MATH 481, FALL 2024

PROJECT 2
DETECTING ART FORGERIES

YOUR NAME HERE

Abstract. We formulate a system of differential equations corresponding to evolution in time of a mixture of radioisotopes uranium-238, radium-226, lead-210, and their eventual conversion to lead-206 which is not radioactive. We explain how our analysis may be applied to detect a forged oil painting.

1. Introduction

Say what this article is about. You may want to refer to the presentation in Martin Braun's book [1], the *Wikipedia* articles [2, 3], and a plethora of articles on Johannes Vermeer and Han van Meegeren on the web.

Please note that a recounting of the subject's history may be useful to establish a context, but that is certainly inadequate as an introduction to the subject of the current project. An article's introduction should tell the reader what is to be found *in this article*, as in "In this article we explain a method of distinguishing between a genuine and forged oil paintings through the analysis of the ratios of radium-226 and lead-210 isotopes which are generally present in all lead-based oil paints".

A further detailed overview can help, such as: "In Section 2 we derive a system of differential equations that describe the quantities of radium-226 and lead-210 as functions of time." Then "In Section ... we do ...".

2. The mathematical model

A radioactive material, such as uranium-238 (chemical symbol ^{238}U), decays according to:

$$\frac{d}{dt}Q(t) = -\lambda Q(t),$$

where $Q(t)$ is the amount of the radioactive material at time t and λ is the material's *decay rate* or *decay constant*. In the context of this project, the "amount of material" refers to the number of atoms present in the sample.

Ideas for your writeup:

- Explain what a half-life is and derive the relationship between the decay rate λ and half-life τ.
- Why are ^{238}U, ^{226}Ra and ^{210}Pb always found together?
- The half-lives of ^{238}U, ^{226}Ra and ^{210}Pb are 4.5 billion years, 1600 years and 22 years, respectively. Compute their decay rates, λ_u, λ_r, and λ_p.

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- Let $\bar{U}$ be the amount of ^{238}U present in a sample. The decay rate of ^{238}U is so small that over the span of a few thousand years its amount effectively is a constant.
- Derive the differential equation for the amount of radium-226, $R(t)$, in the sample.
- Derive the differential equation for the amount of lead-210, $P(t)$, in the sample.
- Assume that radium-226 and lead-210 in the sample have reached mutual equilibrium with the uranium-238 that generates them. Show that their equilibrium amounts, $\bar{R}$ and $\bar{P}$, are related to the amount of the uranium source, $\bar{U}$, according to

$$\lambda_u \bar{U} = \lambda_r \bar{R} = \lambda_p \bar{P},$$

or equivalently,

$$\frac{\bar{U}}{\tau_u} = \frac{\bar{R}}{\tau_r} = \frac{\bar{P}}{\tau_p}, \tag{1}$$

where τ_u, τ_r, and τ_p are the half-lives of ^{238}U, ^{226}Ra, and ^{210}Pb, respectively.

3. Detecting a forgery

Consider lead ore that has been in an undisturbed state for many thousands of years. Then the amounts $\bar{U}$, $\bar{R}$, and $\bar{P}$ of ^{238}U, ^{226}Ra and ^{210}Pb will be interrelated as in the equilibrium equations (1).

The ore is taken to the smeltery to extract its lead. Essentially all of lead-210 and lead-206 is recovered but most of uranium and radium goes out as waste. Thus the equilibrium among ^{238}U, ^{226}Ra and ^{210}Pb in the resulting product is lost.

Let's say that only a fraction α of the uranium and radium remains in the processed lead. For numerical experiments take $\alpha = 10^{-4}$. Write a new set of differential equations and initial conditions that models the decay of the isotopes in the new product. Solve the system of differential equation for $R(t)$ and $P(t)$. Find the expression for the ratio $\rho(t) = P(t)/R(t)$. Verify that the ratio converges to $\bar{P}/\bar{R}$ as $t \to \infty$. Plot $\rho(t)$ over a span of 300 years. Explain the relevance of this to detecting an art forgery.

You may have use for Figure 1 as you develop your article's mathematical model. Note that there is no external graphcs. The entire graphics is defined within the LaTeX file.

References

[1] Martin Braun. *Differential equations and their applications*. 2nd ed. An introduction to applied mathematics; Applied Mathematical Sciences, Vol. 15. Springer-Verlag, 1978.

[2] Wikipedia. *Johannes Vermeer*. Sept. 2024. URL: `http://en.wikipedia.org/wiki/Johannes_Vermeer`.

[3] Wikipedia. *Uranium-238*. Sept. 2024. URL: `http://en.wikipedia.org/wiki/Uranium-238`.

Department of Mathematics and Statistics, UMBC
Email address: `my-email@umbc.edu`

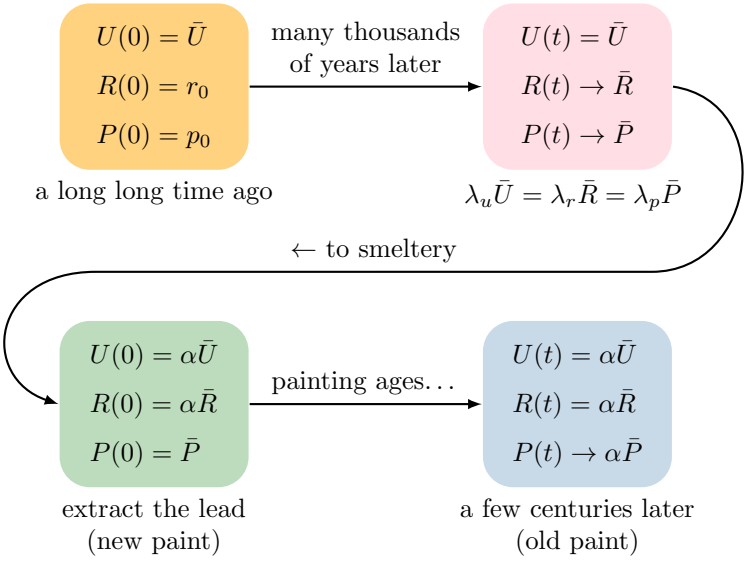

FIGURE 1. Your caption here.