

Math 225, Fall 2025

Quiz #5

Name: _____

Solve the initial value problem $y'' + y = 2 \cos x$, $y(0) = 1$, $y'(0) = 0$. You may choose to do that through the method of judicious guessing or through variation of parameters.

Solution: [Like exercises 23–26 of section 3.7]

The general solution is the sum of homogeneous and particular solutions, as in $y(x) = y_h(x) + y_p(x)$. We begin with solving the homogeneous equation $y_h'' + y_h = 0$. The characteristic equation $r^2 + 1 = 0$ has roots $r_1 = -i$, $r_2 = i$, and therefore

$$y_h(x) = c_1 \cos x + c_2 \sin x. \quad [1 \text{ pt}]$$

As to $y_p(x)$, we may find it via either of the following two methods.

Finding $y_p(x)$ via the method of judicious guessing

We observe that the DE's right-hand side is a special case of the template

$$e^{\alpha x} [P_n(x) \cos \beta x + Q_n(x) \sin \beta x],$$

where

$$\alpha = 0, \quad \beta = 1, \quad n = 0.$$

Therefore, we look for a particular solution of the form

$$y_p(x) = x^s [A_0 \cos x + B_0 \sin x]$$

To determine the exponent s , we form the (generally complex) number $z = \alpha + i\beta = i$. Since z equals r_2 but not r_1 , we conclude that $s = 1$, and the candidate for $y_p(x)$ reduces to

$$y_p(x) = x [A_0 \cos x + B_0 \sin x]. \quad [3 \text{ pts}]$$

We need $y_p''(x)$. We may calculate that either by differentiating $y_p(x)$ twice, or through the differentiation formula $(u(x)v(x))'' = u''(x)v(x) + 2u'(x)v'(x) + u(x)v''(x)$ which we have seen in class a few times, which is quicker:

$$y_p''(x) = 2[-A_0 \sin x + B_0 \cos x] + x[-A_0 \cos x - B_0 \sin x]$$

Plugging this into the DE we get

$$2[-A_0 \sin x + B_0 \cos x] + x[-A_0 \cos x - B_0 \sin x] + x[A_0 \cos x + B_0 \sin x] = 2 \cos x,$$

which simplifies to

$$2B_0 \cos x - 2A_0 \sin x = 2 \cos x.$$

It follows that $A_0 = 0$ and $B_0 = 1$, and therefore

$$y_p(x) = x \sin x. \quad [3 \text{ pts}]$$

We thus arrive at the general solution

$$y(x) = c_1 \cos x + c_2 \sin x + x \sin x.$$

We see that

$$y'(x) = -c_1 \sin x + c_2 \cos x + \sin x + x \cos x,$$

and therefore the initial conditions imply that

$$1 = c_1, \quad 0 = c_2.$$

We conclude that the solution of the initial value problem is

$$y(x) = \cos x + x \sin x. \quad [3 \text{ pts}]$$

Finding $y_p(x)$ via the method of variation of parameters

The solution $y_h(x)$ obtained earlier is a linear combination of $y_1(x) = \cos x$ and $y_2(x) = \sin x$. According to the method of variation of parameters, a particular solution is

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

where

$$v_1'(x) = -\frac{y_2(x)f(x)}{a(x)W(y_1, y_2)}, \quad v_2'(x) = \frac{y_1(x)f(x)}{a(x)W(y_1, y_2)}, \quad [2 \text{ pts}]$$

and where $f(x) = 2 \cos x$ is the DE's right-hand side, $a(x) = 1$ is the coefficient of the 2nd order derivative, and $W(y_1, y_2)$ is the Wronskian

$$W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos^2 x + \sin^2 x = 1. \quad [1 \text{ pts}]$$

We thus arrive at

$$v_1'(x) = -2 \sin x \cos x, \quad v_2'(x) = 2 \cos^2 x = 1 + \cos 2x.$$

We integrate to get

$$v_1(x) = \cos^2 x, \quad v_2(x) = x + \frac{1}{2} \sin 2x = x + \sin x \cos x. \quad [3 \text{ pts}]$$

We conclude that

$$\begin{aligned} y_p(x) &= v_1(x)y_1(x) + v_2(x)y_2(x) = \cos^2 x \cos x + (x + \sin x \cos x) \sin x \\ &= (\cos^2 x + \sin^2 x) \cos x + x \sin x = \cos x + x \sin x, \end{aligned}$$

where we have suppressed the integration constants.

Note that this $y_p(x)$ is *different from* the $y_p(x)$ obtained earlier. That's not an error. Particular solutions are not unique. Both are acceptable. The current $y_p(x)$ leads to the general solution

$$y(x) = y_h(x) + y_p(x) = c_1 \cos x + c_2 \sin x + \cos x + x \sin x.$$

The next-to-last $\cos x$ term may be combined with $c_1 \cos x$, as in

$$y(x) = (1 + c_1) \cos x + c_2 \sin x + x \sin x.$$

Since c_1 is arbitrary, so is $1 + c_1$. We rename it to c_1 and arrive at the general solution

$$y(x) = c_1 \cos x + c_2 \sin x + x \sin x,$$

which agrees with the previously obtained general solution.

Then we apply the initial conditions as before and obtain $c_1 = 1$, $c_2 = 0$, and therefore

$$y(x) = \cos x + x \sin x. \quad [3 \text{ pts}]$$

Remark: Compare the solutions obtained through the two different methods. Which is easier?