Appendix B
Rvachev's R-functions

B.1 ▪ Introduction

The expression

$$\phi(x,y) = \frac{1}{2R}(R^2 - x^2 - y^2) \tag{B.1}$$

characterizes the disk of radius R centered at the origin in the Cartesian xy coordinate plane in the sense that $\phi(x,y)$ is positive inside the disk, negative outside the disk, and zero on the disk's boundary. Similarly, the expression

$$\phi(x,y) = \frac{1}{2a}(a^2 - x^2) \tag{B.2}$$

characterizes the strip $-a < x < a$ in the Cartesian coordinate plane in the sense that $\phi(x,y)$ is positive inside the strip, negative outside the strip, and zero on the strip's boundary. The normalizing factors $1/2R$ and $1/2a$ are chosen so that the derivatives of ϕ in the *inward normal directions* at the boundary are $+1$. This normalization is not absolutely necessary but helps simplify some of our calculations.

Generalizing the above, we say that the continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an *indicator function*[104] of the open domain Ω in the Cartesian xy coordinate plane, if

i. ϕ is infinitely differentiable everywhere in $\mathbb{R}^2$, with the possible exception of the points where the boundary of Ω is non-smooth, e.g., at Ω's corners;

ii. for all $(x,y) \in \mathbb{R}^2$ we have

$$\phi(x,y) = \begin{cases} \text{positive} & \text{if } (x,y) \text{ is inside } \Omega, \\ \text{zero} & \text{if } (x,y) \text{ is on } \Omega\text{'s boundary}, \\ \text{negative} & \text{otherwise;} \end{cases} \tag{B.3}$$

iii. at the points of Ω's boundary where ϕ is differentiable, the derivative of ϕ in the inward normal direction is $+1$.

[104]The phrase *indicator function* used in the current context should not be confused with the same phrase used with an entirely different meaning in other areas of analysis, such as measure theory. There, the indicator function ϕ of the subset of a set takes on the value 1 inside the set and 0 outside the set.

Given such a ϕ, we may readily sketch the domain Ω by hand, or perhaps with the help of a graphing software. The converse, that is, *finding an indicator function ϕ for a given* Ω, is quite nontrivial, and that's the subject of this appendix. That task is accomplished through a technique that was introduced by Rvachev [57] in 1963; see also [58] for an expanded treatment and an extensive bibliography. We limit our treatment of Rvachev's method to domains in $\mathbb{R}^2$, but much of what is said here generalizes to higher-dimensional domains.

We have already seen examples of indicator functions for a disk and a strip in (B.1) and (B.2) above. Constructing an indicator function for the square

$$S = \{(x, y) : \; -1 < x, y < 1\},$$

however, imposes some challenges. Here are a few attempts:

1. Take $\phi : \; (x, y) \mapsto 1 - \max(|x|, |y|)$. This function is continuous in $\mathbb{R}^2$, positive in S, and negative in the complement of the closure of S, but it fails to be differentiable along the lines $x \pm y = 0$.

2. Take $\phi : \; (x, y) \mapsto (1 - x^2)(1 - y^2)$. This function is continuous and differentiable. However it fails to be negative outside of S. For instance, $(2, 2)$ is outside of S yet $\phi(2, 2) = +9$. Furthermore, its gradient vanishes at the corners of S. This can lead to numerical instabilities in some applications.

3. Take $\phi : \; (x, y) \mapsto \sqrt{(1 - x^2)(1 - y^2)}$. This function is continuous and differentiable in the interior of S but it's undefined outside of S. Furthermore, the normal derivatives on the square's edges do not exist.

Rvachev's technique, introduced in the subsequent sections, enables us to construct a good indicator function for a square, as well as for significantly more complex geometric shapes, as we shall see.

B.2 ▪ Rvachev's R-functions

A function $\rho : \; \mathbb{R}^n \to \mathbb{R}$ is called an *R-function* if the *signs* of its arguments, *not their values*, determine the sign of the function's value. For instance, $f(x) = x$ is an R-function since a positive x yields a positive $f(x)$, and a negative x yields a negative $f(x)$, regardless of the magnitude of x. As another example, $f(x, y) = xy$ is an R-function since the sign of $f(x, y)$ is determined solely by the signs of x and y, and not their magnitudes. In contrast, the $f(x, y) = x + y$ *is not* an R-function, since the sign of $f(x, y)$ depends on the magnitudes of x and y, and not solely on their signs. For instance, $f(3, -2) = +1$ while $f(2, -4) = -1$.

Of the infinitely many possible R-functions, the following three play key roles in our subsequent developments:

$$R^c(u) \equiv -u, \tag{B.4a}$$

$$R^{\cap}(u, v) \equiv u + v - \sqrt{u^2 + v^2}, \tag{B.4b}$$

$$R^{\cup}(u, v) \equiv u + v + \sqrt{u^2 + v^2}. \tag{B.4c}$$

The reason for the peculiar symbols/notation for these functions will become clear in the sequel. It should be immediately obvious that R^c is an R-function since a positive u yields a negative value, and vice versa. That the other two are also R-functions may not be immediately obvious but upon a close inspection one may see that their signs are also determined solely by the signs of their arguments; see Figure B.1 for their graphs.

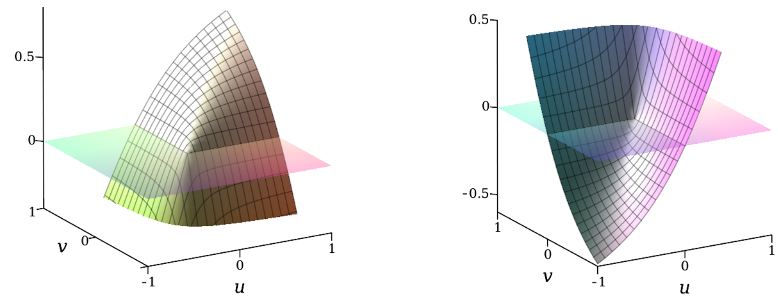


Figure B.1: The graphs of the R-functions $R^{\cap}$ on the left and $R^{\cup}$ on the right, plotted along with zero-level planes. Observe that $R^{\cap}$ is positive only if both u and v are positive, and $R^{\cup}$ is positive only if at least one of u and v are positive.

$u \backslash v$	+	−
+	+	−
−	−	−

(a) The signs of $R^{\cap}(u, v)$

$u \backslash v$	+	−
+	+	+
−	+	−

(b) The signs of $R^{\cup}(u, v)$

Table B.1: The sign of $R^{\cap}(u, v)$ is positive if and only if both u and v are positive. The sign of $R^{\cup}(u, v)$ is positive if and only if at least one of u and v is positive.

Table B.1 summarize the dependence of the signs of the R-functions $R^{\cap}$ and $R^{\cup}$ on the signs of their arguments. The crucial observation here is that those tables are identical with the truth tables of Boolean algebra, where "+" and "−" correspond to "true" and "false", respectively. In that sense, the functions $R^{\cap}$ and $R^{\cup}$ correspond to logical "and" and "or", and R^{c} corresponds to logical negation, or "not".

Now, let us see how one may use the three R-function defined in (B.4) to construct indicator functions of domains of varying complexity.

Consider a pair of overlapping domains Ω_1 and Ω_2 in $\mathbb{R}^2$, and let ϕ_1 and ϕ_2 be their indicator functions. We claim that $R^{\cap}\big(\phi_1(x, y), \phi_2(x, y)\big)$ is the indicator function of the intersection $\Omega_1 \cap \Omega_2$. To see that, consider a point $P(x, y)$ in the intersection. Since P is both in Ω_1 and Ω_2, we have $\phi_1(x, y) > 0$ and $\phi_2(x, y) > 0$. Then, referring to Table B.1(a), we see that $R^{\cap}\big(\phi_1(x, y), \phi_2(x, y)\big) > 0$. On the other hand, if a point $Q(x, y)$ is *not* in the intersection, then one or both of $\phi_1(x, y)$ and $\phi_2(x, y)$ are negative, and therefore, referring to Table B.1(a) again, we see that $R^{\cap}\big(\phi_1(x, y), \phi_2(x, y)\big) < 0$. We conclude that $(x, y) \mapsto R^{\cap}\big(\phi_1(x, y), \phi_2(x, y)\big)$ is the indicator function of the intersection $\Omega_1 \cap \Omega_2$ of the domains Ω_1 and Ω_2. This explains the notation $R^{\cap}$ for this R-function.

A similar reasoning shows that $(x, y) \mapsto R^{\cup}\big(\phi_1(x, y), \phi_2(x, y)\big)$ is the indicator function of the union $\Omega_1 \cup \Omega_2$ of the domains Ω_1 and Ω_2. Finally, if ϕ is the indicator function of a domain $\Omega \in \mathbb{R}^2$, then $(x, y) \mapsto R^{c}\big(\phi(x, y)\big)$ is the indicator function of the complement Ω^c of the Ω in $\mathbb{R}^2$. Figure B.2 summarizes the preceding conclusions.

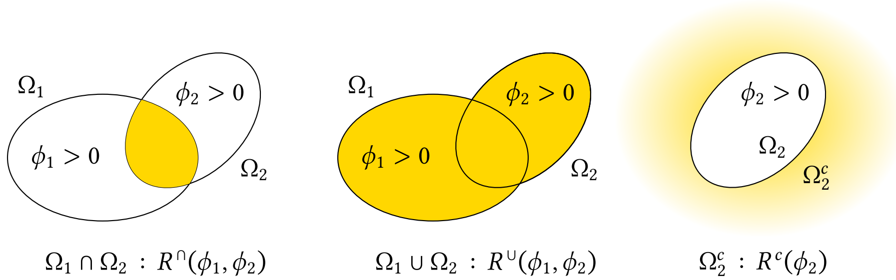


Figure B.2: The indicator function of the intersection of the domains Ω_1 and Ω_2 is $R^\cap(\phi_1, \phi_2)$. That of their union is $R^\cup(\phi_1, \phi_2)$, and that of the complement Ω_2 is $R^c(\phi_2)$.

B.3 ▪ Complex geometries from simple shapes

The purpose of this section is to demonstrate that indicator functions of fairly complex domains may be constructed through expressing the domains as the unions, intersections, and complements of simple domains.

B.3.1 ▪ Strips

The vertical strip $\{(x,y) \ : \ -a < x < a\}$ may be viewed as the intersection of the half-spaces corresponding to the indicator functions $\phi_1(x,y) = x + a$ and $\phi_2(x,y) = a - x$. Thus, we may apply the intersection function $R^\cap$ defined in (B.4b) to obtain the indicator function of the strip:

$$\begin{aligned}
\phi_{\text{v-strip}}(x,y;a) &\equiv R^\cap\big(\phi_1(x,y), \phi_2(x,y)\big) \\
&= \phi_1(x,y) + \phi_2(x,y) - \sqrt{\phi_1(x,y)^2 + \phi_2(x,y)^2} \\
&= (x+a) + (a-x) - \sqrt{(x+a)^2 + (a-x)^2} \\
&= 2a - \sqrt{2(x^2+a^2)}.
\end{aligned}$$

Although this is a perfectly fine indicator function for the strip, it is not the simplest. The ad hoc function

$$\phi_{\text{v-strip}}(x,y;a) = \frac{1}{2a}(a^2 - x^2),$$

is also an indicator function for that strip. In our future constructions we prefer to use the latter since it is algebraically simpler. Figure B.3 shows the graphs of these two indicator functions, and Figure B.4 illustrates the construction of the strip as the intersection of two half-spaces.

As to the horizontal strip $\{(x,y) \ : \ -b < y < b\}$, we will use the indicator function

$$\phi_{\text{h-strip}}(x,y;b) = \frac{1}{2b}(b^2 - y^2).$$

B.3.2 ▪ A square

The square $\{(x,y) : \ -1 < x, y < 1\}$ may be viewed as the intersection of the strips characterized through the $\phi_{\text{v-strip}}(x,y;1)$ and $\phi_{\text{h-strip}}(x,y;1)$ indicator functions constructed

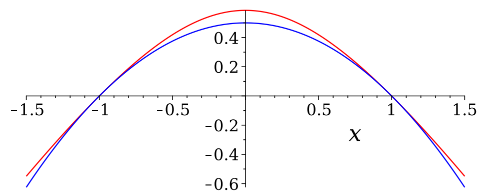


Figure B.3: The strip $\{(x, y) : -1 < x < 1\}$ may be characterized by the indicator function $2 - \sqrt{2(x^2 + 1)}$ (a hyperbola, plotted in red) or $\frac{1}{2}(1 - x^2)$ (a parabola, plotted in blue). Both satisfy the requirements of an indicator function but we prefer the parabola since it is algebraically simpler.

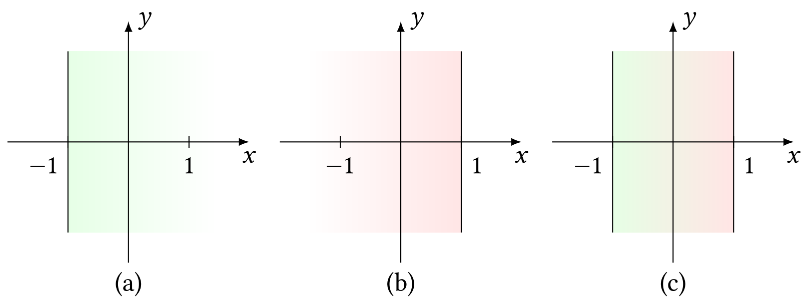


Figure B.4: The intersection of the half-spaces $x > -1$ (left) and $x < 1$ (center) results in the strip $-1 < x < 1$ shown on the right.

earlier. The indicator function $\phi_{\text{square}}(x, y)$ of the square is obtained by applying the R-function $R^\cap$ defined in (B.4b) to the two strips:

$$\begin{aligned}\phi_{\text{square}}(x, y) &= R^\cap\big(\phi_{\text{v-strip}}(x, y; 1), \phi_{\text{h-strip}}(x, y; 1)\big)\\ &= \frac{1}{2}(1 - x^2) + \frac{1}{2}(1 - y^2) - \sqrt{\Big(\frac{1}{2}(1 - x^2)\Big)^2 + \Big(\frac{1}{2}(1 - y^2)\Big)^2}\\ &= \frac{1}{2}\Big[(1 - x^2) + (1 - y^2) - \sqrt{(1 - x^2)^2 + (1 - y^2)^2}\Big].\end{aligned} \tag{B.5}$$

Figure B.5 illustrates the construction of the square as the intersection of the two strips. The graph of the indicator function ϕ_{square} is shown in Figure B.8(a).

B.3.3 ▪ A square with a hole

Consider the square $\Omega_1 = \{(x, y) : -1 < x, y < 1\}$ and the disk $\Omega_2 = \{(x, y) : x^2 + y^2 < 1/4\}$ of radius $1/2$ centered at the origin. Figure B.6 shows the domain $\Omega = \Omega_1 \cap \Omega_2^c$ obtained by removing the disk Ω_2 from the square Ω_1.

The indicator function ϕ_{square} of Ω_1 was constructed in the previous section. The indicator function of Ω_2 is $\phi_{\text{disk}}(x, y) = \frac{1}{4} - x^2 - y^2$; see equation (B.1). Therefore, the indicator

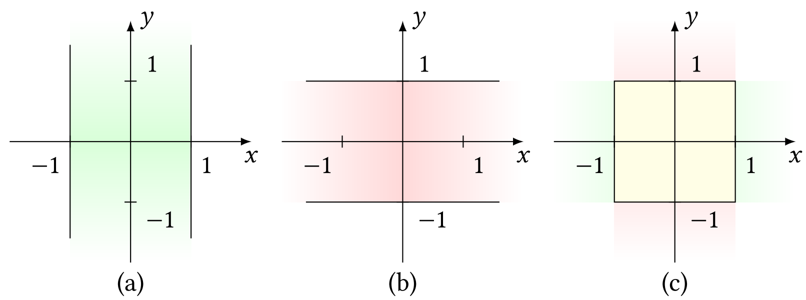


Figure B.5: The intersection of the strips $-1 < x < 1$ (left) and $-1 < y < 1$ (center) and results in the square shown on the right.

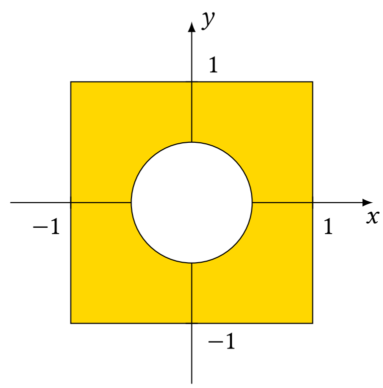


Figure B.6: The square with a hole removed is the domain $\Omega = \Omega_1 \cap \Omega_2^c$, where $\Omega_1 = \{(x, y) : -1 < x, y < 1\}$ and $\Omega_2 = \{(x, y) : x^2 + y^2 < 1/4\}$.

function $\phi_{\text{square-hole}}$ of Ω is calculated as

$$\begin{aligned}
\phi_{\text{square-hole}}(x, y) &= R^{\cap}\Big(\phi_{\text{square}}(x, y), R^{c}\big(\phi_{\text{disk}}(x, y)\big)\Big) \\
&= R^{\cap}\Big(\phi_{\text{square}}(x, y), -\phi_{\text{disk}}(x, y)\Big) \\
&= \phi_{\text{square}}(x, y) - \phi_{\text{disk}}(x, y) - \sqrt{\phi_{\text{square}}(x, y)^2 + \phi_{\text{disk}}(x, y)^2}.
\end{aligned}$$

The graph of the indicator function $\phi_{\text{square-hole}}$ is shown in Figure B.8(c).

Remark B.1. The ad hoc designed function

$$\phi_{\text{square-hole-alt}}(x, y) = \phi_{\text{square}}(x, y) \times \big(-\phi_{\text{disk}}(x, y)\big)$$

can also serve as an indicator function for the domain Ω. Indeed, the first factor is positive inside the square, and the second factor is positive outside the hole. It has the advantage over $\phi_{\text{square-hole}}$ in having a significantly simpler algebraic form. Its drowback is that unlike the cut-off functions constructed through R-functions, its inward normal derivatives on the domain's bounary are not constants, but that is not a concern if we are going to use this merely as a cut-off function in our neural network applications.

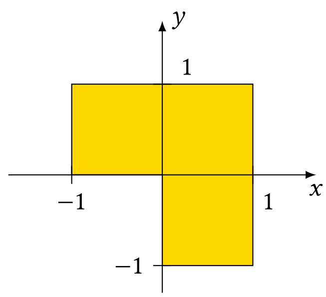


Figure B.7: The L-shaped domain is the intersection of the square $\{(x,y) : -1 < x, y < 1\}$ with the union of the half-spaces $\{(x,y) : x > 0\}$ and $\{(x,y) : y > 0\}$.

B.3.4 ▪ An L-shaped domain

There are various ways of constructing the indicator function of the L-shaped domain shown in Figure B.7. The simplest is to begin with the union of the half-spaces $x > 0$ and $y > 0$ which encompasses the entire $\mathbb{R}^2$ minus the third quadrant. Then we intersect the result with the square whose indicator function ϕ_{square} was constructed in section B.3.2. Writing $\phi_1(x,y) = x$ and $\phi_2(x,y) = y$ for the indicator functions of the two half-spaces, we conclude that the indicator function $\phi_{\text{L-shaped}}$ of the L-shaped domain is

$$\phi_{\text{L-shaped}}(x,y) = R^{\cap}\Big(\phi_{\text{square}}(x,y), R^{\cup}\big(\phi_1(x,y), \phi_2(x,y)\big)\Big).$$

In view of the definition $R^{\cup}$ in (B.4c) we have

$$\begin{aligned} &R^{\cup}\big(\phi_1(x,y), \phi_2(x,y)\big) \\ &\qquad = \phi_1(x,y) + \phi_2(x,y) + \sqrt{\phi_1(x,y)^2 + \phi_2(x,y)^2} \\ &\qquad = x + y + \sqrt{x^2 + y^2}, \end{aligned}$$

and therefore

$$\begin{aligned} \phi_{\text{L-shaped}}(x,y) = \phi_{\text{square}}(x,y) &+ \Big(x + y + \sqrt{x^2+y^2}\Big) \\ &- \sqrt{\phi_{\text{square}}(x,y)^2 + \Big(x + y + \sqrt{x^2+y^2}\Big)^2}, \end{aligned} \tag{B.6}$$

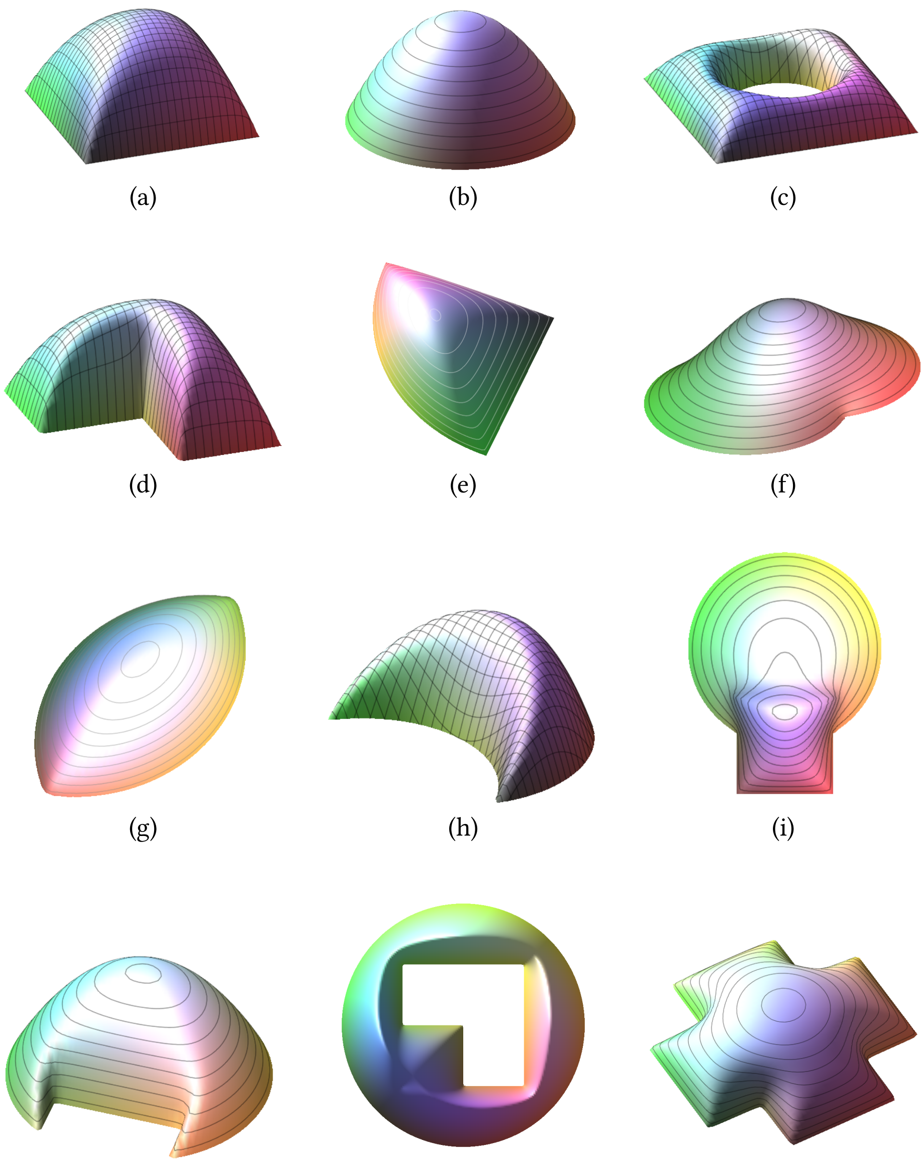
where $\phi_{\text{square}}(x,y)$ is given in (B.5). Here is the expanded version of that expression:

$$\begin{aligned} \phi_{\text{L-shaped}}(x,y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}\sqrt{(x^2+1)^2 + (y^2+1)^2} + x + y + \sqrt{x^2+y^2} \\ - \sqrt{\Big(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}\sqrt{(x^2+1)^2+(y^2+1)^2}\Big)^2 + \Big(x + y + \sqrt{x^2+y^2}\Big)^2}. \end{aligned}$$

The graph of $\phi_{\text{L-shaped}}$ is shown in Figure B.8(d).

B.3.5 ▪ A quarter-disk

The quarter-disk $\Omega = \{(x,y) : x^2 + y^2 < 1,\ x > 0,\ y > 0\}$ is the intersection of of the full disk $\Omega_0 = \{(x,y) : x^2 + y^2 < 1\}$ and the half-spaces $\Omega_1 = \{(x,y) : x > 0\}$ and



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

Figure B.8: A gallery of indicator functions.

$\Omega_2 = \{(x, y) : y > 0\}$. The domain $\Omega = \Omega_0 \cap \Omega_1 \cap \Omega_2$ is shown in Figure B.9. Letting $\phi_{\text{quarter-disk}}(x, y)$, $\phi_0(x, y)$, $\phi_1(x, y)$, $\phi_2(x, y)$ denote the indicator functions of the domain

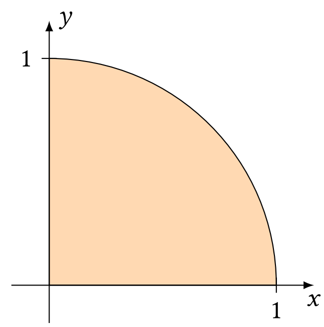


Figure B.9: The quarter-disk $\{(x,y): x^2+y^2<1,\ x>0,\ y>0\}$ is the intersection of the full disk $\{(x,y) : x^2+y^2<1\}$ and the half-spaces $\{(x,y): x>0\}$ and $\{(x,y): y>0\}$.

Ω, Ω_0, Ω_1, Ω_2, respectively, we have

$$\begin{aligned}
\phi_{\text{quarter-disk}}(x,y) &= R^{\cap}\Big(\phi_0(x,y), R^{\cap}\big(\phi_1(x,y),\phi_2(x,y)\big)\Big)\\
&= R^{\cap}\big(\tfrac{1}{2}(1-x^2-y^2), R^{\cap}(x,y)\big)\\
&= R^{\cap}\Big(\frac{1}{2}(1-x^2-y^2), x+y-\sqrt{x^2+y^2}\Big)\\
&= \frac{1}{2}(1-x^2-y^2)+x+y-\sqrt{x^2+y^2}\\
&\quad -\sqrt{\Big[\frac{1}{2}(1-x^2-y^2)\Big]^2+\Big[x+y-\sqrt{x^2+y^2}\Big]^2}.
\end{aligned}$$

The graph of the indicator function $\phi_{\text{quarter-disk}}$ is shown in Figure B.8(e).

B.3.6 ▪ Combining disks

Consider the two disks

$$\Omega_1 = \{(x,y) : (x+1)^2+y^2<2^2\},\quad \Omega_2 = \{(x,y) : (x-1)^2+y^2<2^2\},$$

whose indicator functions, according to (B.1), are

$$\phi_1(x,y) = \frac{1}{4}\big(4-(x+1)^2-y^2\big),\qquad \phi_2(x,y) = \frac{1}{4}\big(4-(x-1)^2-y^2\big).$$

Figure B.10 shows the domains $\Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2$, and $\Omega_1^c\cap\Omega_2$. The corresponding indicator functions may be obtained readily. For instance, the indicator function of the domain $\Omega_1^c\cap\Omega_2$ is

$$\begin{aligned}
\phi(x,y) &= R^{\cap}\Big(R^c\big(\phi_1(x,y)\big),\phi_2(x,y)\Big)\\
&= R^{\cap}\big(-\phi_1(x,y),\phi_2(x,y)\big)\\
&= -\phi_1(x,y)+\phi_2(x,y)-\sqrt{\phi_1(x,y)^2+\phi_2(x,y)^2},
\end{aligned}$$

where ϕ_1 and ϕ_2 are given above. The graphs of the indicator functions of the domains $\Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2$, and $\Omega_1^c\cap\Omega_2$ are shown in Figure B.8(f), B.8(g), and B.8(h), respectively.

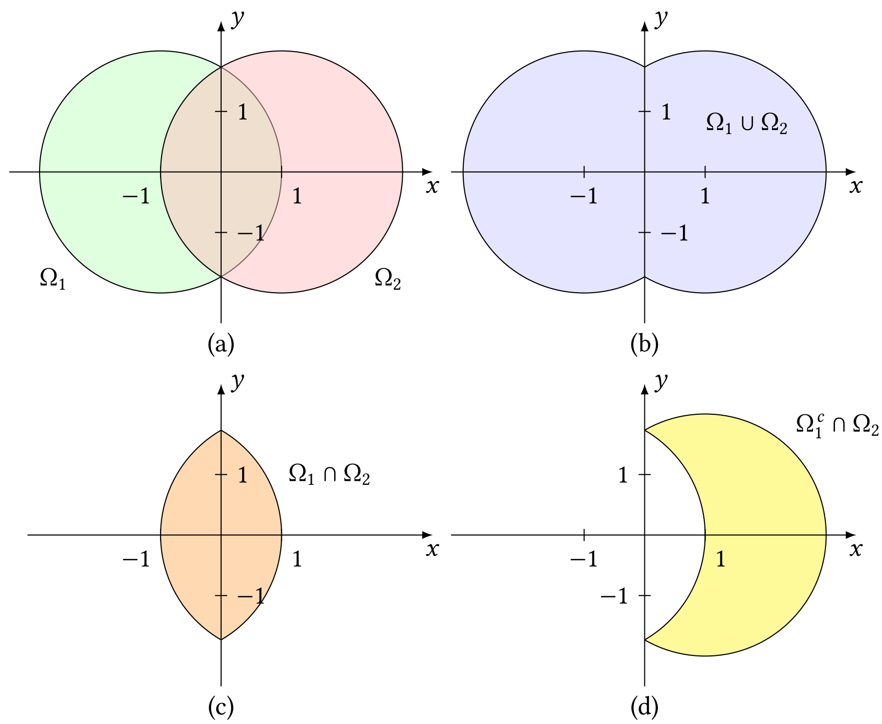


Figure B.10: The domains Ω_1 and Ω_2 are disks of radius 2 each, centered at $(-1, 0)$ and $(1, 0)$ (above left). The domains $\Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2$, and $\Omega_1^c \cap \Omega_2$ are shown in he remaining subfigures.

B.3.7 ▪ A disk and a square

Consider the disk $\Omega_1 = \{(x, y) : x^2 + (y-2)^2 < 2^2\}$ and the square $\Omega_2 = \{(x, y) : -1 < x, y < 1\}$. These, and the domains $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2^c$ are shown in Figure B.11. The indicator function of Ω_1 is

$$\phi_1 = \frac{1}{4}\left(4 - x^2 - (y-2)^2\right),$$

while the indicator function of Ω_2 is given in (B.5). The reader should be able to construct the indicator functions of the domains $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2^c$ in a manner similar to the foregoing illustrations. The graphs of the indicator functions of the domains $\Omega_1 \cup \Omega_2$, and $\Omega_1 \cap \Omega_2^c$ are shown in Figure B.8(i) and B.8(j), respectively.

B.3.8 ▪ A disk with an L-shaped hole

The domain Ω shown in Figure B.12 consists of a disk, Ω_1, with an L-shaped hole, Ω_2 removed from it. We have $\Omega = \Omega_1 \cap \Omega_2^c$. The indicator function of the disk is $\phi_1(x, y) = \frac{1}{4}(4 - x^2 - y^2)$. The indicator function ϕ_2 of the L-shaped hole was constructed in sec-

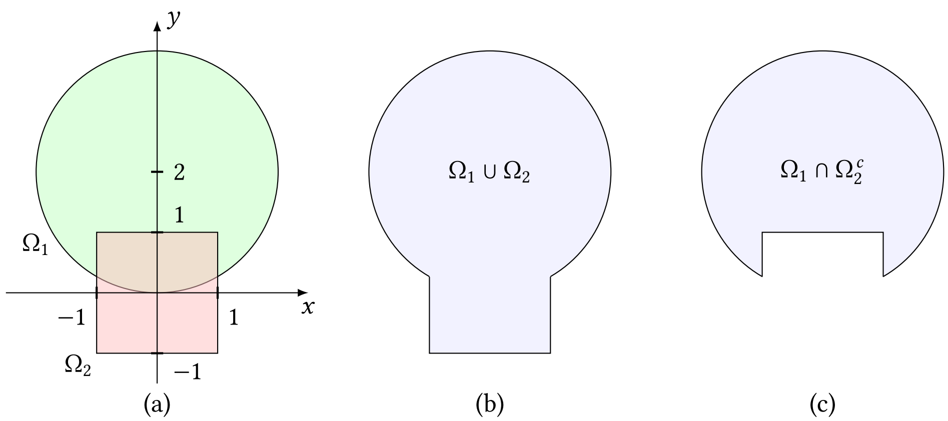


Figure B.11: The domains Ω_1 (a disk) and Ω_2 (a square) are shown on the left. The domains $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2^c$ are shown in lavender.

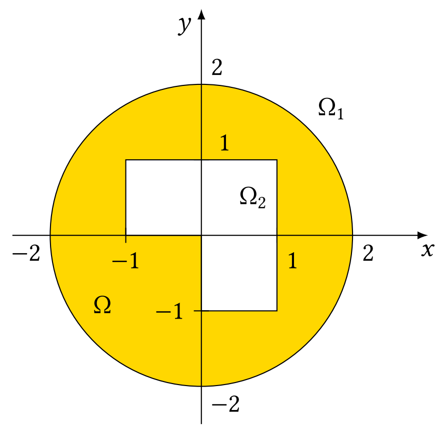


Figure B.12: The shaded domain, Ω, consists of a disk, Ω_1, with an L-shaped hole, Ω_2, removed from it. We have $\Omega = \Omega_1 \cap \Omega_2^c$.

tion B.3.4. Therefore the indicator function of Ω is

$$\begin{aligned}
\phi(x, y) &= R^{\cap}\Big(\phi_1(x, y), R^c\big(\phi_2(x, y)\big)\Big) \\
&= R^{\cap}\Big(\phi_1(x, y), -\phi_2(x, y)\Big) \\
&= \phi_1(x, y) - \phi_2(x, y) - \sqrt{\phi_1(x, y)^2 + \phi_2(x, y)^2}.
\end{aligned}$$

The graph of the indicator functions of Ω is shown in Figure B.8(k).

B.4 ▪ The need for symbolic calculus software

A striking aspect of the indicator functions constructed in the previous sections is that their algebraic expressions tend to be quite complex. What is of concern is that in our applications we need not only the indicator function ϕ of a domain, but also the derivatives $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial x \partial y}, \frac{\partial^2 \phi}{\partial y^2}$. In anything other than the most trivial cases, the calculation of those derivatives can be tedious and error-prone, and the task is best delegated to computer software dedicated to symbolic calculus. As of this writing, the most popular such software are MAPLE and MATHEMATICA. In this section we will see how one may perform the calculation in MAPLE[105] and translate the result to C code.

The calculation of the partial derivatives and the translation of the result to C code is performed through the following general-purpose MAPLE procedure:

```
to_C := proc(expr)
    local phi, ans := Array(0..2, 0..2);
    ans[0,0] := expr:
    ans[1,0] := diff(expr, x):
    ans[0,1] := diff(expr, y):
    ans[2,0] := diff(expr, x, x):
    ans[1,1] := diff(expr, x, y):
    ans[0,2] := diff(expr, y, y):
    CodeGeneration:-C(ans, declare=[x::numeric, y::numeric],
        optimize, resultname=phi);
end proc:
```

The procedure `to_C()` takes an expression in the x and y variables and prints to the screen the C code for evaluating that expression as well as the five partial derivatives noted above. For instance, executing `to_C(sqrt(x^2 + y^2))` in MAPLE, prints

```
t1 = y * y;
t2 = x * x;
t3 = t2 + t1;
t4 = pow(t3, -0.3e1 / 0.2e1);
t5 = t3 * t4;
phi[0][0] = pow(t3, 0.2e1) * t4;
phi[0][1] = t5 * y;
phi[0][2] = -t1 * t4 + t5;
phi[1][0] = t5 * x;
phi[1][1] = -t4 * x * y;
phi[1][2] = 0;
phi[2][0] = -t2 * t4 + t5;
phi[2][1] = 0;
phi[2][2] = 0;
```

We then copy that output and paste it into our C programs. The auxiliary variables `t1`, `t2`, etc., introduced here need to be declared of the type `double` in the C code. Therefore we edit/modify MAPLE's output by declaring their types. The resulting C function will look like this:

```
void my_phi(double x, double y, double phi[3][3])
{
    double t1 = y * y;
    double t2 = x * x;
    double t3 = t2 + t1;
    double t4 = pow(t3, -0.3e1 / 0.2e1);
    double t5 = t3 * t4;
```

[105] The calculation may be performed in MATHEMATICA as well, but as of this writing I don't know enough about MATHEMATICA to provide a useful guide.

```
    phi[0][0] = pow(t3, 0.2e1) * t4;
    phi[0][1] = t5 * y;
    phi[0][2] = -t1 * t4 + t5;
    phi[1][0] = t5 * x;
    phi[1][1] = -t4 * x * y;
    phi[1][2] = 0;
    phi[2][0] = -t2 * t4 + t5;
    phi[2][1] = 0;
    phi[2][2] = 0;
}
```

The function receives the values of `x` and `y`, and a pointer to the 3×3 array `phi`. It evaluates ϕ and its derivatives at (x, y), and places those values in the array `phi` according to the scheme

$$\begin{bmatrix} \texttt{[phi[0][0]} & \texttt{[phi[0][1]} & \texttt{[phi[0][2]} \\ \texttt{[phi[1][0]} & \texttt{[phi[1][1]} & \texttt{[phi[1][2]} \\ \texttt{[phi[2][0]} & \texttt{[phi[2][1]} & \texttt{[phi[2][2]} \end{bmatrix} = \begin{bmatrix} \phi & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial^2 \phi}{\partial y^2} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial^2 \phi}{\partial x \partial y} & 0 \\ \dfrac{\partial^2 \phi}{\partial x^2} & 0 & 0 \end{bmatrix}.$$

B.5 ▪ Exercises

1. Let $\Omega_1 = \{(x, y) : x^2 + y^2 < 4\}$ and $\Omega_2 = \{(x, y) : -2 < x < 2, -1 < y < 1\}$. Find the indicator function of the domain $\Omega_1 \cap \Omega_2^c$.

2. Let

$$\Omega_0 = \{(x, y) : -2 < x < 2, -2 < y < 2\},$$
$$\Omega_1 = \{(x, y) : -1 < x < 1\},$$
$$\Omega_2 = \{(x, y) : -1 < y < 1\}.$$

Find the indicator function of the domain $\Omega_0 \cap (\Omega_1 \cup \Omega_2)$.

3. Let Ω_1 and Ω_2 be disks of radius $1/4$ each, centered at $(-1/2, 0)$ and $(1/2, 0)$, and let Ω_0 be the unit disk centered at $(0, 0)$. Find the indicator function of the domain formed by removing the "holes" Ω_1 and Ω_2 from Ω_0. Here is the graph of the indicator function.