Bouncing on a slope

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In Figure 1 we see a point-mass projectile launched with velocity $v(0)$ that follows a parabolic arc in a vertical plane, hits a slanted surface, and rebounds. We neglect air resistance, thus the trajectory consists of parabolic arcs. We assume no energy loss, thus the magnitude of velocity is the same just before and after impact. We neglect surface friction, thus the component of projectile’s momentum parallel to the surface does not change at the moment of impact. It follows that the incident and reflected arcs form equal angles with the surface’s normal at the point of impact, just like light rays do when reflecting on a polished surface.

Figure 1: Projectile launched with velocity $v(0)$. At impact, the path’s incident and reflected arcs make equal angles with the normal to the slanted surface.

In this note we investigate the projectile’s behavior through successive bounces. The projectile cannot climb indefinitely because its energy is conserved, thus it will have to reverse its motion and “fall back”. What triggers the reversal? How many forward bounces are there before the reversal? Does the dynamics allow infinitely many forward bounces, in the style of Zeno’s tortoise, that accumulate at a point as shown in Figure 2? Does the projective retrace its path upon reversal or does it follow a different path?

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1 Equation of motion

Let us get an important detail out of the way before we begin. In this article we focus on motions that take place in a single vertical plane. This amounts to saying that the initial velocity vector \( v(0) \) lies in the vertical plane formed by the slanted surface’s normal vector and the vector of acceleration of gravity. Thus in effect the problem is strictly two-dimensional and the “slanted surface” is in effect a “slanted line” although we continue referring to it as a surface. The more general situation where \( v(0) \) points to an arbitrary direction in the three dimensional space is discussed briefly in Section 5; see Figure 7.

The analysis of the motion in this article is entirely coordinate-free; there are no references to a Cartesian coordinate system.\(^1\) With this in mind, let \( \{i, j\} \) and \( \{m, n\} \) be two pairs of orthogonal unit vectors as shown in Figure 3. The vector \( n \) is an outward normal to the surface, the acceleration of gravity is \(-g j\), and the vectors \( m \) and \( i \) are in the plane of \( n \) and \( j \). Additionally:

\[
m \cdot j > 0, \quad m \cdot i > 0, \quad n \cdot j > 0, \quad n \cdot i < 0, \quad v^{(0)} \cdot n > 0, \quad v^{(0)} \cdot i > 0. \tag{1}
\]

The first inequality says that \( m \) points uphill relative to the gravitational field. The second says that \( i \) points toward the surface’s ascending direction. The third says that the surface faces up as a roof, not down as a ceiling. The fourth inequality is implied by the first three. The fifth inequality says that we are shooting away from the surface, not into it. The last inequality says that the projectile’s first impact will be up the hill relative to the launch point.

Between bounces, the projectile’s equation of motion is given by Newton’s law, \( \ddot{x}(t) = -g j \), where \( x(t) \) be the projectile’s position vector at time \( t \), the dots indicate derivatives with respect to \( t \), and \( g \) is the acceleration of gravity. Let

\(^1\)This purely vector-based approach turns out to be especially rewarding when extended to the fully three dimensional case which is very briefly sketched in Section 5.
\{t_k\}, k = 0, 1, 2, \ldots be the sequence of instances of impact of the projectile with the slanted surface, with \(t_0\) corresponding to the initial launch time. Let \(x^{(k)}\) be the corresponding position vectors, and \(v^{(k)}\) be the velocity immediately after the \(k\)th impact; we call it the \textit{rebound velocity}; see Figure 4. Upon integrating the equation of motion we obtain

\[ \dot{x}(t) = v^{(k)} - (t - t_k)gj \]  \hspace{1cm} (2)

and

\[ x(t) = x^{(k)} + (t - t_k)v^{(k)} - \frac{1}{2}(t - t_k)^2gj. \]  \hspace{1cm} (3)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{One parabolic arc between rebounds. The impact velocity \(\tilde{v}^{(k+1)}\) and rebound velocity \(v^{(k+1)}\) are mirror images relative to the vector \(m\).}
\end{figure}

\textbf{Lemma 1.} \textit{The consecutive rebound velocities} \(v^{(k)}\) \textit{and} \(v^{(k+1)}\) \textit{are related through}

\[ v^{(k+1)} = v^{(k)} + 2\frac{i \cdot n}{j \cdot n} (v^{(k)} \cdot n)m. \]  \hspace{1cm} (4)

\textit{Proof.} Consider the motion between times \(t_k\) and \(t_{k+1}\). From equation (3) we see that

\[ x^{(k+1)} = x^{(k)} + (t_{k+1} - t_k)v^{(k)} - \frac{1}{2}(t_{k+1} - t_k)^2gj. \]

Since both \(x^{(k+1)}\) and \(x^{(k)}\) are on the hill’s surface, then \(n \cdot (x^{(k+1)} - x^{(k)}) = 0\), therefore

\[ (t_{k+1} - t_k)v^{(k)} \cdot n - \frac{1}{2}(t_{k+1} - t_k)^2gj \cdot n = 0, \]

which then yields,

\[ t_{k+1} = t_k + \frac{2v^{(k)} \cdot n}{gj \cdot n}. \]  \hspace{1cm} (5)

Substituting this value of \(t_{k+1}\) for \(t\) in equation (2) yields the \textit{impact velocity}, \(\tilde{v}^{(k+1)}\), at time \(t_{k+1}\):

\[ \tilde{v}^{(k+1)} = v^{(k)} - \frac{2v^{(k)} \cdot n}{j \cdot n}j. \]
The rebound velocity, \( v^{(k+1)} \) at time \( t_{k+1} \) is obtained from the conditions that upon impact the \( m \) component of the momentum does not change and the energy is conserved. It is easy to verify that these imply that the \( n \) component of the momentum reverses, that is:

\[
v^{(k+1)} \cdot m = \tilde{v}^{(k+1)} \cdot m, \quad v^{(k+1)} \cdot n = -\tilde{v}^{(k+1)} \cdot n.
\]

It follows that:

\[
v^{(k+1)} = (v^{(k+1)} \cdot m)m + (v^{(k+1)} \cdot n)n
\]

\[
= (\tilde{v}^{(k+1)} \cdot m)m - (\tilde{v}^{(k+1)} \cdot n)n
\]

\[
= \left[ (v^{(k)} - \frac{2v^{(k)} \cdot n}{j \cdot n}j) \cdot m \right] m - \left[ (v^{(k)} - \frac{2v^{(k)} \cdot n}{j \cdot n}j) \cdot n \right] n
\]

\[
= \left[ (v^{(k)} \cdot m)m + (v^{(k)} \cdot n)n \right] - 2(v^{(k)} \cdot n)n - \frac{2v^{(k)} \cdot n}{j \cdot n}((j \cdot m)m - (j \cdot n)n)
\]

\[
= v^{(k)} - \frac{2v^{(k)} \cdot n}{j \cdot n}j \cdot m,
\]

whence the lemma’s assertion follows because \( j \cdot m = -i \cdot n \).

There is more to this lemma than meets the eye. By dot-multiplying both sides of equation (4) by \( n \) we see that \( v^{(k+1)} \cdot n = v^{(k)} \cdot n \) which is quite remarkable—it says that all rebound velocities have the same normal component! Consequently

\[
v^{(k)} \cdot n = v^{(0)} \cdot n, \quad k = 0, 1, 2, \ldots (6)
\]

This, in conjunction with equation (5) says that the bounces occur in equal time intervals:

\[
t_{k+1} - t_k = \frac{2v^{(0)} \cdot n}{gj \cdot n}.
\]

In particular, the Zeno-like infinitely many bounces in finite time, shown in Figure 2, is not possible.

Furthermore, in view of equation (6) and Lemma 1, we obtain an explicit expression for the rebound velocity at time \( t_k \):

\[
v^{(k)} = v^{(0)} + 2k \frac{i \cdot n}{j \cdot n} (v^{(0)} \cdot n)m.
\]

2 Number of hops

The inequalities in (1) imply that the coefficient of \( m \) in equation (7) is negative thus a negative multiple of \( m \) is added to \( v^{(k)} \) on each bounce. Consequently, after a finite number of bounces, the horizontal projection of \( v^{(k)} \), that is \( v^{(k)} \cdot i \), becomes negative and the motion turns retrograde relative to the ground. To
obtain the point of reversal, we set \( v^{(k)} \cdot i = 0 \) in equation (7) and calculate \( k \), noting that \( m \cdot i = n \cdot j \). We obtain:

\[
\text{number of forward hops} = \left\lceil -\frac{v^{(0)} \cdot i}{2(i \cdot n)(v^{(0)} \cdot n)} \right\rceil,
\]

where \( \lceil \cdots \rceil \) denotes the smallest integer which is less or equal the enclosed quantity. This is the exact count of forward hops in the projectile’s trajectory.

Remark 1. Let \( 0 < \alpha < \pi/2 \) be the angle of the slanted surface relative to the horizon, and let \( 0 \leq \beta \leq \pi \) be the angle of the initial velocity vector \( v^{(0)} \) relative to the slanted surface. Then \( m = i \cos \alpha + j \sin \alpha \), \( n = -i \sin \alpha + j \cos \alpha \), and \( v^{(0)} = m \cos \beta + n \sin \beta \). Then the expression for the number of forward hops reduces to

\[
\text{number of forward hops} = \left\lceil \frac{\cos(\alpha + \beta)}{2 \sin \alpha \sin \beta} \right\rceil = \left\lceil \frac{1}{2} \left( \cot \alpha \cot \beta - 1 \right) \right\rceil.
\]

Interestingly, the number of forward hops is determined solely in terms of angles \( \alpha \) and \( \beta \). It depends neither on the magnitude of the initial velocity, \( v^{(0)} \), nor the acceleration of gravity, \( g \). In hindsight, the lack of dependence on \( v^{(0)} \) and \( g \) could have been anticipated on the basis of dimensional analysis.

We see that the number of uphill hops may be made arbitrarily large by making either \( \alpha \) or \( \beta \) sufficiently small. Thus, if the projectile is shot almost parallel to the hill, the number of hops would be very large. In the limiting case of \( \beta = 0 \), the projectile’s uphill glide may be viewed as a sequence of infinitely many infinitesimal hops.

Similarly, there will be infinitely many hops when \( \alpha = 0 \), \( \beta > 0 \). This corresponds to bouncing on a horizontal surface. The projectile hops forward along a horizontal line in a sequence of identical parabolic arcs and runs away to infinity, except when \( \beta = \pi/2 \) in which cases it bounces up and down on a vertical line indefinitely.

### 3 The envelope of the arcs

How far does the projectile rise above the hill at each bounce? This amounts to finding the maximum of \( x(t) \cdot n \) where \( x(t) \) is given in (3). We have:

\[
(x(t) - x^{(k)}) \cdot n = (t - t_k)(v^{(0)} \cdot n) - \frac{1}{2}(t - t_k)^2g(j \cdot n),
\]

where we have set \( v^{(k)} \cdot n = v^{(0)} \cdot n \) in accordance with equation (6). The time \( t^* \) when the maximum distance from the plane is achieved is obtained by solving \( \frac{dx}{dt}((x(t) - x^{(k)}) \cdot n) = 0 \) for \( t \). We obtain: \( t^* - t_k = \frac{v^{(0)} \cdot n}{g(j \cdot n)} \). Then the value of the maximum is expressed as:

\[
(x(t^*) - x^{(k)}) \cdot n = \frac{(v^{(0)} \cdot n)^2}{2g(j \cdot n)}.
\]
This value is independent of \( k \), thus all the parabolic arcs are tangent to the line that lies parallel and above the slanted line at the distance \( \left( \mathbf{v}(0) \cdot \mathbf{n} \right)^2/(2g(j \cdot \mathbf{n})) \); the envelope of the family of the parabolic arcs. The envelope is shown as a dotted line in Figures 5 and 6.

![Diagram of projectile trajectory](image)

Figure 5: The projectile’s trajectory corresponding to surface slant of \( \alpha = \pi/12 \) and launch angle \( \beta = \pi/4 \). Solid arches are forward hops. Dashed arches are backward hops. Number of forward hops is

\[
\left\lfloor \frac{1}{2} \left( \cot \frac{\pi}{12} \cot \frac{\pi}{4} - 1 \right) \right\rfloor = \left\lfloor \frac{1+\sqrt{2}}{2} \right\rfloor = 2.
\]

4 When is the forward path retraced?

As we see in Figure 5, the forward and reverse trajectories of the projectile are not necessarily the same. Are there situations where the forward and reverse trajectories coincide? The answer is yes:

**Lemma 2.** The projectile’s forward and retrograde trajectories coincide if and only if \( \cot \alpha \cot \beta \) is a positive integer.

In fact, there are two distinct circumstances when the trajectory is retraced, corresponding to the integer value in the lemma being odd or even. We will deal with these in the following two subsections.

4.1 Retraced trajectory: Case 1

In the left drawing in Figure 6 the projectile strikes the surface perpendicularly on the third impact. Consequently it rebounds in the perpendicular direction and retraces its incoming path. In general, a perpendicular impact will occur when \( \mathbf{v}(k) \cdot \mathbf{m} = 0 \) for some \( k \). Then from equation (7) we obtain

\[
\mathbf{v}(0) \cdot \mathbf{m} + 2k \frac{i \cdot \mathbf{n}}{j \cdot \mathbf{n}} (\mathbf{v}(0) \cdot \mathbf{n}) = 0,
\]

which, after expressing the dot products in terms of the angles \( \alpha \) and \( \beta \) as before, reduces to

\[
\cot \alpha \cot \beta = 2k.
\]
When $\alpha$ and $\beta$ are related as above where $k$ is any positive integer, the $k$th impact is perpendicular to the surface and the forward path is retraced upon reversal.

![Figure 6: Two cases of retraced trajectories. The surface slant is $\alpha = \pi/12$ radians in both cases. On the left, the launch angle $\beta$ is selected according to equation (9) with $k = 3$, thus $\beta = \arctan \frac{1}{6} \cot \frac{\pi}{12} \approx 31.882^\circ$. Consequently the third impact is perpendicular to the surface and the forward path is retraced upon return. On the right, the launch angle $\beta$ is selected according to equation (10) with $k = 2$, thus $\beta = \arctan \frac{1}{5} \cot \frac{\pi}{12} \approx 36.738^\circ$. Consequently the second rebound is in the vertical direction and the forward path is retraced upon return. Note that the third parabolic arc has degenerated into a vertical line segment.](image)

### 4.2 Retraced trajectory: Case 2

In the right drawing in Figure 6 the projectile rebounds in the vertical direction upon the second impact. After reaching maximum height, it falls vertically and its subsequent motion retraces the incoming path. In general, a vertically rising path will occur when $v^{(k)} \cdot i = 0$ for some $k$. Then from equation (7) we obtain

$$v^{(0)} \cdot i + 2k \frac{i \cdot n}{j \cdot n} (v^{(0)} \cdot n)(m \cdot i) = 0,$$

which, after expressing the dot products in terms of the angles $\alpha$ and $\beta$ as before, reduces to

$$\cot \alpha \cot \beta = 2k + 1.$$  \hfill (10)

When $\alpha$ and $\beta$ are related as above where $k$ is any positive integer, the $k$th rebound is in the vertical direction and the forward path is retraced upon reversal.

### 5 The full three dimensional case

Figure 7 illustrates a generic path in the full three dimensional setting. To extend the previous analysis to this case, we extend the vector basis $\{i, j\}$ to the right-handed orthonormal triad $\{i, j, k\}$ with $k = i \times j$. Since the pairs $\{i, j\}$...
and \( \{m, n\} \) are coplanar, then \( \{m, n, k\} \) also forms a right-handed orthonormal triad.

The differential equation of motion is \( \ddot{x}(t) = -g_j \), as before. It can be shown that equations (2) through (8) remain as they are. The expressions for the number of forward hops remain unchanged as well, once the meaning of forward hops is clarified appropriately. The envelope of the family of arcs, which was a straight line in the two dimensional case, is replaced by a plane. We leave the details as an exercise for the interested reader.