbf{N}$, we get:

$$\mathbf{Q}(s) = -\mathbf{N}(s)\cos\tau s + \mathbf{B}(s)\sin\tau s.$$

Note that $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are of frequency $1/\sqrt{a^2+b^2}$ and $\sin\tau s$ and $\cos\tau s$ are of frequency $b/(a^2+b^2)$. Therefore the vectors $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ are not periodic in s in general. Figure 4 shows the components of the vector $\mathbf{P}(s)$ over the range $0 \le s \le 8\pi$ for the helix in (7) with $a = b = 1$.

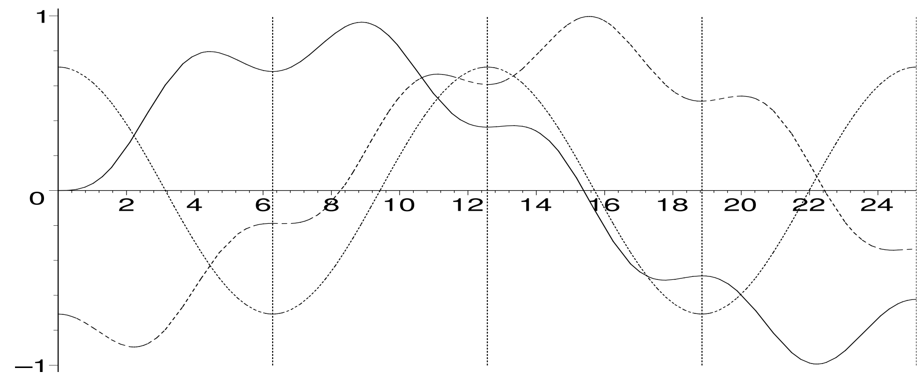

FIGURE 4. The graphs of the three components of the vector $\mathbf{P} = (p_1, p_2, p_3)$ of the triad $\{\mathbf{T}, \mathbf{P}, \mathbf{Q}\}$ on the helix of equation (7) as the triad makes four turns around the helix. The graphs of p_1, p_2 and p_3 are drawn as solid, dashed, and dotted lines, respectively. Vertical lines indicate divisions of 2π intervals.

3. Numerical implementation

DYNAGRAPH is a graphics software for rendering high-quality three dimensional curves and surfaces, with lighting, smooth shading, and interactive rotation and zooming. DYNAGRAPH is implemented in **C** using the OPENGL graphics library in the UNIX environment for the X WINDOW SYSTEM. Details about DYNAGRAPH, including its on-line manual and source code are available on the web.[2]

One of DYNAGRAPH's functions provides a means for drawing tubes given a tube radius and its centerline curve. For instance the sequence of commands:[3]

```
u := -10*cos(t) - 2*cos(5*t) + 15*sin(2*t);
v := -15*cos(2*t) + 10*sin(t) - 2*sin(5*t);
w := 10*cos(3*t);
tubeplot([u,v,w], t= 0..2*Pi,
        tuberadius=4,tubepoints=20, numpoints=100);
```

plots the trefoil knot shown in Figure 3. Internally the tube is generated according to the parametric representation in (6) employing the tube triad $\{\mathbf{T}, \mathbf{P}, \mathbf{Q}\}$ as described in the previous sections.

DYNAGRAPH accepts curve descriptions as symbolic equations, as in the sample input shown above, or as numeric data, in the form of a list of points in $\mathbf{R}^3$. If the input is in the symbolic format, it is converted immediately to a list of points. In what follows we denote this list by $\mathbf{x}_j, j = 1, 2, \dots, n$. The argument `numpoints=100` in the sample input above corresponds to $n = 100$.

There is some flexibility in defining $\mathbf{T}_j$, the discretized unit tangent vector $\mathbf{T}_j$ at node j. The choice in practice is dictated by the algorithmic accuracy, computational efficiency, and aesthetic considerations. In DYNAGRAPH we pretend that the curve is traced at constant speed in $\mathbf{R}^3$. We compute a unit vector representing the velocity on approaching the node j, and another unit vector representing the

[2]URL: `http://www.math.umbc.edu/~rouben/dynagraph/gallery/`

[3]DYNAGRAPH's syntax is almost identical to that of MAPLE, a symbolic computation program produced by *Waterloo Maple Inc.*

velocity leaving the node j. We average the two vectors and normalize to obtain $\mathbf{T}_j$. Algorithmically this is represented as:

$$\begin{cases} \mathbf{a}_j = \mathbf{x}_j - \mathbf{x}_{j-1}, \\ \mathbf{b}_j = \mathbf{x}_{j+1} - \mathbf{x}_j, \\ \mathbf{c}_j = \dfrac{1}{2}\Big(\dfrac{\mathbf{a}_j}{\|\mathbf{a}_j\|} + \dfrac{\mathbf{b}_j}{\|\mathbf{b}_j\|}\Big), \\ \mathbf{T}_j = \dfrac{\mathbf{c}_j}{\|\mathbf{c}_j\|}. \end{cases} \tag{8}$$

With this choice of $\mathbf{T}_j$, the curve segments $\mathbf{x}_j - \mathbf{x}_{j-1}$ and $\mathbf{x}_{j+1} - \mathbf{x}_j$ make equal angles with the plane which is perpendicular to $\mathbf{T}_j$ at $\mathbf{x}_j$, that is, the normal plane at each node bisects the angle between the adjacent curve segments.

The normal vectors $\mathbf{P}_j$ and $\mathbf{Q}_j$ of the tube triad are defined through recurrence:

$$\begin{cases} \mathbf{d}_j = \mathbf{T}_j \times \mathbf{P}_{j-1} \\ \mathbf{Q}_j = \dfrac{\mathbf{d}_j}{\|\mathbf{d}_j\|}, \\ \mathbf{P}_j = \mathbf{Q}_j \times \mathbf{T}_j. \end{cases} \tag{9}$$

To see the correspondence of this recurrence to the tube triad $\{\mathbf{T}, \mathbf{P}, \mathbf{Q}\}$ of Definition 2 we recall the equation $\mathbf{P}' = -\alpha\mathbf{T}$ of Theorem 2 and discretize it:

$$\frac{\mathbf{P}_j - \mathbf{P}_{j-1}}{\Delta s} = -\alpha\mathbf{T}_j + o(1),$$

then form the cross-product of this with $\mathbf{T}_j$:

$$\mathbf{T}_j \times \mathbf{P}_j \; - \; \mathbf{T}_j \times \mathbf{P}_{j-1} = -\alpha\,\Delta s\,\mathbf{T}_j \times \mathbf{T}_j + \Delta s\, o(1) = o(\Delta s).$$

But $\mathbf{Q} = \mathbf{T} \times \mathbf{P}$, therefore $\mathbf{Q}_j = \mathbf{T}_j \times \mathbf{P}_j = \mathbf{T}_j \times \mathbf{P}_{j-1} + o(\Delta s)$ showing that if Δs is small then $\mathbf{T}_j \times \mathbf{P}_{j-1}$ is a good approximation to $\mathbf{Q}_j$. The extra normalization in (9) is to prevent accumulation and propagation of incremental errors through the recursion.

The initialization of the recursion takes different forms depending on whether the curve is open or closed. We say a curve is *closed* if its end points coincide. Otherwise it is *open*.

3.1. **Initializing an open curve.** The computation of the discretized tangent vector $\mathbf{T}_j$ at each node j involves the values of the three consecutive nodes $\mathbf{x}_{j-1}$, $\mathbf{x}_j$ and $\mathbf{x}_{j+1}$. To treat the end nodes $\mathbf{x}_1$ and $\mathbf{x}_n$ in the same manner as the interior nodes, we extend the definition of the curve by adjoining two additional nodes, $\mathbf{x}_0$ and $\mathbf{x}_{n+1}$, by defining

$$\mathbf{x}_0 = \mathbf{x}_1 - (\mathbf{x}_2 - \mathbf{x}_1), \qquad \mathbf{x}_{n+1} = \mathbf{x}_n + (\mathbf{x}_n - \mathbf{x}_{n-1}).$$

In other words, $\mathbf{x}_1 = (\mathbf{x}_0 + \mathbf{x}_2)/2$ and $\mathbf{x}_n = (\mathbf{x}_{n-1} + \mathbf{x}_{n+1})/2$. Then (8) determines $\mathbf{T}_j$ for all $j = 1, 2, \ldots, n$. Our choice of the extension points $\mathbf{x}_0$ and $\mathbf{x}_{n+1}$ results in a tube where the end sections are perpendicular to the centerline curve.

We initialize the vectors $\mathbf{P}_j$ and $\mathbf{Q}_j$ at the node $j = 1$ by taking any pair of vectors $\mathbf{P}_1$, $\mathbf{Q}_1$ such that $\{\mathbf{T}_1, \mathbf{P}_1, \mathbf{Q}_1\}$ is a positively oriented orthonormal set. Then the recurrence in (9) determines $\mathbf{P}_j$ and $\mathbf{Q}_j$ for $j = 2, 3, \ldots, n$.

3.2. **Initializing a closed curve.** In this case we also extend the curve by adjoining two additional nodes, $\mathbf{x}_0$ and $\mathbf{x}_{n+1}$, by defining

$$\mathbf{x}_0 = \mathbf{x}_{n-1}, \qquad \mathbf{x}_{n+1} = \mathbf{x}_2.$$

Then (8) determines $\mathbf{T}_j$ for all $j = 1, 2, \ldots, n$. Our choice of the extension points $\mathbf{x}_0$ and $\mathbf{x}_{n+1}$ removes any distinction between the interior and end nodes of the curve. The segments $\mathbf{x}_j - \mathbf{x}_{j-1}$ and $\mathbf{x}_{j+1} - \mathbf{x}_j$ make equal angles with the curve's normal section at $\mathbf{x}_j$, at all nodes including the end nodes, thus making the tube smoothly close onto itself and and giving it an appearance of one continuous, seamless surface with no boundary.

We initialize the vectors $\mathbf{P}_j$ and $\mathbf{Q}_j$ at node $j = 1$ by taking any pair of vectors $\mathbf{P}_1$, $\mathbf{Q}_1$ such that $\{\mathbf{T}_1, \mathbf{P}_1, \mathbf{Q}_1\}$ is a positively oriented orthonormal set. Then the recurrence in (9) determines $\mathbf{P}_j$ and $\mathbf{Q}_j$ for $j = 2, 3, \ldots, n$.

4. Further considerations

We noted in Section 2 that even for as simple a curve as a helix the tube triad is not periodic in general. By the same token the triad does not necessarily return to its starting position on a closed curve as it goes a full round around the curve. This produces unacceptable results when the tube is displayed as a meshed surface because the mesh does not line up where the ends of the tube join.[4]

To draw closed tubes, Dynagraph computes the angle between the starting and ending positions of the triad, then it retroactively adjusts the rotation of the triad about the $\mathbf{T}$ vector by distributing the amount of mismatch uniformly along the curve. The mesh in Figure 3, which has been produced this way, shows the seamless closure of the grid lines.

References

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[2] Jack K. Hale. *Ordinary differential equations.* Robert E. Krieger Publishing Co. Inc., Huntington, N.Y., second edition, 1980.

[3] Barrett O'Neill. *Elementary differential geometry.* Academic Press, New York, 1966.

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[4]This problem does not exist with the Frenet-Serret system because the Frenet-Serret triad is a local property of the curve therefore its vectors necessarily coincide where the curve smoothly closes upon itself.