please write name legibly

Math 225, Fall 2025 **Quiz #4** **Name:** ____________________

Find the general solution of the nonhomogeneous differential equation $y'' - y = 4xe^{-x}$.

Solution: [Like exercise 20 of section 3.7]

The general solution is the sum of homogeneous and particular solutions, as in $y(x) = y_h(x) + y_p(x)$. We begin by solving the homogeneous equation $y'' - y = 0$. The characteristic equation $r^2 - 1 = 0$ factorizes as $(r+1)(r-1) = 0$ and yields the roots

$$r_1 = -1, \quad r_2 = 1,$$

whence

$$y_h(x) = c_1 e^{-x} + c_2 e^{x}. \qquad \text{[2 pts]}$$

As to $y_p(x)$, we observe that the DE's right-hand side is a special case of the template

$$e^{\alpha x}\Big[P_n(x)\cos\beta x + Q_n(x)\sin\beta x\Big],$$

where

$$\alpha = -1, \quad \beta = 0, \quad n = 1.$$

Therefore, we look for a particular solution of the form

$$y_p(x) = x^s e^{-x}(A_0 + A_1 x). \qquad \text{[2 pts]}$$

To determine the exponent s, we form the (generally complex) number $z = \alpha + i\beta = -1$. Since z equals r_1 but not r_2, we conclude that $s = 1$, [2pts] and the candidate for $y_p(x)$ reduces to

$$y_p(x) = xe^{-x}(A_0 + A_1 x),$$

that is

$$y_p(x) = e^{-x}(A_0 x + A_1 x^2).$$

We calculate the first derivative of $y_p(x)$ through the usual product formula and we get

$$\begin{aligned} y_p'(x) &= -e^{-x}(A_0 x + A_1 x^2) + e^{-x}(A_0 + 2A_1 x) \\ &= e^{-x}\Big[A_0 + (2A_1 - A_0)x - A_1 x^2\Big], \end{aligned}$$

then we differentiate once more:

$$\begin{aligned} y_p''(x) &= -e^{-x}\Big[A_0 + (2A_1 - A_0)x - A_1 x^2\Big] + e^{-x}\Big[(2A_1 - A_0) - 2A_1 x\Big] \\ &= e^{-x}\Big[(2A_1 - 2A_0) + (A_0 - 4A_1)x + A_1 x^2\Big]. \end{aligned}$$

Substituting these into the DE we get

$$e^{-x}\Big[(2A_1 - 2A_0) + (A_0 - 4A_1)x + A_1 x^2\Big] - e^{-x}(A_0 x + A_1 x^2) = 4xe^{-x}.$$

We divide through by the common factor e^{-x} and then group the remaining terms:

$$(2A_1 - 2A_0) - 4A_1 x = 4x. \qquad \text{[3 pts]}$$

It follows that

$$\begin{cases} 2A_1 - 2A_0 = 0, \\ \qquad\; -4A_1 = 4. \end{cases}$$

and consequently $A = B = -1$. We conclude that

$$y_p(x) = -x(x+1)e^{-x},$$

and therefore the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{x} - x(x+1)e^{-x}. \qquad \text{[1 pt]}$$