

COUETTE AND POISEUILLE FLOW IN NON-NEWTONIAN FLUIDS

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ABSTRACT

Weissenberg and Merrington have done certain experiments on Couette and Poiseuille flow with highly viscous fluids. Rivlin, Reiner and Serrin attempted to explain these flows theoretically by taking the coefficient of viscosity and cross-viscosity as constants. In general the coefficients of viscosity and cross-viscosity are functions of the scalar invariants of D , the rate of deformation tensor and of the thermodynamic state of fluid. In this paper we deal with the Couette and Poiseuille flows by taking the coefficient of viscosity and cross-viscosity as functions of second invariant of D . The present investigation supports the conclusion of Serrin in the case of Couette flow that the strangulation of liquid can be explained in terms of cross-viscosity.

1. Couette flow

Weissenberg first demonstrated a very strange phenomenon in the hydrodynamic behaviour of certain highly viscous liquids. When such a liquid is sheared between two co-axial rotating cylinders both of which move at different but with steady rotational speeds, the liquid is drawn inwards against the action of centrifugal forces and upwards against the forces of gravity so that the whole arrangement forming a sort of 'centripetal pump'. Weissenberg attributes this centripetal pump effect to the elasticity of the fluid. But Reiner deduced from mathematical principles that there is a theoretical possibility that the most general viscous liquid might show such a behaviour. He established the most general relation between the stress tensor and the rate of deformation tensor (1.1) and showed that the appearance of γ in (1.1) gives rise to the phenomenon of 'cross-viscosity' or in other words the phenomenon leading to the 'centripetal pump effect'.

In recent years, several authors have examined the steady motion of such fluids (Rivlin, 1948; Truesdell, 1952; Braun and Reiner, 1952; Serrin, 1959; Oldroyd, 1958). Serrin (1959) has discussed the strangulation of the liquid in the space between the two cylinders by considering the coefficients of viscosity and cross-viscosity as function of the material constants. In general they are functions of second and third invariants of the rate of deformation tensor, D , besides the temperature of the fluid. In this paper the physical nature of Rivlin's solution is clarified by taking the coefficients of viscosity and cross-viscosity as functions of second and third invariants of D .

The theory of non-Newtonian fluids is based on the stress-deformation relation in the form

$$P = \alpha . I + \beta . D + \gamma . D^2 . \quad \dots \quad \dots \quad \dots \quad (1.1)$$

where P is the stress tensor, D is the rate of deformation tensor and I is the unit tensor. α , β , γ are certain scalar coefficients. We shall be concerned here with incompressible fluids, so that α is identified with the negative of the dynamic pressure, i.e. $\alpha = -p$. The coefficients β and γ are functions of the second invariant of D besides the fluid temperature.

We shall now consider the steady rotatory motion of an incompressible liquid between two co-axial cylinders of radii r_1 and r_2 ($r_2 > r_1$). The angular velocities of the outer and inner cylinders being ω_2 and ω_1 respectively.

If the angular velocity of the liquid at a distance r from the axis of the cylinders is ω then

$$\begin{cases} \omega = \omega_1 & \text{when } r = r_1, \\ \omega = \omega_2 & \text{when } r = r_2. \end{cases} \quad \dots \quad \dots \quad \dots \quad (1.2)$$

The fluid motion is governed by the law

$$\rho \frac{DV}{Dt} = \rho \underline{f} + \text{div } P, \quad \dots \quad \dots \quad \dots \quad (1.3)$$

$$\text{div } \underline{V} = 0. \quad \dots \quad \dots \quad \dots \quad (1.4)$$

where \underline{V} is the velocity vector, \underline{f} is the gravitational force and P is given by (1.1). Taking the cylindrical polar co-ordinate system (r, θ, z) , the z -axis of which coincides with the axis of the cylinders, we have

$$u_r = 0, \quad u_\theta = r\omega(r), \quad u_z = 0 \quad \dots \quad \dots \quad \dots \quad (1.5)$$

The rate of deformation tensor becomes

$$D = \frac{1}{2} \begin{pmatrix} 0 & r\omega' & 0 \\ r\omega' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P = -p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \beta r \omega' \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4} \gamma r^2 \omega'^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1.6)$$

where

$$\begin{aligned} \beta &= \beta(-\frac{1}{4} r^2 \omega'^2, T), \\ \gamma &= \gamma(-\frac{1}{4} r^2 \omega'^2, T). \end{aligned}$$

Assuming that β and γ do not involve T explicitly or T is a function of r only, we take

$$\beta = \beta_0 (r^2 \omega'^2)^n, \quad n > 0, \quad \dots \quad \dots \quad \dots \quad (1.7)$$

$$\gamma = \gamma_0 + \gamma_1 (r^2 \omega'^2) + \gamma_2 (r^2 \omega'^2)^2 + \dots \quad \dots \quad \dots \quad (1.8)$$

where $\beta_0, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \dots$ are constants.

The momentum equations are:

$$\left. \begin{aligned} -\rho r \omega^2 &= -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r} \left(\frac{1}{4} r^2 \omega'^2 \gamma \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{2} \beta \omega' \right), \\ 0 &= -\frac{\partial p}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{2} r^3 \beta \omega' \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{4} r^3 \gamma \omega'^2 \right), \\ 0 &= -\rho g - \frac{\partial p}{\partial z}. \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad (1.9)$$

These equations have the integrals

$$r^3 \beta \omega' = C, \quad \dots \quad \dots \quad \dots \quad (1.10)$$

$$p = -\rho g z + f(r) + k, \quad \dots \quad \dots \quad \dots \quad (1.11)$$

where C and K are constants and

$$f(r) = \frac{1}{4} r^2 \omega'^2 \gamma + \rho \int r \omega^2 dr. \quad \dots \quad (1.12)$$

Assuming axial symmetry, we shall take $\frac{\partial}{\partial \theta} \equiv 0$.

Using the values of β and γ as given by (1.7) and (1.8) we have the solution of the equation of motion as

$$\omega = \frac{A}{m} r^{-m} + B, \quad \dots \quad (1.13)$$

where

$$A = - \left(\frac{C}{\beta_0} \right)^{m/2} \text{ and } m = \frac{2}{2n+1}.$$

The constants A and B are evaluated by applying the boundary conditions (1.2) :

$$\left. \begin{aligned} A &= - \frac{m(\omega_2 - \omega_1)(r_1 r_2)^m}{r_2^m - r_1^m}, \\ B &= \frac{\omega_2 r_2^m - \omega_1 r_1^m}{r_2^m - r_1^m}, \end{aligned} \right\} \quad \dots \quad (1.14)$$

so that

$$u_\theta = \frac{r}{r_2^m - r_1^m} \left[(\omega_2 r_2^m - \omega_1 r_1^m) - (\omega_2 - \omega_1) \left(\frac{r_1 r_2}{r} \right)^m \right] \quad \dots \quad (1.15)$$

It is clear that the radial distribution of the angular velocity is unaffected by γ .

From the equation (1.11) we have

$$\begin{aligned} p &= \frac{1}{4} \{ \gamma_0 A^2 r^{-2m} + \gamma_1 (A^2 r^{-2m})^2 + \gamma_2 (A^2 r^{-2m})^3 + \dots \} + \\ &\quad \frac{\rho r^2}{m^2} \left\{ \frac{A^2 r^{-2m}}{2(1-m)} + \frac{B^2 m^2}{2} + \frac{2ABm}{2-m} r^{-m} \right\} - \rho g z + k \quad \dots \quad (1.16) \end{aligned}$$

To complete the solution we find the stress components for the state of motion considered :

$$\left. \begin{aligned} p_{rr} &= p_{\theta\theta} = - \frac{\rho r^2}{m^2} \left\{ \frac{A^2 r^{-2m}}{2(1-m)} + \frac{B^2 m^2}{2} + \frac{2ABm}{2-m} r^{-m} \right\} + \rho g z - k, \\ p_{r\theta} &= \frac{1}{2} C r^2, \\ p_{rz} &= p_{z\theta} = 0, \end{aligned} \right\} \quad \dots \quad (1.17)$$

$$\begin{aligned} p_{zz} &= -p = \rho g z - k - \frac{1}{4} \{ \gamma_0 A^2 r^{-2m} + \gamma_1 (A^2 r^{-2m})^2 + \gamma_2 (A^2 r^{-2m})^3 + \dots \} \\ &\quad - \frac{\rho r^2}{m^2} \left\{ \frac{A^2 r^{-2m}}{2(1-m)} + \frac{B^2 m^2}{2} + \frac{2ABm}{2-m} r^{-m} \right\} \quad \dots \quad (1.18) \end{aligned}$$

We observe that the only stress component whose value is affected by γ is p_{zz} . The flow is taking place between two concentric rotating cylinders with the upper surface of the fluid left open to the atmosphere. The shape of this free surface is given by

$$\begin{aligned} z &= \frac{A^2 r^{-2m}}{4\rho g} \{ \gamma_0 + \gamma_1 A^2 r^{-2m} + \gamma_2 (A^2 r^{-2m})^2 + \gamma_3 (A^2 r^{-2m})^3 \dots \} \\ &\quad + \frac{r^2}{gm^2} \left\{ \frac{A^2 r^{-2m}}{2(1-m)} + \frac{B^2 m^2}{2} + \frac{2ABm}{2-m} r^{-m} \right\} + \frac{k - \pi}{\rho g}, \quad \dots \quad (1.19) \end{aligned}$$

where z is measured vertically upwards from a convenient level.

The slope of the meridian section of the surface is given by

$$\frac{dz}{dr} = \frac{A^2 r^{-(2m+1)}}{m^2 g} \left[\{r(1+\Theta)\}^2 - \frac{2m^3}{4\rho} \{ \gamma_0 + 2\gamma_1 A^2 r^{-2m} + 3\gamma_2 (A^2 r^{-2m})^2 + \dots \} \right], \quad (1.20)$$

where

$$\Theta = \frac{B}{A} m r^m.$$

When $m = 2$ and $\gamma = \gamma_0$, a constant, we get, as a particular case, Serrin's result :

$$\frac{dz}{dr} = \frac{A^2}{g r^5} \left[\{r(1+\Theta)\}^2 - \frac{4\gamma}{\rho} \right]. \quad \dots \quad (1.21)$$

We shall now consider two cases in detail.

Case (i): The inner cylinder ($r = r_1$) remains at rest, so that

$$\Theta = - \left(\frac{r}{r_1} \right)^m.$$

Case (ii): The outer cylinder ($r = r_2$) remains at rest, so that

$$\Theta = - \left(\frac{r}{r_2} \right)^m.$$

We discuss the flow of liquid in the space between the two co-axial cylinders as follows :

(a) *The case with no cross-viscosity :*

If $\gamma = 0$, (1.2) gives

$$\frac{dz}{dr} = \frac{A^2 r^{-(2m+1)}}{m^2 g} \{r(1+\Theta)\}^2. \quad \dots \quad (1.22)$$

The slope of the meridian section of the free surface in case (i) varies from zero at the inner cylinder to a positive value at the outer cylinder, while in case (ii) it varies from a positive value at the inner cylinder to zero at the outer cylinder.

(b) *The case with cross-viscosity :*

When the cross-viscosity is present, let γ be given by (1.8) which must be a positive definite form in the indeterminate $(\omega'r)^2$. Since $(\omega'r)$ occurs in even powers, a sufficient condition for (1.8) to be positive definite is that all $\gamma_0, \gamma_1, \gamma_2, \dots$ are positive.

Case (i): The liquid will tend to climb the inner cylinder and will generally also climb the outer cylinder if

$$2\rho \left[r_2 \left\{ \left(\frac{r_2}{r_1} \right)^m - 1 \right\} \right]^2 > m^3 \left\{ \gamma_0 + 2\gamma_1 A^2 r_2^{-2m} + 3\gamma_2 (A^2 r_2^{-2m})^2 + \dots \right\}. \quad \dots \quad (1.23)$$

Case (ii): The liquid will tend to fall slightly at the outer cylinder. At the inner cylinder the liquid will rise or fall according as

$$m^3 \left\{ \gamma_0 + 2\gamma_1 A^2 r_1^{-2m} + 3\gamma_2 (A^2 r_1^{-2m})^2 + \dots \right\} \geq 2 \left[r_1 \left\{ 1 - \left(\frac{r_1}{r_2} \right)^m \right\} \right]^2 \rho. \quad (1.24)$$

The case when $r_1 \ll r_2$

This leads to the criterion that the liquid will tend to climb the inner cylinder if

$$r_1 < \sqrt{\frac{m^3}{2\rho} \{ \gamma_0 + 2\gamma_1 A^2 r_1^{-2m} + 3\gamma_2 (A^2 r_1^{-2m})^2 + \dots \}} \quad \dots \quad (1.25)$$

and fall away from the inner cylinder otherwise.

(c) Let now $\gamma = \gamma_0 + \gamma_1 (\omega' r)^2, \quad \dots \quad \dots \quad \dots \quad (1.26)$

where $\gamma_0 > 0, \gamma_1 < 0,$
and γ_1 is so small that γ is positive definite,

i.e. $\gamma_0 > |\gamma_1| A^2 r^{-2m}, \quad \dots \quad \dots \quad \dots \quad (1.27)$

Case (i): In this case the liquid will tend to rise at the inner cylinder if

$$\gamma_0 > 2\gamma_1 A^2 r_1^{-2m},$$

and will tend to fall if

$$\gamma_1 A^2 r_1^{-2m} < \gamma_0 < 2\gamma_1 A^2 r_1^{-2m}. \quad \dots \quad \dots \quad (1.28)$$

The liquid will climb at the outer cylinder also if

$$2\rho \left[r_2 \left\{ \left(\frac{r_2}{r_1} \right)^m - 1 \right\} \right]^2 > m^3 \{ \gamma_0 - 2\gamma_1 A^2 r_2^{-2m} \}. \quad \dots \quad (1.29)$$

Case (ii): In this case the liquid will tend to fall slightly at the outer cylinder if

$$\gamma_0 > 2\gamma_1 A^2 r_2^{-2m}, \quad \dots \quad \dots \quad \dots \quad (1.30)$$

and will tend to rise if

$$\gamma_1 A^2 r_2^{-2m} < \gamma_0 < 2\gamma_1 A^2 r_2^{-2m}. \quad \dots \quad \dots \quad (1.31)$$

At the inner cylinder the liquid will rise or fall according as

$$m^3 \{ \gamma_0 - 2\gamma_1 A^2 r_1^{-2m} \} \geq 2\rho \left[r_1 \left\{ 1 - \left(\frac{r_1}{r_2} \right)^m \right\} \right]^2. \quad \dots \quad (1.32)$$

The case when $r_1 \ll r_2$

The liquid will tend to climb at the inner cylinder if

$$r_1 < \sqrt{\frac{m^3}{2\rho} (\gamma_0 - 2\gamma_1 A^2 r_1^{-2m})} \quad \dots \quad \dots \quad (1.33)$$

and fall at the inner cylinder otherwise.

The above conclusions lead us to confirm that the 'centripetal pump effect' is entirely due to cross-viscosity of the liquid and the present investigation supports the conclusions of Serrin in the case of Couette flow that the strangulation of liquid can be explained in terms of cross-viscosity.

2. Poiseuille flow

We consider a steady laminar flow through a long straight pipe of circular cross-section of diameter $2a$. We shall use the cylindrical polar co-ordinate system (r, θ, z) where z -axis is taken along the axis of the pipe which is placed in a vertical

position and r is the perpendicular distance from this axis. Assuming axial symmetry we shall take $\frac{\partial}{\partial \theta} = 0$. For a flow parallel to the axis of the pipe we have

$$u_r = 0, u_\theta = 0, u_z = \omega(r, z). \quad \dots \quad (2.1)$$

In view of the equation of continuity, ω is a function of r only,

i.e.
$$u_z = \omega(r).$$

For such a motion the rate of deformation tensor D takes the form

$$D = \frac{1}{2} \begin{pmatrix} 0 & 0 & \omega' \\ 0 & 0 & 0 \\ \omega' & 0 & 0 \end{pmatrix},$$

so that

$$P = -p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \beta \omega' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{4} \gamma \omega'^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \dots \quad (2.1)$$

where β and γ are functions of the second invariant of the rate of deformation tensor D and the temperature of the fluid,

i.e.
$$\beta = \beta(-\frac{1}{4}\omega'^2, T),$$

and
$$\gamma = \gamma(-\frac{1}{4}\omega'^2, T).$$

Assuming that β and γ do not involve T explicitly or T is a function of r only, we take

$$\beta = \beta_0(\omega'^2)^n, \quad n > 0, \quad \dots \quad (2.3)$$

$$\gamma = \gamma_0 + \gamma_1(\omega'^2) + \gamma_2(\omega'^2)^2 + \dots, \quad \dots \quad (2.4)$$

where $\beta_0, \gamma_0, \gamma_1, \gamma_2, \dots$ are constants.

The momentum equations are :

$$0 = -\frac{\partial p}{\partial r} + \frac{1}{r} \left\{ \frac{\partial}{\partial r} (\frac{1}{4} \gamma r \omega'^2) + \frac{\partial}{\partial z} (\frac{1}{2} r \beta \omega') \right\}, \quad \dots \quad (2.5)$$

$$\rho g = -\frac{\partial p}{\partial z} + \frac{1}{r} \left\{ \frac{\partial}{\partial r} (\frac{1}{2} r \beta \omega') + \frac{\partial}{\partial z} (\frac{1}{4} r \gamma \omega'^2) \right\}. \quad \dots \quad (2.6)$$

These equations have the integrals

$$\beta \omega' = cr, \quad \dots \quad (2.7)$$

$$p = (c - \rho g)z + f(r), \quad \dots \quad (2.8)$$

where

$$f(r) = \frac{c^2}{4} \left\{ \frac{\gamma r^2}{\beta^2} + \int \frac{\gamma r}{\beta^2} dr \right\}.$$

Using the values of β and γ as given by (2.3) and (2.4) we have the solution in the form

$$\omega = \frac{A}{m} r^m + K, \quad \dots \quad (2.9)$$

where

$$A = \left(\frac{c}{\beta_0}\right)^{m-1} \quad \text{and} \quad m = \frac{2n+2}{2n+1}.$$

The constant K is evaluated by applying the boundary condition $\omega(a) = 0$.

We get

$$\omega = -\frac{A}{m}(a^m - r^m), \quad \dots \dots \dots \dots \quad (2.10)$$

and

$$p = \frac{A^2 r^{2(m-1)}}{16(m-1)} [2\gamma_0(2m-1) + \gamma_1(4m-3)A^2 r^{2(m-1)} + \dots] + (c - \rho g)z + B. \quad (2.11)$$

To complete the solution, we find the stress components for the state of motion considered :

$$p_{rr} = p_{zz} = (\rho g - c)z - \frac{A^2}{16(m-1)} r^{2(m-1)} \left\{ 2\gamma_0 + \gamma_1 A^2 r^{2(m-1)} + \dots \right\} - B, \quad (2.12)$$

$$p_{\theta\theta} = (\rho g - c)z - \frac{A^2}{16(m-1)} r^{2(m-1)} \left\{ 2\gamma_0(2m-1) + \gamma_1(4m-3)A^2 r^{2(m-1)} + \dots \right\} - B, \quad (2.13)$$

$$p_{r\theta} = p_{\theta z} = 0,$$

$$p_{rz} = \frac{1}{2} cr.$$

The total mass flux is

$$M = - \int \int \rho \omega r dr d\theta = \frac{\pi \rho A a^{m+2}}{m+2} \dots \dots \dots \dots \quad (2.14)$$

We notice that the normal stress on the pipe wall varies linearly along the length of the pipe.

The effect of γ is felt in the non-uniform distribution of pressure across a section of the pipe. To investigate the effect of γ , we may suppose the fluid to issue from the pipe into the atmosphere at pressure p_0 , the latter exerting a force on the output cross-section of the amount $\pi a^2 p_0$, such that

$$\pi a^2 p_0 = - \int_0^a p_{zz} \cdot 2\pi r dr \dots \dots \dots \dots \quad (2.15)$$

Using this equation to eliminate the constant B in (2.11) we get the formula

$$p_{zz} = p_{rr} = (\rho g - c)z + \frac{A^2}{16(m-1)} \left[2\gamma_0 \left\{ \frac{a^{2(m-1)}}{m} - r^{2(m-1)} \right\} + \gamma_1 A^2 \left\{ \frac{a^{4(m-1)}}{2m-1} - r^{4(m-1)} \right\} + \dots \right] - p_0. \quad (2.16)$$

Let P be the force per unit area which the fluid exerts on the pipe walls, so that $P = -p_{rr}$ evaluated at the wall.

From (2.16) we have

$$P - p_0^* = cz + \left(\frac{\Gamma}{a}\right)^2 \left[\frac{\gamma_0}{8m} + \frac{\gamma_1}{8(2m-1)} (m+2)^2 \left(\frac{\Gamma}{a}\right)^2 + \dots \right] (m+2)^2, \quad (2.17)$$

where

$$p^* = p_0 - \rho gz$$

and

$$\Gamma = \frac{M}{\pi a^2 \rho} = \frac{A a^m}{m+2} = \text{volume of the flow per second per unit cross-section area.} \quad \dots (2.18)$$

According to (2.17) the excess pressure at the exit section is

$$P - p_0 = \left(\frac{\Gamma}{a}\right)^2 \left\{ \frac{\gamma_0}{8m} + \frac{\gamma_1}{8(2m-1)} (m+2)^2 \left(\frac{\Gamma}{a}\right)^2 + \dots \right\} (m+2)^2. \quad \dots (2.19)$$

When the fluid emerges from the tube, the result will be a 'swelling' of the emergent column of liquid as has been observed by Merrington (1943). Equation (2.19) shows that, at the exit section, the excess of pressure in the fluid can be caused by the presence of $\gamma_0, \gamma_1, \gamma_2, \dots$ if we assume them to be positive which implies that γ must be positive definite in the indeterminate $(\omega')^2$. Therefore the equation (2.19) implies that the swelling should be emphasized by high flux and small pipe radius.

Merrington attributes this swelling of the fluid to the elasticity of the liquid, but the present work supports Reiner's viewpoint that the effect must be entirely due to cross-viscosity of the liquid.

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