6. $\dot{x}=y+x(x^2+y^2-1)(x^2+y^2-2)$
$\dot{y}=-x+y(x^2+y^2-1)(x^2+y^2-2)$

7. $\dot{x}=xy+x\cos(x^2+y^2)$
$\dot{y}=-x^2+y\cos(x^2+y^2)$

8. (a) Show that the system

$$\dot{x}=y+xf(r)/r, \qquad \dot{y}=-x+yf(r)/r \qquad (r^2=x^2+y^2) \qquad (*)$$

has limit cycles corresponding to the zeros of $f(r)$. What is the direction of motion on these curves?

(b) Determine all limit cycles of (*) and discuss their stability if $f(r)=(r-3)^2(r^2-5r+4)$.

Use the Poincaré–Bendixson Theorem to prove the existence of a nontrivial periodic solution of each of the following differential equations.

9. $\ddot{z}+(z^2+\dot{z}^4-2)z=0$

10. $\ddot{z}+[\ln(z^2+4\dot{z}^2)]\dot{z}+z=0$

11. (a) According to Green's theorem in the plane, if C is a closed curve which is sufficiently "smooth," and if f and g are continuous and have continuous first partial derivatives, then

$$\oint_C [f(x,y)\,dy-g(x,y)\,dx]=\iint_R [f_x(x,y)+g_y(x,y)]\,dx\,dy$$

where R is the region enclosed by C. Assume that $x(t)$, $y(t)$ is a periodic solution of $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$, and let C be the orbit of this solution. Show that for this curve, the line integral above is zero.

(b) Suppose that f_x+g_y has the same sign throughout a simply connected region D in the $x-y$ plane. Show that the system of equations $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$ can have no periodic solution which is entirely in D.

12. Show that the system of differential equations

$$\dot{x}=x+y^2+x^3, \qquad \dot{y}=-x+y+yx^2$$

has no nontrivial periodic solution.

13. Show that the system of differential equations

$$\dot{x}=x-xy^2+y^3, \qquad \dot{y}=3y-yx^2+x^3$$

has no nontrivial periodic solution which lies inside the circle $x^2+y^2=4$.

14. (a) Show that $x=0, y=\psi(t)$ is a solution of (2) for any function $\psi(t)$ satisfying $\dot{\psi}=-c\psi-f\psi^2$.

(b) Choose $\psi(t_0)>0$. Show that the orbit of $x=0, y=\psi(t)$ (for all t for which ψ exists) is the positive y axis.

4.9 Predator–prey problems; or why the percentage of sharks caught in the Mediterranean Sea rose dramatically during World War I

In the mid 1920's the Italian biologist Umberto D'Ancona was studying the population variations of various species of fish that interact with each other. In the course of his research, he came across some data on per-

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centages-of-total-catch of several species of fish that were brought into different Mediterranean ports in the years that spanned World War I. In particular, the data gave the percentage-of-total-catch of selachians, (sharks, skates, rays, etc.) which are not very desirable as food fish. The data for the port of Fiume, Italy, during the years 1914–1923 is given below.

1914	1915	1916	1917	1918
11.9%	21.4%	22.1%	21.2%	36.4%
1919	1920	1921	1922	1923
27.3%	16.0%	15.9%	14.8%	10.7%

D'Ancona was puzzled by the very large increase in the percentage of selachians during the period of the war. Obviously, he reasoned, the increase in the percentage of selachians was due to the greatly reduced level of fishing during this period. But how does the intensity of fishing affect the fish populations? The answer to this question was of great concern to D'Ancona in his research on the struggle for existence between competing species. It was also of concern to the fishing industry, since it would have obvious implications for the way fishing should be done.

Now, what distinguishes the selachians from the food fish is that the selachians are predators, while the food fish are their prey; the selachians depend on the food fish for their survival. At first, D'Ancona thought that this accounted for the large increase of selachians during the war. Since the level of fishing was greatly reduced during this period, there were more prey available to the selachians, who therefore thrived and multiplied rapidly. However, this explanation does not hold any water since there were also more food fish during this period. D'Ancona's theory only shows that there are more selachians when the level of fishing is reduced; it does not explain why a reduced level of fishing is *more* beneficial to the predators than to their prey.

After exhausting all possible biological explanations of this phenomenon, D'Ancona turned to his colleague, the famous Italian mathematician Vito Volterra. Hopefully, Volterra would formulate a mathematical model of the growth of the selachians and their prey, the food fish, and this model would provide the answer to D'Ancona's question. Volterra began his analysis of this problem by separating all the fish into the prey population $x(t)$ and the predator population $y(t)$. Then, he reasoned that the food fish do not compete very intensively among themselves for their food supply since this is very abundant, and the fish population is not very dense. Hence, in the absence of the selachians, the food fish would grow according to the Malthusian law of population growth $\dot{x}=ax$, for some positive constant a. Next, reasoned Volterra, the number of contacts per unit time between predators and prey is bxy, for some positive constant b. Hence, $\dot{x}=ax-bxy$. Similarly, Volterra concluded that the predators have a natural rate of decrease $-cy$ proportional to their present number, and