please write name legibly

Math 225, Fall 2025 **Quiz #10** **Name:** ____________________

Find the general solution of $\vec{x}\,' = A\vec{x}$, where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$.

Solution: [Like Exercises 1–5 of section 6.5]

We begin with solving the eigenvalue problem $A\vec{\xi} = \lambda\vec{\xi}$. The eigenvalues are solution to the algebraic equation $\det(A - \lambda I) = 0$. We have

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{bmatrix},$$

and therefore

$$\det(A - \lambda I) = (-1-\lambda)(-1-\lambda) + 9 = (1+\lambda)^2 + 9.$$

It follows that $1 + \lambda = \pm 3i$, and therefore the eigenvalues are $\lambda = -1 \pm 3i$.

Next, we calculate the eigenvector $\vec{\xi}$ corresponding to the eigenvalue $\lambda = -1 + 3i$ by solving the linear system $(A - \lambda I)\vec{\xi} = \vec{0}$. We have

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 3 \\ -3 & -1-\lambda \end{bmatrix} = \begin{bmatrix} -1-(-1+3i) & 3 \\ -3 & -1-(-1+3i) \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix},$$

and therefore $(A - \lambda I)\vec{\xi} = \vec{0}$ takes the form

$$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which amounts to $-3i\xi_1 + 3\xi_2 = 0$, that is, $i\xi_1 = \xi_2$. We pick $\xi_1 = 1$, $\xi_2 = i$, and arrive at

$$\vec{\xi} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{a} + i\,\vec{b},$$

where

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then, the DE's general solution is expressed as

$$\vec{x}(t) = c_1 e^{-t}\left[\vec{a}\cos 3t - \vec{b}\sin 3t\right] + c_2 e^{-t}\left[\vec{a}\sin 3t + \vec{b}\cos 3t\right],$$

which (optionally) may be written in the explicit form

$$\vec{x}(t) = c_1 e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos 3t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 3t\right) + c_2 e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin 3t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 3t\right),$$

which collapses to

$$\vec{x}(t) = e^{-t}\begin{bmatrix} c_1\cos 3t + c_2\sin 3t \\ -c_1\sin 3t + c_2\cos 3t \end{bmatrix}.$$

Yet another way is to express the solution in terms of the fundamental matrix

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t}\cos 3t & e^{-t}\sin 3t \\ -e^{-t}\sin 3t & e^{-t}\cos 3t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$