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★**The mathematical theory of elasticity.**

Second edition.
CRC Press, *Boca Raton*, *FL*, 2011. *xxvi*+800 *pp*. $119.95.
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This book offers a passionate account of the theories of linear elasticity and thermoelasticity and fills a significant niche, since finding both theories in one volume is rather rare.

The book seems to be addressed to theoretically inclined engineers. There is much to be learned from it by graduate students and researchers who wish to gain deeper understanding of the subject than is usually offered under the umbrella of "strength of materials" or "mechanics of materials". A large number of exercises at the end of each chapter provide opportunities for testing and extending one's understanding.

The elasticity part is firmly grounded in M. E. Gurtin's monograph [in *Handbuch der Physik. Band VIa/2*, 1–295, Springer, Berlin, 1972; see MR0347187 (49 #11907)] but there are significant additions in the treatment of elastodynamics. The thermoelasticity part shares some elements with D. E. Carlson's [in *Handbuch der Physik. Band VIa/2*, 297–345, Springer, Berlin, 1972; see MR0347187 (49 #11907)]. An applied mathematician would find the book under review somewhat too wordy and insufficiently precise. Gurtin's and Carlson's monographs are more concise and accurate but they lack illustrative examples and exercises.

Chapter 1 is a moderately long (16 pages) discussion of the history of linear elasticity, told in the form of capsule biographies of men who contributed to the creation and development of the theory. I found that compendium interesting and informative.

Chapter 2 introduces the mathematical notations and tools used in the rest of the book. Tensors are viewed as coordinate-free linear operators, which is the right approach for this subject. Coordinate-dependent index notation is also given in each instance for the benefit of those who find the coordinate-free concept too abstract. A good treatment of the Laplace transformation and convolutions is also included; these are used later in the formulation and analysis of initial value problems of elastodynamics.

Chapter 3 contains the core material of the subject. The notions of strain, stress, and constitutive equations are introduced and the equations of equilibrium and motion are derived. Thermoelasticity is also introduced here and treated as an integral part of the theory.

A nice feature is the explicit display of the elasticity tensors (as 6×6 symmetric matrices) for orthotropic, transversely isotropic, and isotropic materials in terms of Young's modulus, Poisson's ratio, etc.

Chapters 4 and 5 deal with the analysis of initial and boundary value problems. Chapters 6 through 8 deal with variational formulations and reductions related to two-dimensional geometry.

The second half of the book, Chapters 9 through 13, presents an extensive collection of initial and/or boundary value problems of elasticity that are solvable in (more or less) closed form. In most cases the derivations of the solutions are given in minute details. This can be a valuable resource for graduate students interested in developing their skills in analytical techniques pertaining to elasticity in particular, and mathematics in general.

My initial impression of this book was quite positive. I even began imagining it as a textbook for an interesting course for graduate students of applied mathematics. Unfortunately I had to change my mind upon closer examination of the book's details. In the rest of this review I will list some of the concerns that led me to that conclusion. Most of these would not be difficult to fix and I would hope that they will be fixed in a future edition.

- The book's title, "The Mathematical Theory of Elasticity", is somewhat misleading since finite elasticity receives hardly a mention at all—the book is strictly on linear theories. A better title would have been "Theories of Linear Elasticity and Thermoelasticity".
- As far as I could detect, no formal definition of an elastic body is given anywhere. The opening sentence in Section 1.1 comes close: "A body is called elastic if it returns to its original shape upon the removal of applied forces." But according to this, a rubber hemispherical shell is not an elastic body because it has at least two force-free equilibrium states. The fix requires the concept of infinitesimal deformations which doesn't appear until Chapter 3.
- In Section 3.1.1 we have: "The condition $\det(\nabla \boldsymbol{\kappa}) > 0$ means that the mapping $\boldsymbol{\kappa}$ is uniquely invertible." This is not entirely true; it is quite easy to construct a noninvertible $\boldsymbol{\kappa}$ with $\det(\nabla \boldsymbol{\kappa}) > 0$. The inequality $\det(\nabla \boldsymbol{\kappa}) > 0$ is a condition of *local* invertibility.
- In Section 3.1 the concepts of finite and infinitesimal deformations are intertwined in an unpleasant way and present quite a muddy picture. This material forms the mathematical foundation of the subject and deserves a more lucid treatment.

- The finite strain tensor corresponding to the deformation gradient $\boldsymbol{F}$ is defined as $\boldsymbol{D} = \frac{1}{2}\big(\boldsymbol{F}^{\mathsf{T}}\boldsymbol{F} - \mathbf{1}\big)$ on page 76 with no explanation or motivation. Consequently, the statement on page 78 that the deformation field $\boldsymbol{\kappa}(\boldsymbol{x}) = \boldsymbol{y}_0 + \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0)$ (where $\boldsymbol{Q}$ is an orthogonal tensor) is a rigid rotation *because* [my emphasis] the strain tensor $\boldsymbol{D}$ vanishes, makes little sense. The *polar decomposition* of tensors, which is the missing key here, is not mentioned at all in the book.
- An *infinitesimal rigid displacement* is defined on page 78 as a field of the form $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \boldsymbol{W}(\boldsymbol{x} - \boldsymbol{x}_0)$, where $\boldsymbol{W}$ is a skew tensor. The qualifier "infinitesimal" is dropped afterward without a notice or warning. For example, on page 81 the terms "infinitesimal rigid displacement" and "rigid displacement" are used interchangeably.
- Mathematical symbols for vectors and tensors are set in serif fonts within the main text, e.g., $\mathbf{u}$ and $\mathbf{D}$, but inexplicably and annoyingly switch to sans serif, e.g., u and D, in all examples throughout the book. This is probably due to a glitch in a typesetting macro.
- The definition of a *linearly elastic material* is introduced on page 137 through the linear constitutive equation $\boldsymbol{S} = \mathsf{C}[\boldsymbol{E}]$. It is assumed right from the beginning that the elasticity tensor C is symmetric, that is, $\boldsymbol{A}\cdot\mathsf{C}[\boldsymbol{B}] = \boldsymbol{B}\cdot\mathsf{C}[\boldsymbol{A}]$ for all symmetric second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. It would have been appropriate to note at this point that a material with such a symmetry is called *hyperelastic*. A good part of the theory of elasticity may be developed without the assumption of hyperelasticity.
- On page 139 we have the definition: "By an *anisotropic elastic body* we mean a body for which the tensor C possesses in general 21 different components." I know what the authors *intend* to say here but what they say is ambiguous at best and wrong at worst. For instance, what does this definition imply about a material whose C has only three independent components? Does it make it isotropic? What about a material whose C has only two independent components? The way to avoid the problem is to define an *isotropic* material first, then say that anything else is anisotropic.
- The proposition that the symmetry of the elasticity tensor is equivalent to the existence of a stored energy function is presented in a rambling argument that takes up three pages (152–154). See [M. E. Gurtin, op. cit.] for a concise and transparent proof.

- Section 3.3.5 introduces the constitutive equation $\mathbf{S} = \mathsf{C}[\boldsymbol{E}] + T\boldsymbol{M}$ for a thermoelastic body, where T is the temperature and $\boldsymbol{M}$ is the *stress-temperature* tensor. It then states that $\boldsymbol{M} = -(3\lambda + 2\mu)\alpha\mathbf{1}$ for an isotropic material (μ and λ are the Lamé moduli). Some elaboration of the reasons for this would not have been out of place.
- Section 4.1.5 states a theorem on the uniqueness of solutions of the mixed boundary value problems of elastostatics. Specifically, suppose that displacements and tractions are specified on the disjoint subsets ∂B_1 and ∂B_2 of the boundary, respectively. The theorem states that if ∂B_1 is nonempty, then the solution is unique. This does not pass careful scrutiny. For example, if ∂B_1 is a straight line, then the mixed boundary value problem can have infinitely many solutions that pivot about that line. The flaw in the statement becomes apparent upon an examination of the "proof".
- Section 5.1.2 introduces the method of Rayleigh-Ritz to approximate the displacements $\boldsymbol{u}$ of mixed boundary value problems of elastostatics by sums of the form $\boldsymbol{u}^{(N)} = \widehat{\boldsymbol{u}}^{(N)} + \sum_{n=1}^{N} a_n \boldsymbol{f}_n$, where $\widehat{\boldsymbol{u}}^{(N)}$ is chosen to capture the displacement boundary conditions on part ∂B_1 of the boundary and each $\boldsymbol{f}_n$ vanishes on ∂B_1. It is stated that: "If $N \to \infty$ the approximate displacement $\boldsymbol{u}^{(N)}$ converges to the displacement corresponding to the exact solution." This is not true in the absence of further assumptions on $\boldsymbol{f}_n$s, e.g., linear independence, completeness, orthogonality.

Despite these issues, the book still is of value as a quick reference for concepts and formulae, and especially for its large collection of exactly solvable problems. I intend to keep it at close reach on my bookshelf.

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