MATH 481, SPRING 2019

PROJECT 3
POLLUTION IN LAKES

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Abstract. We present a mathematical model of two interconnected reservoirs and solutes flowing through them. We derive a system of differential equation for the time-dependent concentrations of the solutes in the two reservoirs. We apply our model to simulate the concentrations of pollutants in lakes Erie and Ontario, and investigate the time that it takes to stabilize the lakes to a lower pollution level after cutting down the inflow of pollutants.

1. Introduction

Say what this article is about. You will find a wealth of information in Wikipedia under "Great Lakes", "Lake Erie", and "Lake Ontario".

2. The model

The schematic diagram in Figure 1 represents two interconnected and well-stirred reservoirs of volumes V_1 and V_2. A solute of concentration $c_{1,\text{in}}$ enters the first reservoir at the volumetric flow rate of $Q_{1,\text{in}}$. The output of the first reservoir flows into the second reservoir where it joins another inflow of the same solute at concentration $c_{2,\text{in}}$ at the flow rate of $Q_{2,\text{in}}$, then the mixture flows out at the rate of $Q_{2,\text{out}}$. The concentrations of the solutes in the two reservoirs are $c_1(t)$ and $c_2(t)$.

In this case study we will assume that the volumes V_1 and V_2, the flow rates $Q_{1,\text{in}}$ and $Q_{2,\text{in}}$, and the input concentrations $c_{1,\text{in}}$ and $c_{2,\text{in}}$ are constants, that is, they they do not change with time. Consequently, $Q_{1,\text{out}} = Q_{1,\text{in}}$ and $Q_{2,\text{out}} = Q_{1,\text{out}} + Q_{2,\text{in}}$.

3. The equations of balance of mass

Explain how the differential equations for for $c_1(t)$ and $c_2(t)$ are derived. Express them in terms of the *residence times* of the two reservoirs. The residence times are defined as $T_1 = V_1/Q_{1,\text{out}}$ and $T_2 = V_2/Q_{2,\text{out}}$. It would be good if you say a few words about what a residence time signifies.

Solve the system with initial conditions $c_1(0) = c_{1,0}$ and $c_2(0) = c_{2,0}$. You will find that

$$c_1(t) = c_{1,0}e^{-t/T_1} + c_{1,\text{in}}(1 - e^{-t/T_1})$$

but don't take my word for it. You will also find that $c_2(t)$ looks like this:

$$c_2(t) = c_{1,0}(\cdots) + c_{1,\text{in}}(\cdots) + c_{2,0}(\cdots) + c_{2,\text{in}}(\cdots).$$

Fill in the details. Then find the equilibrium values $\bar{c}_1$ and $\bar{c}_2$ of $c_1(t)$ and $c_2(t)$.

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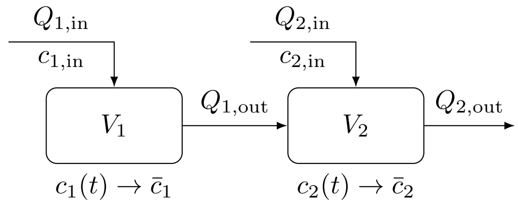

FIGURE 1. A schematic diagram of two interconnected reservoirs. If the volumes V_1 and V_2, the flow rates $Q_{1,\text{in}}$ and $Q_{2,\text{in}}$, and the input concentrations $c_{1,\text{in}}$ and $c_{2,\text{in}}$ are constants, then the concentrations $c_1(t)$ and $c_2(t)$ in the reservoirs tend to equilibrium values $\bar{c}_1$ and $\bar{c}_2$ in the long term.

4. Disturbing the system

Suppose that we reduce the concentrations $c_{1,\text{in}}$ and $c_{2,\text{in}}$ of the solutes entering the reservoirs to $\alpha c_{1,\text{in}}$ and $\beta c_{2,\text{in}}$, respectively, where $0 \le \alpha, \beta < 1$.

What are the differential equations for $c_1(t)$ and $c_2(t)$ in the new regime? Solve them with the previously computed equilibrium values $\bar{c}_1$ and $\bar{c}_2$ as initial conditions. Compute the new long time limits, $\tilde{c}_1$ and $\tilde{c}_2$.

Explain the motivation behind defining the quantities $\delta(t) = c_2(t) - \tilde{c}_2$ and $\rho(t) = \delta/\tilde{c}_2$. You will find that ρ looks like this:

$$\rho(t) = \frac{(\cdots)c_{1,\text{in}} + (\cdots)c_{2,\text{in}}}{(\cdots)c_{1,\text{in}} + (\cdots)c_{2,\text{in}}} = \frac{(\cdots) + (\cdots)\frac{c_{2,\text{in}}}{c_{1,\text{in}}}}{(\cdots) + (\cdots)\frac{c_{2,\text{in}}}{c_{1,\text{in}}}}.$$

Thus, $\rho(t)$ depends on $c_{1,\text{in}}$ and $c_{2,\text{in}}$ only through their ratio $c_{2,\text{in}}/c_{1,\text{in}}$.

5. Lakes Erie and Ontario

Suppose the states around the Great Lakes region are considering tough regulations to clean up the lakes. As a "what if" scenario, examine the rather drastic measure corresponding to setting $\alpha = 0.5$ and $\beta = 0$, where α and β are defined in Section 4.

Apply the analysis of the previous sections to determine the time that it will take for the pollution in Lake Ontario to stabilize to its new equilibrium state. We consider that this is achieved when $c_2(t)$ is within 5% of the $\tilde{c}_2$, that is, when $\rho(t) = 0.05$. For this, you will need the value of the ratio $c_{2,\text{in}}/c_{1,\text{in}}$. It turns out that the answer is not very sensitive to the choice of that value. Try a wide range of guesses, such as 0.3, 0.5, 0.7, to demonstrate this. Plot the corresponding $\rho(t)$ functions and superpose their graphs.

Ultimately what is achieved by these regulations and efforts? That is, how does $\tilde{c}_2$ compare against $\bar{c}_2$?

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