Rolling Dynamics of an Inhomogeneous Ball on an Inclined Track

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Experiments and computer simulations show that when a rigid ball with inhomogeneous but symmetric density distribution rolls on an inclined track, it tends to adjust its orientation to rotate about its major axis of inertia. We present tentative arguments toward explaining that behavior and sketch ideas that may lead to rigorous analysis and better understanding of the phenomenon.

I. INTRODUCTION

In this note we report on experimental results and computer simulations of an interesting phenomenon observed in a simple rigid body motion.

We consider a rigid spherical ball of radius $R$ with an inhomogeneous mass distribution. We assume that its mass is distributed symmetrically about the ball’s center, therefore the center of mass coincides with the geometric center. However, the three principal moments of inertia $I_1 \leq I_2 \leq I_3$ (relative to the center) are not necessarily equal.

We construct a “track” consisting of a pair of parallel slender rigid cylinders (rails) of radius $r$ each, with their axes set at a distance of $a$ apart. The track is inclined and makes an angle of $\alpha$ relative to the ground; see Figure 1. The ball is placed on the track and allowed to roll down. We assume that $a > 2r$ so that the rails do not overlap and $2R > a - 2r$ so that the ball does not fall through the gap between the tracks.

The contact between the ball and the rails is assumed to be of the Coulomb type, i.e., dry friction. The ball will slip at the contact points if the tangential contact force exceeds what the dry friction can support. The ball is allowed to lift off the tracks, losing contact with one or both rails, if the dynamics so dictates.

The initial conditions of the motion may be quite generic and is not instrumental to the discussion that follows. In the simplest case, we start with the ball in contact with both rails, with zero linear and angular velocities, and its principal axes of moment of inertia oriented in a random way.

As the ball rolls down the track, it picks up speed and spins progressively faster. In the early stages of the motion it maintains no-slip contact with both rails. However as its angular velocity increases, the inhomogeneous distribution of mass results in dynamic imbalance (due to unequal principal moments of inertia) and a sequence of events ensues. Depending on the situation, one or both contacts may begin to slip. It is also possible that dynamic forces may cause the ball to lift off from one or even both rails (see the illustrations in Figure 2). Subsequently, the ball may recover contact with one or both rails or it may completely fall off the track, never to return. The latter case is more likely to happen if the distance $a$ between tracks is small relative to the ball’s radius $R$. The falling off case is of no immediate interest to us; in this article we focus on those cases where the ball does not fall off the tracks.

The behavior just described is strictly a three-dimensional phenomenon. The imbalance is due to torques, resulting from centrifugal forces, that tend to twist the ball in a plane perpendicular to the direction of motion. Otherwise the ball is statically balanced about its center because of the symmetric distribution of mass. If we dispense with that symmetry, then a two-dimensional counterpart exist in the classic “hopping hoop” (cf. Littlewood12, Tokieda22) where a point mass
FIG. 1. These schematic diagrams depict three views of the ball of radius \( R \) riding on an inclined track consisting of two parallel cylindrical rails of radius \( r \) each, set a distance \( a \) apart. We assume \( 2R > a - 2r \), so that the ball doesn’t fall through the gap between the tracks.

FIG. 2. Diagrams of the ball in views down the track. During the motion the ball may be in contact with both rails, only one rail, or have no contact with rails at all. The modeling of the dynamics of the one-contact cases is quite challenging—the dynamics in nonholonomic, the contact constrained is unilateral, and the stick-slip nature of Coulomb friction introduces discontinuities in the differential equations of motion.

is attached to a weightless hoop which rolls without slipping in a vertical plane. The hoop lifts off (hops) when the radius vector to the weight becomes horizontal.

A. Rolling auto-orientation

Extensive laboratory experiments, observations with high-speed cameras, and detailed computer simulations point to an interesting phenomenon: As the ball rolls down the track, it tends to orient itself so that in the long run it spins about the major axis of inertia, that is, the axis of largest moment of inertia. We refer to this behavior as rolling auto-orientation.

We have no convincing explanation for rolling auto-orientation in terms of basic principles of mechanics. The familiar minimum energy argument of Section V, which works well for unconstrained motion, does not really apply here. The stochastic analysis approach outlined in Section VI holds some promise but it requires substantial development to yield useful information. The purpose of this Note is to bring this phenomenon to the applied mechanics community in the hope that better explanations than ours may emerge as result.

B. An overview of the sections

This note is organized as follows. In Section II we describe the expected qualitative behavior of the rolling ball and identify nine modes of motion. We describe obstacles in solving numerically the differential equations of motion in four modes that involve nonholonomic constraints. In Section III we outline our approach to simulating the motion using the computer software MSC ADAMS and show that simulations confirm the rolling auto-orientation. In Section IV we describe the methods and results of several experiments which again confirm auto-orientation. Section V describes a well-known mechanical system which exhibits auto-orientation but the reasoning that justifies that behavior does not apply to our system. Finally, Section VI notes a scarcely explored but promising approach to the analysis of the rolling auto-orientation phenomenon from the stochastic point of view.

C. A note on the origins of this study

The study reported here had its beginnings in the analysis of a prototype device (whose patent is pending) developed for the US Department of Agriculture whose purpose is to optically scan apples for contaminants and blemishes before they are shipped to the market. Apples are rolled down inclined tracks in an assembly line fashion and are scanned as they go past optical scan-
ning devices. Experiments show that most varieties of apples orient themselves after a brief amount of rolling, and regardless of the initial conditions, they spin about the stem-calyx axis in the long run. Such automatic orientation is essential for proper scanning because otherwise the optical devices may confuse the stem or calyx for external contaminants. See\textsuperscript{9,14–16} for reports of these experiments and description of instrumentation.

The perfect sphere model presented in this note removes geometric complications due to a real apple’s irregular shape but retains the unequal principal moments of inertia.

II. INTEGRATING THE EQUATIONS OF MOTION

In a typical case, after the ball picks up sufficient angular speed, it loses contact with one rail and rolls for a while with a single point contact with the other rail; see representative schematic diagrams in Figure 2. During the motion with one-point contact, the ball has extra degrees of freedom compared with the case of two-point contact. Computer simulations and experimental observations with high-speed cameras show that the ball tends to pivot about that single point of contact while continuing its motion down the track. In due course the center of the ball drops due to the combined action of gravity and other dynamical influences and the ball regains contact with both rails. This lifting and regaining of contact may happen a few times in quick succession. The ball reorients itself during those one-point contact phases, each time coming closer to the final orientation where it will be spinning about the major axis of inertia.

Since rotation about a principal axis of inertia is free of dynamical imbalance, once the ball settles into rotating about a principal axis, the orientation is no longer disturbed; it continues a simple accelerated rolling motion down the track.

The equations of motions of the ball may be obtained from the general principles of rigid body dynamics. They are quite involved, however, because accounting for the unequal principal moments of inertia requires tracking the orientation of the ball through its Euler angles or the equivalent. The imposition of the unilateral constraints at contact points and dry friction introduce further complications.

The rolling motion of the ball may be divided into 9 distinct modes. In mode 1, the simplest of all cases, the ball rolls with no-slip contact with both rails. In modes 2, 3, and 4 the ball retains contact with both rails however it slips against one or the other or both rails. In modes 5 and 6 the ball is in no-slip contact with \textit{only one rail}. Modes 7 and 8 are like modes 5 and 6 but the ball slips at the contact point. In mode 9 the ball detaches from both rails and is in free flight; see illustrations in Figure 2.

The differential equations of motion are quite different in different modes. The motion in mode 1, for example, is expressed as a simple ordinary differential equation that may be solved quite trivially by hand. Equations for mode 9 are Euler’s differential equations for a freely rotating rigid body. The equations for the remaining modes turn out to be quite complicated—the nonholonomic nature of the one-point contact leads to a system of 12 coupled \textit{differential-algebraic equations} (DAEs) involving Euler angles, their derivatives, and other kinematic variables. The recent books by Holm\textsuperscript{6,7} and monographs on nonholonomic systems by Bloch\textsuperscript{1}, Chirikjian and Kyatkin\textsuperscript{3}, and Neimark and Fufaev\textsuperscript{17}, have a wealth of information on nonholonomic constraints and methods for handling them. Neimark and Fufaev’s monograph, in particular, has a detailed study of the rolling of a ball on a general curved surface with a one-point, no-slip, contact, but the ball there is homogeneous, therefore complications from tracking the Euler’s angles do not arise.

We used the computer algebra system MAPLE\textsuperscript{24} to organize the derivations of the equations of motion and perform some of the unwieldy calculations. We used MAPLE’s differential equations solver (in the numeric, not symbolic, mode) to solve and visualize the results. Switching among the nine modes would be done through a master controlling procedure that would monitor the constraint reactions at every time step and switch from one mode to another, as necessary.

Unfortunately the computations were not always successful. We ran into an inherent difficulty with the equations of motion of our system.

As noted above, the effect of presence of nonholonomic constraints in modes 5–8 is that the equations of motion change from a system of ordinary differential equations (ODEs) to a set of differential algebraic equations (DAEs). The theory and algorithms for numerical solvers for DAEs are not fully developed. The state of the art is described in the book of Brenan, Campbell and Petzold\textsuperscript{2}. The theory of DAEs of index 1 is quite well-developed. The article by Shampine, Reichelt, and Kierzenka\textsuperscript{19} is a survey of numerical techniques for solving index 1 DAEs and their implementation as built into MATLAB\textsuperscript{25}. The current knowledge of DAEs of index 2 is somewhat spotty. There is hardly anything useful known for DAEs of higher index.

MAPLE’s DAE solver is limited to DAEs of index 1 or 2. It determines that our equations are of higher index and gives up. Porting the equations to MATLAB did not help either, because MATLAB’s DAE solvers can handle only index 1. We are not aware of general algorithms or software for solving DAEs of index higher than 2. Our conclusion is that developing a numerical integrator for the ball’s equations of motion would be a challenging but worthwhile undertaking. It calls for innovative ideas and techniques which may be useful in other contexts as well.

At this point we should point out yet another unpleasant aspect of the mathematical model of a ball on two
Thus the vertical component of each support is the weight of the ball resting on the rails. Figure 3 depicts the situation. The horizontal components of the reaction are statically indeterminate; they can be any pair of equal and opposite forces. The indeterminacy may be removed by assuming that the resultant of the reaction force at each support is normal to the ball's surface. This assumption applies if the surfaces of the ball and the rails are polished therefore the contact is frictionless, otherwise the horizontal components of the force are not determined from considerations of statics alone. To obtain a feel for the situation, imagine holding up an orange with two or three fingers, similar to the drawing in Figure 3, and squeezing it gently. You should be able to modulate the horizontal components of reactions within a limited range, if your fingers are not too slippery.

The indeterminacy of the horizontal components of reaction forces is a serious issue in modeling the equations of motion of the rolling ball. As we have noted above, the motion faces nine possible modes, each described by a distinct set of differential equations. The motion switches from a non-slip to slip mode when the tangential component of the reaction force exceeds what Coulomb friction can support. Since the tangential component is undefined if the horizontal reactions are indeterminate, then the entire scheme of mode-switching breaks down.

One way to remove the indeterminacy of the reactions is to model the ball and the rails as elastic bodies and account for elastic deformations when computing the reactions. This is similar to the situation in the analysis of statically indeterminate trusses where the indeterminacy is resolved by accounting for the elastic properties of the truss' members.

The method described in the next section accounts for elastic deformation at contacts therefore does not suffer from the problems that we have recounted above.

### III. MOTION SIMULATION

The approach taken by the MSC ADAMS motion simulator overcomes the problems described in the previous section by allowing local deformation of objects upon contact according to an extended Hertz contact model. To describe the idea, consider an instant where the ball is lifted momentarily off both tracks, as in the rightmost diagram in Figure 2. The subsequent motion of the ball is a free fall under the influence of gravity and the integration of the differential equations of motion is straightforward. The numerical integrator takes small time steps and at each step updates the ball's phase space variables, that is, the location and orientation of the ball, and the corresponding linear and angular momenta. At each step the integrator checks for collisions, which in our case is a matter of computing the distance $d$ of the ball's center from either track. If $d > R + r$, then there is no collision and the free fall continues to the next time step.

If $d < R + r$, then the ball has penetrated the rail. The Hertz model assumes that the rail reacts by repelling the ball with a force of magnitude $F = k \delta^\epsilon$, where $\delta = R + r - d$ is the depth of penetration, and the constants $k$ and $\epsilon$ encapsulate in a very primitive, but still meaningful way, the complex events that occur during the impact. There is no general rule for selecting their values other than trial and error, some judicious guessing, and accumulated experience. In the contact of smooth objects, such as a sphere and a cylinder, the choice $\epsilon = 2$ amounts to assuming that the repulsion force $F$ is proportional to the volume of the overlap region. MSC ADAMS' user’s guide suggests $\epsilon = 2.1$.

To account for possible plastic deformation and energy loss, MSC ADAMS augments the repulsion force $F$ by a damping term which is a certain nonlinear function of the depth and speed of penetration. If the objects have a sliding motion relative to each other during the collision, a further term is added to account for dry friction. The resultant of the reaction forces is used then to determine the change in linear and angular momenta of the ball, the phase space variables are updated accordingly, and the integrator proceeds to the next time step.

The graphs in Figure 4, produced by MSC ADAMS, show typical results of the simulation. The graph on the left corresponds to the principal moments of inertia selected as $I_1/I_3 = I_2/I_3 = 0.9$. The graph on the right corresponds to the principal moments of inertia selected as $I_1/I_3 = I_2/I_3 = 1.1$. In both cases the ball starts rolling from rest, in contact with both rails, and its major axis of inertia making a 45 degree angle with the vertical. The slope of the track is 20 degrees. The graphs show that in both cases the ball orient itself to roll about the major axis of inertia.

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**FIG. 3.** A ball resting on parallel horizontal rails. If the contact is frictional, then the support reactions are statically indeterminate.
vector that points along the principal inertia axis corresponding to orientation is measured by the angle between a fixed horizontal vector which is perpendicular to the tracks and a variable of the ball. The total mass of each configuration was

\[ M + 12 \text{ grams} \]

of the ball. We created three loaded balls, according to the conjecture on the automatic alignment of a rolling ball. Each coin has a mass of 100 grams. Its two hemispheres could be split open and snapped back. We modified its moments of inertia by affixing stacks of US quarter-dollar coins to the inside of the hollow shell. Each coin has a mass of \( m = 5.67 \text{ grams} \), radius of 12.13 millimeters, and thickness of 1.75 millimeters. We created three loaded balls, according to the configurations below, described in terms of an \( xyz \) Cartesian system of coordinates centered at the center of the ball. The total mass of each configuration was \( M + 12m = 168 \text{ grams} \). The principal moments of inertia, \( I_x, I_y, I_z \), of each configuration were computable from the given data.

a. Configuration 1: Two diametrically opposite stacks of 6 coins each on the \( z \) axis. This corresponds to \( I_x/I_z = I_y/I_z = 1.61 \), thus \( I_z \) is the smaller of the three moments of inertia. The left column in Figure 6 shows a composite image extracted from a high speed camera movie of the ball rolling down the track. Two large dots mark the north and south poles (the locations of the coins). We see that after just a few revolutions, the ball has oriented itself to spin about an axis perpendicular to the \( z \)-axis.

b. Configuration 2: Two diametrically opposite stacks of 3 coins each on the \( x \) axes plus two diametrically opposite stacks of 3 coins each on the \( y \) axes. This corresponds to \( I_x/I_z = I_y/I_z = 0.786 \), thus \( I_z \) is the larger of the three moments of inertia. The middle column in Figure 6 shows a composite image extracted from a high speed camera movie. The equators (the \( xy \) plane) is marked by a dark circle. We see that after just a few revolutions, the ball has oriented itself to spin about the \( z \)-axis.

c. Configuration 3: Two diametrically opposite stacks of 3 coins each on the \( x \) axes plus two diametrically opposite stacks of 2 coins each on the \( y \) axes plus a pair of diametrically opposite coins on the \( z \) axes. This corresponds to \( I_x/I_z = 0.858 \), \( I_y/I_z = 0.924 \), thus \( I_z \) is the larger of the three moments of inertia. The right column in Figure 6 shows a composite image extracted from a high speed camera movie. The equators (the \( xy \) plane) is marked by a dark circle. We see that after just a few revolutions, the ball has oriented itself to spin about the \( z \)-axis.

In each experiment, the initial conditions consisted of zero linear and angular velocities, the ball was in contact with both rails, and it was oriented so that one of the principal moments of inertia was parallel to the track and another one made an approximate 45 degree angle with the track in our experiment consisted of 3.5 meter long parallel wooden rails, set at an angle of 20 degrees relative to the ground. The diagram in Figure 5 shows a scale drawing of the track's cross-section and the ball resting on the track.

The thin hollow plastic ball used in our experiments had a radius of 30 millimeters and a mass of \( M = 100 \text{ grams} \). Its two hemispheres could be split open and snapped back. We modified its moments of inertia by affixing stacks of US quarter-dollar coins to the inside of the hollow shell. Each coin has a mass of \( m = 5.67 \text{ grams} \), radius of 12.13 millimeters, and thickness of 1.75 millimeters. We created three loaded balls, according to the configurations below, described in terms of an \( xyz \) Cartesian system of coordinates centered at the center of the ball. The total mass of each configuration was \( M + 12m = 168 \text{ grams} \). The principal moments of inertia, \( I_x, I_y, I_z \), of each configuration were computable from the given data.

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In each experiment, the initial conditions consisted of zero linear and angular velocities, the ball was in contact with both rails, and it was oriented so that one of the principal moments of inertia was parallel to the track and another one made an approximate 45 degree angle with

![Diagram of ball on track](image1)

FIG. 4. Graphs produced by MSC Adams show the orientation of the ball versus length traveled down the track. The orientation is measured by the angle between a fixed horizontal vector which is perpendicular to the tracks and a variable vector that points along the principal inertia axis corresponding to \( I_z \). The graph on the left corresponds to the moments of inertia \( I_1/I_3 = I_2/I_3 = 0.9 \). The graph on the right corresponds to the moments of inertia \( I_1/I_3 = I_2/I_3 = 1.1 \). In either case, the ball orients itself to roll about the major axis of inertia.

![Scale diagram of ball on track](image2)

FIG. 5. A scale diagram of the ball resting on the experimental wooden tracks. The overall length of the track is 3.5 meters but typically the ball achieved its final orientation within 1 meter of its starting point.

IV. EXPERIMENTS

We performed a series of experiments to confirm our conjecture on the automatic alignment of a rolling ball. The track in our experiment consisted of 3.5 meter long parallel wooden rails, set at an angle of 20 degrees relative to the ground. The diagram in Figure 5 shows a scale drawing of the track’s cross-section and the ball resting on the track.

The thin hollow plastic ball used in our experiments had a radius of 30 millimeters and a mass of \( M = 100 \text{ grams} \). Its two hemispheres could be split open and snapped back. We modified its moments of inertia by affixing stacks of US quarter-dollar coins to the inside of the hollow shell. Each coin has a mass of \( m = 5.67 \text{ grams} \), radius of 12.13 millimeters, and thickness of 1.75 millimeters. We created three loaded balls, according to the configurations below, described in terms of an \( xyz \) Cartesian system of coordinates centered at the center of the ball. The total mass of each configuration was \( M + 12m = 168 \text{ grams} \). The principal moments of inertia, \( I_x, I_y, I_z \), of each configuration were computable from the given data.

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In each experiment, the initial conditions consisted of zero linear and angular velocities, the ball was in contact with both rails, and it was oriented so that one of the principal moments of inertia was parallel to the track and another one made an approximate 45 degree angle with

![Graphs of orientation](image3)

FIG. 4. Graphs produced by MSC Adams show the orientation of the ball versus length traveled down the track. The orientation is measured by the angle between a fixed horizontal vector which is perpendicular to the tracks and a variable vector that points along the principal inertia axis corresponding to \( I_z \). The graph on the left corresponds to the moments of inertia \( I_1/I_3 = I_2/I_3 = 0.9 \). The graph on the right corresponds to the moments of inertia \( I_1/I_3 = I_2/I_3 = 1.1 \). In either case, the ball orients itself to roll about the major axis of inertia.
"exposure" effect. The left column, corresponding to Configuration 1, has internal masses concentrated near the north and south poles. These are marked by the large dots. The middle column, corresponding to Configuration 2, has equal masses placed in 90 degree intervals around the equator which is marked by a dark circle. The right column, corresponding to Configuration 3, has unequal pairs of masses placed in three orthogonal directions. The equator is marked by a dark circle. In all cases, the ball quickly orients itself to rotate about an axis of maximal moment of inertia.

the plane of the track. The outcomes of the experiments were not sensitive to the initial orientation; within a short distance the ball oriented itself to spin about the major axis of inertia.

V. AN ENERGY ARGUMENT

It is tempting to explain away the auto-orientation phenomenon through a familiar energy argument. It is said that abandoned orbiting satellites tend to orient themselves in the long run to spin about the major axis of inertia. There is an elementary explanation for it (see, e.g., page 371 of Greenwood): away from the drag of the atmosphere and other non-conservative external influences, its angular momentum, $H$, remains constant. Suppose an internal mechanism dissipates rotational kinetic energy into heat. If the magnitude of the angular velocity is $\omega$ and the object spins about the principal moment of inertia axis $I_i$, the magnitude of the angular momentum is $H = I_i \omega$ and the kinetic energy is $K = I_i \omega^2 / 2 = (I_i \omega)^2 / (2I_i) = H^2 / (2I_i)$. The minimum of the kinetic energy, subject to constraint of constant angular momentum, is achieved when $I_i$ the largest, thus in its final state the object will be spinning about the major axis of inertia.

Such argument is not directly applicable to the ball on the track. The angular momentum is certainly not constant and the interaction with the track plays an essential role and cannot be ignored. There is no fundamental law of mechanics that we know of that implies the observed large time behavior.

VI. STOCHASTIC DYNAMICS

Stochastic dynamics offers a very different approach to the analysis of the large time behavior of the rolling ball and may hold the key to the rigorous explanation of the rolling auto-orientation phenomenon.

Consider a ball with unequal principal moments of inertia, as we have in the previous sections. Let us consider the ball in isolation—no tracks or gravity—as it would be the case if it were floating freely in the outer space.

Now imagine the surface of the ball under continuous bombardment of random torques in the form of isotropic white noise. Specifically, let us look at the limiting case where the frequency of the bombardments goes to infinity and their amplitude goes to zero. How will the ball behave in the long run? We expect that the ball will settle into a (rotational) Brownian motion in the space of special orthogonal group $SO(3)$. There is a large amount of literature on the Brownian motion of a point mass but very little on Brownian motion of rigid bodies. The book by McConnell and articles by Liao and Wang are among the few that address the issue. Bounds for the expected values of deviation from steady-state rotation about the major or minor axes of inertia are derived in.

To connect this to the auto-orientation problem, we may view the reactions of the rails as "random bombardment" of forces that toss the rolling ball about in unpredictable ways. If under such random influences the ball happens to come close to rotating about the major axis of inertia, then the bombardment ceases, because rotation about a principal axis of inertia is free of dynamic imbalance. Thereafter the ball rolls down the track without further disturbance. The preference of the random dynamics to select the major, rather than the minor, axis of inertia may be attributable to the wider basin of attraction (deeper energy well) corresponding to rotation about the major axis of inertia.

In a sense, the situation here is akin to the atomistic versus phenomenological view of matter. From the atomistic point of view, an ideal gas consists of a very large number of point masses flying about in random directions and velocities within the confines of a rigid boundary. Each particle behaves according to the Newtonian laws of motion. But tracking the motion of individual particles is rather pointless. At the phenomenological
level, the gas is viewed as a continuum, characterized by its state variables: pressure, temperature and density. In statistical mechanics one reconciles the two points of view by relating the state variables to certain averages of the point mass dynamics.

The stochastic approach outlined in this section takes the view that the rolling auto-orientation is a phenomenological effect in the sense that it is the result of averaging of fine (and unknowable) details of the random interaction of the ball with the track.

In this connection, it is interesting to note that mathematical tools developed for the study of quantum mechanics may be brought to bear some understanding of the effects of random initial conditions on the evolution of the ball’s motion.

To illustrate, assume that the ball starts out with random initial orientation and angular velocity, both with uniform probability distributions. Thus the set of all possible initial conditions may be viewed as an isotropic spherical “cloud” in the probability space. From our experiments and simulations we know that as the ball rolls down the track, it prefers to take on an orientation whereby it spins about its major axis of inertia. If that is so, then as the ball moves down the track, the cloud of all possible configurations elongates in one direction, losing its isotropy, and giving a greater probability to spinning about the major axis.

The evolution of the cloud is governed by the Fokker-Plank partial differential equation. (See Chirikjian and Kyatkin\(^3\) and Kadanoff\(^8\) for the theory and application of the Fokker-Plank equations.) Thus the analysis of the orientation of the rolling ball reduces to solving a partial differential equation and determining the evolution of the probability cloud as time goes to infinity. Working out the details of such an analysis is nontrivial because determining the coefficients of the Fokker-Plank equation requires a close analysis of the ball’s dynamics.

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