please write name legibly

Math 225, Fall 2025 **Exam #2** **Name:** ____________________

- Please make an effort to write neatly, and *insert a few words where necessary* to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words.
  I will award up to 2 bonus points if I find your work well-documented and easy to read and understand.
- *No books, notes, calculators or other electronic devices on this exam.*
- Use the reverse sides of the pages or the extra blank sheets at the end if you need them.
- There are four problems, each worth 10 points.

The method of undetermined coefficients. Consider the second order, linear, constant coefficients, nonhomogeneous equation

$$ay'' + by' + cy = e^{\alpha x}\Big[P_n(x)\cos\beta x + Q_n(x)\sin\beta x\Big], \tag{1}$$

where P_n and Q_n are polynomials of up to nth degree. Then the differential equation has a particular solution of the form

$$y_p(x) = x^s e^{\alpha x}\Big[(A_0 + A_1 x + \cdots + A_n x^n)\cos\beta x + (B_0 + B_1 x + \cdots + B_n x^n)\sin\beta x\Big], \tag{2}$$

where A_0 through A_n and B_0 through B_n are constants to be determined.

As to the exponent s, let r_1 and r_2 be the roots of the characteristic equation, and $z \equiv \alpha + i\beta$. If z equals neither of the roots r_1 and r_2, then $s = 0$. If z equals only one of the roots, then $s = 1$. If z equals both roots, then $s = 2$.

Variation of parameters

If the linearly independent functions $y_1(x)$ and $y_2(x)$ are solutions of the *homogeneous* differential equation $a(x)y'' + b(x)y' + c(x)y = 0$, then a particular solution of the *nonhomogeneous* equation $a(x)y'' + b(x)y' + c(x)y = f(x)$, may be obtained through $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$, where

$$v_1'(x) = \frac{-y_2(x)f(x)}{a(x)W(y_1, y_2)}, \quad v_2'(x) = \frac{y_1(x)f(x)}{a(x)W(y_1, y_2)},$$

and where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Cheers!

1. Determine *the form* of the particular solution $y_p(x)$ in each of the following equations. No need find the undermined coefficients. Each of the four questions is worth 2.5 points.

   (a) $y'' + 5y' + 6y = xe^x$

   (b) $y'' + 5y' + 6y = (1 + x^2)e^{-3x}$

   (c) $y'' + 6y' + 9y = (1 + x^2)e^{-3x}$

   (d) $y'' + 6y' + 10y = e^{-3x} \sin x$

   Solution: (a) The characteristic equation $r^2 + 5r + 6 = 0$ has roots $r = -2$ and $r = -3$. The right-hand side matches the general template with $\alpha = 1$, $\beta = 0$, and $n = 1$. We let $z = \alpha + i\beta = 1$. Since z matches neither of the characteristic equation's roots, we have $s = 0$ and therefore a particular solution has the form

$$y_p(x) = x^0(A_0 + A_1 x)e^x = (A_0 + A_1 x)e^x.$$

   (b) The characteristic equation $r^2 + 5r + 6 = 0$ has roots $r = -2$ and $r = -3$. The right-hand side matches the general template with $\alpha = -3$, $\beta = 0$, and $n = 2$. We let $z = \alpha + i\beta = -3$. Since z matches only one of the characteristic equation's roots, we have $s = 1$ and therefore a particular solution has the form

$$y_p(x) = x^1(A_0 + A_1 x + A_2 x^2)e^{-3x} = (A_0 x + A_1 x^2 + A_2 x^3)e^{-3x}.$$

   (c) The characteristic equation $r^2 + 6r + 9 = 0$ has the repeated root $r = -3$. The right-hand side matches the general template with $\alpha = -3$, $\beta = 0$, and $n = 2$. We let $z = \alpha + i\beta = -3$. Since z matches both of the characteristic equation's roots, we have $s = 2$ and therefore a particular solution has the form

$$y_p(x) = x^2(A_0 + A_1 x + A_2 x^2)e^{-3x} = (A_0 x^2 + A_1 x^3 + A_2 x^4)e^{-3x}.$$

   (d) The characteristic equation $r^2 + 6r + 10 = 0$ has roots $r = -3 \pm i$. The right-hand side matches the general template with $\alpha = -3$, $\beta = 1$, and $n = 0$. We let $z = \alpha + i\beta = -3 + i$. Since z matches only one of the characteristic equation's roots, we have $s = 1$ and therefore a particular solution has the form

$$y_p(x) = x^1 e^{-3x}\left[A_0 \cos x + B_0 \sin x\right] = xe^{-3x}\left[A_0 \cos x + B_0 \sin x\right].$$

2. The functions $y_1(x) = x$ and $y_2(x) = x^3$ are solutions of the *homogeneous* differential equation $x^2y'' - 3xy' + 3y = 0$. Apply the method of variation of parameters to find the general solution of the *nonhomogeneous* differential equation $x^2y'' - 3xy' + 3y = x^2 + 3$.

   Solution: [Like Sec 3.8, #14, 15]

   We apply the variation of parameters formula given on the cover sheet to find a particular solution $y_p(x)$ of the nonhomogeneous equation. We begin with calculating the Wronskian of y_1 and y_2:

$$W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} x & x^3 \\ 1 & 3x^2 \end{pmatrix} = 3x^3 - x^3 = 2x^3.$$

   Then, considering that $a(x) = x^2$ and $f(x) = x^2 + 3$, we have

$$\begin{aligned} v_1'(x) &= \frac{-(x^3)(x^2+3)}{(x^2)(2x^3)} = -\frac{1}{2} - \frac{3}{2x^2} = -\frac{1}{2} - \frac{3}{2}x^{-2}, \\ v_2'(x) &= \frac{(x)(x^2+3)}{(x^2)(2x^3)} = \frac{1}{2x^2} + \frac{3}{2x^4} = \frac{1}{2}x^{-2} + \frac{3}{2}x^{-4}. \end{aligned}$$

   It follows that

$$\begin{aligned} v_1(x) &= -\frac{1}{2}x + \frac{3}{2}x^{-1} = -\frac{1}{2}x + \frac{3}{2x}, \\ v_2(x) &= -\frac{1}{2}x^{-1} - \frac{1}{2}x^{-3} = -\frac{1}{2x} - \frac{1}{2x^3}. \end{aligned}$$

   We conclude that

$$\begin{aligned} y_p(x) &= v_1(x)y_1(x) + v_2(x)y_2(x) \\ &= \Big(-\frac{1}{2}x + \frac{3}{2x}\Big)x + \Big(-\frac{1}{2x} - \frac{1}{2x^3}\Big)x^3 = -\frac{1}{2}x^2 + \frac{3}{2} - \frac{1}{2}x^2 - \frac{1}{2} = 1 - x^2, \end{aligned}$$

   and therefore

$$y(x) = c_1x + c_2x^3 + 1 - x^2.$$

3. (a) [5 pts] Evaluate $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{s^2+4s+4}\right\}$

(b) [5 pts] Evaluate $\mathscr{L}^{-1}\left\{\dfrac{4s+6}{s^2+2s+5}\right\}$

Solutions:

(a) [Like exercises 7 and 14 of Section 5.3]

Considering that denominator factorizes as $(s+2)^2$, we split the given expression into partial fractions as

$$\frac{2s+5}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}.$$

We clear the denominators by multiplying through by $(s+2)^2$ and arrive at

$$2s+5 = A(s+2) + B. \tag{$*$}$$

We substitute $s=-2$ into $(*)$ and obtain $\boxed{B=1}$.

To find A, we differentiate $(*)$ with respect to s which results in $2=A$, that is, $\boxed{A=2}$. In conclusion, we have

$$\frac{2s+5}{s^2+4s+4} = \frac{2}{s+2} + \frac{1}{(s+2)^2}. \tag{$**$}$$

The two fractions on the right-hand side of $(**)$ look like shifted (by -2) versions of $\dfrac{2}{s}$ and $\dfrac{1}{s^2}$ whose inverse transforms are 2 and t, respectively. We conclude that

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+4s+4}\right\} = e^{-2t}(2+t).$$

(b) [Like exercises 6, 8, and 13 of Section 5.3]

Completing the square in the denominator and obtain

$$s^2+2s+5 = (s+1)^2+2^2,$$

and observe that this looks like the denominator of the transform of a sine or cosine function sifted by -1. We adjust the numerator accordingly,

$$4s+6 = 4(s+1)+2$$

and therefore

$$\frac{4s+6}{s^2+2s+5} = \frac{4(s+1)}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}.$$

Since

$$\mathscr{L}\{\cos 2t\} = \frac{s}{s^2+2^2}, \quad \mathscr{L}\{\sin 2t\} = \frac{2}{s^2+2^2},$$

and since a shift by -1 in the transform domain amounts to multiplying by e^{-t} in the time domain, we conclude that

$$\mathscr{L}^{-1}\left\{\frac{4s+6}{s^2+2s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{4(s+1)}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right\} = e^{-t}\Big[4\cos 2t + \sin 2t\Big].$$

4. Apply the Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = 2e^{-3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution: [Like exercises 1–14 of Section 5.4]

Applying the Laplace transform to the DE we get

$$\left[s^2 \mathscr{L}\{y(t)\} + sy(0) + y'(0)\right] + 3\left[s\mathscr{L}\{y(t)\} + y(0)\right] + 2\mathscr{L}\{y(t)\} = \frac{2}{s+3}.$$

We substitute for the initial conditions and simplify

$$(s^2 + 3s + 2)\mathscr{L}\{y(t)\} = \frac{2}{s+3},$$

and since $(s^2 + 3s + 2) = (s+1)(s+2)$, we arrive at

$$\mathscr{L}\{y(t)\} = \frac{2}{(s+1)(s+2)(s+3)}.$$

In order to invert the Laplace transform, we split the result into partial fractions:

$$\frac{2}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}. \qquad (*)$$

To find A, we multiply $(*)$ through by $s+1$,

$$\frac{2}{(s+2)(s+3)} = A + (s+1)\cdot\frac{B}{s+2} + (s+1)\cdot\frac{C}{s+3},$$

and set $s = -1$, and arrive at $\dfrac{2}{(-1+2)(-1+3)} = A$, and therefore $A = 1$.

Similarly, to find B, we multiply $(*)$ through by $s+2$,

$$\frac{2}{(s+1)(s+3)} = (s+2)\cdot\frac{A}{s+1} + B + (s+2)\cdot\frac{C}{s+3},$$

and set $s = -2$, and arrive at $\dfrac{2}{(-2+1)(-2+3)} = B$, and therefore $B = -2$.

To find C, we multiply $(*)$ through by $s+3$,

$$\frac{2}{(s+1)(s+2)} = (s+3)\cdot\frac{A}{s+1} + (s+3)\cdot\frac{B}{s+2} + C,$$

and set $s = -3$, and arrive at $\dfrac{2}{(-3+1)(-3+2)} = C$, and therefore $C = 1$.

In conclusion, we have

$$\mathscr{L}\{y(t)\} = \frac{2}{(s+1)(s+2)(s+3)} = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3},$$

and therefore

$$y(t) = e^{-t} - 2e^{-2t} + e^{-3t}.$$