please write name legibly

Math 225, Fall 2025 **Final Exam** **Name:** ____________________

- Please make an effort to write neatly, and *insert a few words where necessary* to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words.
  I will award up to 2 bonus points if I find your work well-documented and easy to read and understand.
- *No books, notes, calculators or other electronic devices on this exam.*
- Use the reverse sides of the pages or the extra blank sheets at the end if you need them.
- There are five problems, each worth 10 points.

Systems with complex eigenvalues. If the matrix A has complex eigenvalues $\alpha \pm i\beta$, and corresponding eigenvectors $\vec{a} \pm i\vec{b}$, then the system $\vec{x}\,' = A\vec{x}$ has a pair of linearly independent solutions

$$\vec{x}^{(1)}(t) = e^{\alpha t}\left[\vec{a}\cos\beta t - \vec{b}\sin\beta t\right],$$
$$\vec{x}^{(2)}(t) = e^{\alpha t}\left[\vec{a}\sin\beta t + \vec{b}\cos\beta t\right].$$

Cheers!

1. Solve the initial value problem

$$y'' + 6y' + 10y = \delta(t-\pi) + \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1,$$

where δ is Dirac's delta.

Solution: [Like Exercise 13 of section 5.7]

We apply the Laplace transform to the equation and obtain

$$\Big[s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0)\Big] + 6\Big[s\mathscr{L}\{y(t)\} - y(0)\Big] + 10\mathscr{L}\{y(t)\} = e^{-\pi s} + e^{-2\pi s}.$$

We account for the given initial conditions and rearrange the result into

$$(s^2 + 6s + 10)\mathscr{L}\{y(t)\} = 1 + e^{-\pi s} + e^{-2\pi s},$$

and therefore

$$\mathscr{L}\{y(t)\} = \frac{1}{s^2 + 2s + 2} + \frac{1}{s^2 + 2s + 2}e^{-\pi s} + \frac{1}{s^2 + 2s + 2}e^{-2\pi s}.$$

It suffices to find the inverse transform of the first term on the right-hand side. The other two terms produce the delayed versions of the that.

We have

$$\frac{1}{s^2 + 6s + 10} = \frac{1}{(s+3)^2 + 1},$$

Considering that $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$, we have $\mathscr{L}\{e^{-3t}\sin t\} = \frac{1}{(s+3)^2+1}$, and therefore

$$y(t) = e^{-3t}\sin t + u(t-\pi)\Big[e^{-3(t-\pi)}\sin(t-\pi)\Big] + u(t-2\pi)\Big[e^{-3(t-2\pi)}\sin(t-2\pi)\Big].$$

2. Find the general solution of the differential equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, where

$$A = \begin{bmatrix} -1 & -3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution: [Like Exercises 13–15 of section 6.4]

We begin with finding the eigenvalues and eigenvectors of A. We have

$$A - \lambda I = \begin{bmatrix} -1-\lambda & -3 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix},$$

We see that $\det(A - \lambda I) = (-1-\lambda)(-2-\lambda)^2$, and therefore the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -2$.

We find an eigenvector $\vec{\xi}^{(1)}$ corresponding to the eigenvalue $\lambda_1 = -1$ through solving the system $(A - \lambda_1 I)\vec{\xi}^{(1)} = \vec{0}$, that is,

$$\begin{bmatrix} 0 & -3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first and second equations say that $\xi_2 = 0$, while the third equation says that $\xi_3 = 0$. We taken $\xi = 1$ and arrive at

$$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next, we look for eigenvectors $\vec{\xi}$ corresponding to λ_2 and λ_3. We have

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

That says $\xi_1 = 3\xi_2$, and that ξ_3 is arbitrary. So we pick

$$\vec{\xi}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{\xi}^{(3)} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

and conclude that the general solution is

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

3. Find the general solution of the system $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Solution: [Like Exercise 5 of section 6.5]

We begin with finding the eigenvalues and eigenvectors of A. We have

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 1 & 0 \\ -4 & -1-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{bmatrix},$$

and

$$\begin{aligned} \det(A - \lambda I) &= (-1-\lambda)^2(-3-\lambda) + 4(-3-\lambda) \\ &= \left[(-1-\lambda)^2 + 4\right](-3-\lambda) = -(\lambda^2 + 2\lambda + 5)(3+\lambda) \end{aligned}$$

We conclude that the eigenvalues are

$$\lambda_1 = -3, \quad \lambda_2 = -1 + 2i, \quad \lambda_3 = -1 - 2i.$$

We find an eigenvector $\vec{\xi}^{(1)}$ corresponding to the eigenvalue $\lambda_1 = -3$ through solving the system $(A - \lambda_1 I)\vec{\xi}^{(1)} = \vec{0}$, that is,

$$\begin{bmatrix} 2 & 1 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that $\xi_1 = \xi_2 = 0$ and ξ_3 is arbitrary, so we take

$$\vec{\xi}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and arrive at a solution

$$\vec{x}^{(1)}(t) = e^{-3t}\,\vec{\xi}^{(1)}.$$

Next, we look for an eigenvector $\vec{\xi}^{(2)}$ corresponding to $\lambda_2 = -1 + 2i$. We have

$$A = \begin{bmatrix} -2i & 1 & 0 \\ -4 & -2i & 0 \\ 0 & 0 & -2+i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

The third equation says $\xi_3 = 0$. The first and second equations both amount to $\xi_2 = 2i\xi_1$, so we pick

$$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 2i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \vec{a} + i\vec{b}.$$

This results in the two solutions

$$\vec{x}^{(2)}(t) = e^{-t}\left[\vec{a}\cos 2t - \vec{b}\sin 2t\right], \qquad \vec{x}^{(3)}(t) = e^{-t}\left[\vec{a}\sin 2t + \vec{b}\cos 2t\right],$$

and we conclude that the system's general solution is

$$\begin{aligned}
\vec{x}(t) &= c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) + c_3\vec{x}^{(3)}(t) \\
&= c_1 e^{-3t}\begin{bmatrix}0\\0\\1\end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cos 2t - \begin{bmatrix}0\\2\\0\end{bmatrix}\sin 2t\right) + c_3 e^{-t}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\sin 2t + \begin{bmatrix}0\\2\\0\end{bmatrix}\cos 2t\right).
\end{aligned}$$

4. Find the general solution of the differential equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{f}(t)$, where

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}, \quad \vec{f}(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

Solution: [Like Exercises 4–8 of section 6.6]

We begin with calculating A's eigenvalues. We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{bmatrix} = (2-\lambda)(-5-\lambda) + 12 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2),$$

and therefore the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$.
The eigenvector $\vec{\xi}^{(1)}$ corresponding to λ_1 is obtained by solving $(A - \lambda_1 I)\vec{\xi}^{(1)} = \vec{0}$, that is,

$$\begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which amounts to the single equation $\xi_1 = 4\xi_2$, so we pick $\vec{\xi}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Similarly, the eigenvector $\vec{\xi}^{(2)}$ corresponding to λ_2 is obtained by solving $(A - \lambda_2 I)\vec{\xi}^{(2)} = \vec{0}$, that is,

$$\begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which amounts to the single equation $\xi_1 = 3\xi_2$, so we pick $\vec{\xi}^{(2)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

We introduce the functions

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t}\vec{\xi}^{(1)} = e^{-t}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \vec{x}^{(2)}(t) = e^{\lambda_2 t}\vec{\xi}^{(2)} = e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

and conclude that the general solution of the *homogeneous system* is

$$\vec{x}^{(h)} = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t).$$

Toward finding a particular solution, we construct the 2×2 fundamental matrix

$$X(t) = \left[\ \vec{x}^{(1)}(t) \ \vdots \ \vec{x}^{(2)}(t)\ \right] = \begin{bmatrix} 4e^{-t} & 3e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix},$$

and search for a particular solution of the *nonhomogeneous system* in he form

$$\vec{x}^{(p)} = X(t)\vec{u}(t)$$

We know that $\vec{u}(t)$ is the solution of $X(t)\vec{u}\,'(t) = \vec{f}(t)$, which expands to

$$\begin{bmatrix} 4e^{-t} & 3e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix},$$

that is,

$$
\begin{aligned}
4e^{-t}u_1'(t) + 3e^{-2t}u_2'(t) &= e^{-t}, \\
e^{-t}u_1'(t) + e^{-2t}u_2'(t) &= 0.
\end{aligned}
$$

From the second equation we get $u_1'(t) = -e^{-t}u_2'(t)$. Plugging this into the first equation we arrive at

$$-4e^{-2t}u_2'(t) + 3e^{-2t}u_2'(t) = e^{-t},$$

whence $u_2'(t) = -e^{t}$, and consequently, $u_1'(t) = 1$.
It follows that

$$u_1(t) = t, \quad u_2(t) = -e^{t}.$$

and therefore

$$\vec{x}^{(p)} = X(t)\vec{u}(t) = \begin{bmatrix} 4e^{-t} & 3e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} t \\ -e^{t} \end{bmatrix} = \begin{bmatrix} 4te^{-t} - 3e^{-t} \\ te^{-t} - e^{-t} \end{bmatrix} = \begin{bmatrix} (4t-3)e^{-t} \\ (t-1)e^{-t} \end{bmatrix}.$$

We conclude that

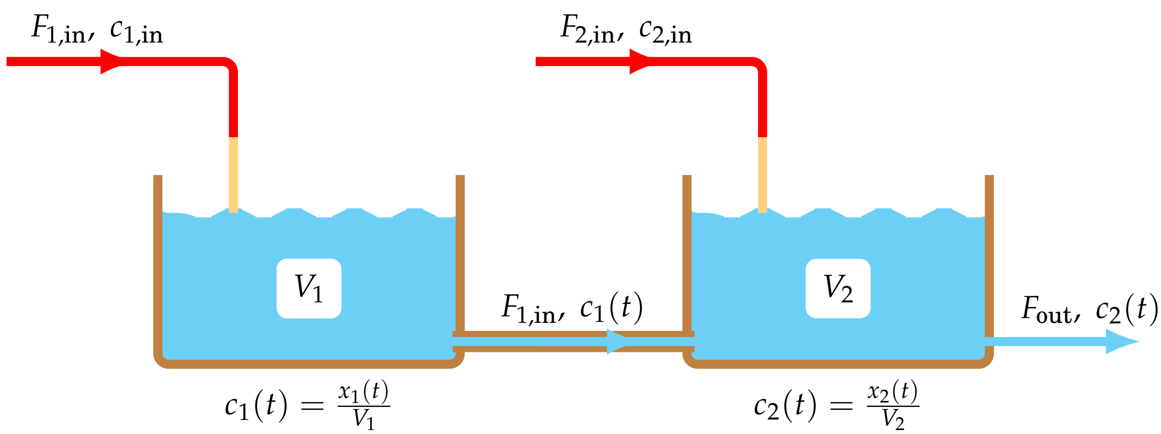
$$\vec{x}(t) = \vec{x}^{(h)} + \vec{x}^{(p)} = c_1e^{-t}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} (4t-3)e^{-t} \\ (t-1)e^{-t} \end{bmatrix}.$$

5. The diagram below depicts two interconnected tanks. Water flows into tanks 1 and 2 at the volumetric rates (that is, volume per unit time) of $F_{1,\text{in}}$ and $F_{2,\text{in}}$, respectively. These carry with them solutes of concentrations (that is, mass per unit volume) of $c_{1,\text{in}}$ and $c_{2,\text{in}}$. Let $x_1(t)$ and $x_2(t)$ be the masses of the solutes in the two tanks at any time t.

   The volumes in the two tanks are kept constant. Thus, the inflow and outflow rates from tank 1 are both $F_{1,\text{in}}$, and the outflow rate F_{out} of tank 2 equals the sum of its inflow rates, that is $F_{\text{out}} = F_{1,\text{in}} + F_{2,\text{in}}$.

   Find the linear system $\vec{x}\,' = A\vec{x} + \vec{f}$ that describes the mass of solute in each tank, where A is a 2×2 matrix and $\vec{f}$ is a column vector. Solving that system is not at all difficult, but you are not required to do it on this exam.

   Hint: Concentrations of the solutes in the two tanks are $c_1(t) = \frac{x_1(t)}{V_1}$ and $c_2(t) = \frac{x_2(t)}{V_2}$.



Solution: Solution: [Like Exercise 5 of section 6.8]

Balance of masses in tank 1.

The mass of solute entering tank 1 per unit time is $F_{1,\text{in}}c_{1,\text{in}}$. The concentration of the solute in the tank is $c_1(t) = x_1(t)/V_1$. Therefore the mass leaving tank 1 per unit time is $F_{1,\text{in}}c_1(t)$, that is, $F_{1,\text{in}}x_1(t)/V_1$. The difference of the rates of inflow and outflow equals the rate of change of the mass of solute in the tank, that is

$$x_1'(t) = F_{1,\text{in}}c_{1,\text{in}} - F_{1,\text{in}}\frac{x_1(t)}{V_1}.$$

Balance of masses in tank 2.

Solute enters from two sources into tank 2. It pours in at the rate of $F_{2,\text{in}}c_{2,\text{in}}$ from the top (see the diagram) and comes in at the rate of $F_{1,\text{in}}c_1(t)$, that is, $F_{1,\text{in}}\frac{x_1(t)}{V_1}$, from the bottom. The concentration of the solute in the tank is $x_2(t)/V_2$. Therefore the mass leaving tank 2 per unit time is $F_{1,\text{in}}c_2(t)$, that is, $F_{\text{out}}x_2(t)/V_2$. The difference of the rates of inflow and outflow equals the rate of change of the mass of solute in the tank, that is

$$x_2'(t) = F_{2,\text{in}}c_{2,\text{in}} + F_{1,\text{in}}\frac{x_1(t)}{V_1} - F_{\text{out}}\frac{x_2(t)}{V_2}.$$

We combine the two differential equations into a system:

$$\frac{d}{dx}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{F_{1,\text{in}}}{V_1} & 0 \\ \frac{F_{1,\text{in}}}{V_1} & -\frac{F_{\text{out}}}{V_1} \end{bmatrix} + \begin{bmatrix} F_{1,\text{in}}c_{1,\text{in}} \\ F_{2,\text{in}}c_{2,\text{in}} \end{bmatrix}.$$